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ON THE BRANCHING OF SOLUTIONS AND SIGNORINI'S PERTURBATION PROCEDURE IN ELASTICITY

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1. Introduction

Let us formally express the traction boundary problem for large deformations of a hyperelastic body in the neighbourhood of a placement $\mathcal{B}_o$ under initial stress S_o as follows

$$
(1.1) \qquad \begin{aligned} - \operatorname{div} S(\nabla p) &= b && \text{in } \mathcal{B}_o , \\ S(\nabla p) n_o &= s && \text{in } \partial \mathcal{B}_o ; \end{aligned}
$$

where b is the force per unit volume and s the surface traction; p is the position vector in the equilibrium placement $\mathcal{B}$ and S is the Piola-Kirchhoff stress tensor,which is expressed by a constitutive relation in terms of the gradient ∇p of p; n_o is the unit vector normal to the boundary $\partial\mathcal{B}_o$ of $\mathcal{B}_o$.

Assume that $\mathcal{B}_o$ (with position vector p_o) is an equilibrium placement under loads (b_o, s_o):

$$
(1.2) \qquad \begin{aligned} - \operatorname{div} S(\nabla p_o) &= b_o && \text{in } \mathcal{B}_o, \\ S(\nabla p_o) n_o &= s_o && \text{in } \partial \mathcal{B}_o, \text{ with } S(\nabla p_o) = S_o . \end{aligned}
$$

Then the perturbation procedure of Signorini starts from the assumption that b and s are dead loads (i.e.,they do not depend on the placement) and can be expressed as power series of a parameter

$$
(1.3) \qquad b = b_o + \sum_{1}^{\infty}{}_h \, \varepsilon^h b_h , \qquad s = s_o + \sum_{1}^{\infty}{}_h \, \varepsilon^h {}_h ,
$$

proceeds with the hypothesis that the solution p of (1.1) is itself developable

$$p = p_o + \sum_1^{\infty} {}_h\, \varepsilon^h u_h \,, \tag{1.4}$$

and admits that the convergence properties of (1.4) are such that u_n is the solution of the appropriate linear problem that can be deduced formally from (1.1), (1.4). Because of the non-linearity of the dependence of S on ∇p, the boundary problem to be satisfied by u_h, although expressed in terms of an operator which does not depend on h, involves "modified" extra loads(b_h^*, s_h^*)

$$(1.5)\qquad \begin{aligned} -\operatorname{div} \mathfrak{S}[H_h] &= b_h^*, && \text{in } \mathcal{B}_o; \\ \mathfrak{S}[H_h]n_o &= s_h^*, && \text{in } \partial\mathcal{B}_o,\ h = 1,2,\ldots \end{aligned}$$

Here $H_h = \nabla u_h$; $\mathfrak{S}$ is the fourth-order tensor of the elasticities

$$\mathfrak{S}[H] = \nabla S \big|_{\nabla p = 1} [H]\,; \tag{1.6}$$

and the starred loads are defined as follows

$$b_h^* = b_h + \operatorname{div} \mathcal{S}_h \,, \qquad s_h^* = s_h - \mathcal{S}_h\, n_o, \tag{1.7}$$

where $\mathcal{S}_h$ can be specified in terms of p_o and H_k (k= 1,2,... h-1) (with $\mathcal{S}_1 \equiv 0$) because it is the quantity entering the expansion

$$S(\nabla p(\varepsilon)) = S_o + \sum_1^{\infty} {}_h (\nabla S \big|_{\nabla p=1} [H_h] + \mathcal{S}_h (p_o; \{H_r\}_1^{h-1}))\, \varepsilon^h. \tag{1.8}$$

When $\mathcal{B}_o$ is a placement at ease (i.e., $S_o \equiv 0$), one can specify conditions on the shape of $\mathcal{B}_o$ and on the function $S(\nabla p)$ so that, when ε is small enough, problem (1.1) has a solution of type (1.4), provided certain quantitative conditions on the loads are satisfied (Stoppelli's theorem). Signorini was mainly concerned with those side conditions, which have an interesting mechanical interpretation and some curious aspects [1]; in [3 , 4] the conditions are also explored at length but without introducing assumptions on S_o. Here the hypothesis that the loads are dead is also abandoned; furthermore, in Sect.4, a dynamic analysis is pursued which clarifies the significance of certain failures of a purely static study.

2. Fredholm conditions.

We abandon here the hypothesis that the loads (b, s) are dead loads and do not exclude that they are of the "follower" type:

$$b = b\,(p, \nabla p) \quad , \quad s = s\,(p, \nabla p), \tag{2.1}$$

so that the developments (1.3) must be substituted by

$$\begin{aligned} b &= b_o + \sum_1^{\infty} {}_h\, \varepsilon^h \Big(\mathfrak{b}[H_h] + Bu_h + \mathfrak{d}_h (p_o, \nabla p_o, \{u_s\}_1^{h-1}, \{H_s\}_1^{h-1}) \Big), \\ s &= s_o + \sum_1^{\infty} {}_h\, \varepsilon^h \Big(\mathfrak{S}[H_h] + \Sigma\, u_h + \mathfrak{s}_h (p_o, \nabla p_o, \{u_s\}_1^{h-1}, \{H_s\}_1^{h-1}) \Big), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \mathfrak{b} &= \operatorname{grad}_{\nabla p} b \;, \quad B = \operatorname{grad}_p b \;, \\ \mathfrak{S} &= \operatorname{grad}_{\nabla p} s \;, \quad \Sigma = \operatorname{grad}_p s. \end{aligned} \tag{2.3}$$

At the same time the systems (1.5) become

$$\begin{aligned} &- \operatorname{div} \mathfrak{S}[H_h] - \mathfrak{b}[H_h] - Bu_h = \bar{b}_h^{*}, && \text{in } \mathcal{B}_o, \\ &\mathfrak{S}[H_h] n_o - \mathfrak{S}[H_n] - \Sigma\, u_h = \bar{s}_h^{*}, && \text{in } \partial\mathcal{B}_o, \end{aligned} \tag{2.4}$$

with

$$\bar{b}_h^{*} = \mathfrak{d}_h + \operatorname{div} \mathcal{S}_h, \quad \bar{s}_h^{*} = \mathfrak{s}_h - \mathcal{S}_h n_o \;, \qquad h = 1, 2, \dots$$

Apart from these qualifications which, at this stage, are of a formal character, a remark of substance is in order here. The loads (b, s) of problem (1.1) are balanced in the final placement

$$\begin{aligned} &\int_{\mathcal{B}_o} b(p, \nabla p)\; d(\mathrm{vol}) + \int_{\partial\mathcal{B}_o} s(p, \nabla p)\; d(\mathrm{surf}) = 0, \\ &\int_{\mathcal{B}_o} p \times b(p, \nabla p)\; d(\mathrm{vol}) + \int_{\partial\mathcal{B}_o} p \times s(p, \nabla p)\; d(\mathrm{surf}) = 0, \end{aligned} \tag{2.5}$$

but these relations do not impose restrictions on (b,s); they are simply an expression of mutual consistency of the (data, solution) pair.Nor need they be verified when p is reduced to p_o in them; though the first reduced one must be true in Signorini's case. Further, if $S_o \equiv 0$, the choice of $\mathcal{B}_o$ is open among isometric placements; then Da Sylva's theorem assures us that among those placement there exists at least one where the moment is balanced. Such freedom in the choice of $\mathcal{B}_o$, however, is absent in the presence of follower loads or when $S_o \neq 0$. In conclusion the balance of loads (b, s) in the starting placement either can be trivially assured, and then it corresponds to a conventional choice of $\mathcal{B}_o$ among the many open choices, or must not be required.

Let us consider now the sequence of linear problems (2.4). The corresponding homogeneous problem

$$(2.6) \qquad \operatorname{div} \mathfrak{S}[H] + \mathfrak{b}[H] + Bu = 0, \quad \text{in } \mathcal{B}_o \qquad \mathfrak{S}[H]\, n_o - \sigma[H] - \Sigma u = 0, \quad \text{in } \partial\mathcal{B}_o,$$

admits a set $\mathcal{S}$ of non-trivial solutions: for instance it is well known that in the presence of dead loads and when $S_o \equiv 0$, $\mathcal{S}$ contains the set $\mathfrak{I}$ of all infinitesimal isometries. In any case the theorem of alternative states a prerequisite for the existence of solutions of (2.4): the loads $(\bar{b}^*_h, \bar{s}^*_h)$ must be orthogonal to all solutions of (2.6)

$$(2.7) \quad \int_{\mathcal{B}_o} v.\bar{b}^*_h\, d(vol) + \int_{\partial\mathcal{B}_o} v.\bar{s}^*_h\, d(surf) = 0, \qquad \forall\, v \in \mathcal{S}.$$

We will consider here only the case when $\mathcal{S}$ is finite-dimensional and non-trivial; after having chosen a basis for $\mathcal{S}$ (v_i (i= 1,2,...r); r = dim $\mathcal{S}$) one can write (2.7) as a set of r equations for $(\bar{b}^*_h, \bar{s}^*_h)$. Notice also that the solution of (2.4) is never unique: adding to any solution $\tilde{u}_h$ a linear combination of v_i yields again a solution. This fact is the basis of the remark that the character of (2.7) for h=1 is completely different from that of (2.7) for $h \geq 2$.

In fact, let us examine first the case h=1; then $\bar{b}^*_1 = \mathfrak{b}_1(p_o, \nabla p_o)$, $\bar{s}^*_1 = \mathfrak{s}_1(p_o, \nabla p_o)$ and therefore

$$(2.8)\qquad \int_{\mathcal{B}_o} v.\,\mathfrak{b}_1\, d(\mathrm{vol}) + \int_{\partial\mathcal{B}_o} v.\,\mathfrak{s}_1\, d(\mathrm{surf}) = 0\,, \quad \forall\ v \in \mathcal{S}\,,$$

represents a condition which is absent in the statement of the original non-linear problem and is of a technical character for the existence of solutions of (1.1) of type (1.4).

Signorini remarked that in his special case, when $\mathcal{S} \equiv \mathfrak{I}$, (2.8) requires the balance of the first order loads in $\mathcal{B}_o$. This interesting remark is, however, misleading to some extent because it leads one to focus exclusive attention on classical balance conditions. An example of Ericksen and Toupin and an example of Bordoni (see [4], Sections 7 and 11) shows explicitly that the "balance conditions" (2.8) required of the first-order loads are much deeper than the classical ones.

3. An analysis of branching of solutions.

Suppose for the moment that (2.8) is satisfied, and a solution $\tilde{u}_1$ of (2.4) for h=1 has been found. Then system $(2.4)_1$ admits also a whole set of solutions

$$(3.1)\qquad u_1 = \tilde{u}_1 + \sum_1^r {}_k\, \gamma_1^k\, v_k,$$

where γ_1^k are r arbitrary real parameters. On the other hand, we have from (2.7) for h=2 the conditions

$$\int_{\mathcal{B}_o} v_k \cdot (\mathfrak{b}_2 + \operatorname{div} \mathfrak{S}_2)\, d(\mathrm{vol}) + \int_{\partial\mathcal{B}_o} v_k \cdot (\mathfrak{s}_2 - \mathfrak{S}_2 n_o)\, d(\mathrm{surf}) = 0,$$

$$k = 1, 2, \ldots. r;$$

here $\mathfrak{b}_2$, $\mathfrak{s}_2$, $\mathfrak{S}_2$ depend on (p_o, ∇p_o and) u_1, ∇u_1, which can be expressed in turn through (3.1). The dependence on u_1, ∇u_1 is algebraic of degree 2, so that finally we come up with an algebraic system of degree 2 for the coefficients γ_1^i :

$$(3.2)\qquad c_{k,1} + \sum_1^r {}_i\, b_{ki}\gamma_1^i + \sum_1^r {}_{ij}\, a_{kij}\, \gamma_1^i\, \gamma_1^j = 0$$

here $\mathcal{C}_{k,1}$, $\mathcal{b}_{ki}$, $\mathcal{a}_{kij}$ have complicated expressions. For instance, in the case of dead loads one has

$$
(3.3)\quad
\begin{aligned}
\mathcal{C}_{k,1} &= \int_{\mathcal{B}_o} (b_2 \cdot v_k - (\nabla^2 s \,|_{\nabla p=1} [\nabla \tilde{u}_1])[\nabla u_1]) \cdot \nabla v_k)) d(vol) + \\
&\qquad + \int_{\partial \mathcal{B}_o} s_2 \cdot v_k \, d(surf), \\
\mathcal{b}_{k,i} &= - \int_{\mathcal{B}_o} ((\nabla^2 s \,|_{\nabla p=1} [\nabla \tilde{u}_1])[\nabla v_i]) \cdot \nabla v_k \, d(vol), \\
\mathcal{a}_{kij} &= - \frac{1}{2} \int_{\mathcal{B}_o} ((\nabla^2 s \,|_{\nabla p=1} [\nabla v_i])[\nabla v_j]) \cdot \nabla v_k d(vol).
\end{aligned}
$$

The set of r algebraic equations (3.2) may be used to determine the parameters γ_1^k. There are cases when such determination can be achieved and is unique; for instance, in Signorini's case all coefficients $\mathcal{a}_{kij}$ vanish because in that case the tensor $\nabla^2 s$ has certain properties of symmetry, whereas ∇v_i is necessarily skew as v_i must represent an infinitesimal isometry; then the system becomes linear and the parameters γ_1^k can be explicitly given, if det $\mathcal{b}_{ki} \neq 0$ (a condition which is equivalent to the requirement that the first order loads do not admit of axes of equilibrium).

In general the search for real solutions of (3.2) is more delicate and the variety of situations reflects the complexities of the cases of branching of solutions in the original non-linear problem. Suffice here to remark that the apparent indetermination observed for u_1 occurs also for all $u_h (h \geq 2)$ and that conditions (2.7) for $h \geq 2$ can be again invoked to overcome such indetermination. On the other hand there are cases where (3.2) cannot be satisfied at all; then the solution of $(2.4)_1$ which seems at stage 1 as a legitimate approximation of first order to a solution of the non-linear problem (1.1) must be rejected on the strength of the Fredholm conditions regarding stage 2; a similar discrepancy may occur at any stage. We devote therefore the next section to an interpretation and an analysis of these cases of "incompatibility".

4. Reformulation of the traction problem in elastodynamics.

Within the limits of a static study, the rejection of an alleged approximation of h-th order because of the failure of the Fredholm conditions at stage h+1 is absolute. Yet we can still give some significance to that approximation, relying on the fact that problem (1.1) can be considered as a special case of a more general dynamic problem (ρ, density in $\mathcal{B}_o$)

$$(4.1)\qquad \begin{aligned} &- \operatorname{div} S(\nabla p) = b - \rho \ddot{p} , && \text{in } \mathcal{B}_o,\\ &S(\nabla p)n_o = s , && \text{in } \partial\mathcal{B}_o,\end{aligned}$$

which requires the assignement of appropriate initial conditions. As we shall see, we will be able to interpret the developments of the preceding Sections as the search for such initial conditions having special properties; our analysis will, at the same time, lead to the complete specification of an "acceptable" approximation (necessarily dynamic) of order h+1.

In fact, suppose that one of the initial conditions to be attached to (4.1) requires the vanishing of the velocity

$$(4.2)\qquad \dot{p}\big|_{t=0} = 0 \qquad\qquad \text{in } \overline{\mathcal{B}}_o;$$

and try to determine $p\big|_{t=0}$ as a function $\tilde{p}$ so that no motion ensues:

$$p \equiv \tilde{p}, \ \forall t.$$

Imagine that determination to proceed in successive stages of approximation, corresponding to the specification (2.2) of the loads.

If $b = b_o$, $s = s_o$, one solution is, by hypothesis, $\tilde{p} = p_o$. When $(b,s) \neq (b_o, s_o)$, to start the process at all, $(\mathfrak{b}_1, \mathfrak{s}_1)$ must satisfy (2.8) as we have already remarked.

But suppose for a moment that condition (2.8) is contravened; then, the following technique may be used to explore the main aspect of the ensuing dynamic phenomenon, and to provide for it an approximate description .

Multiply scalarly the members of

$$\begin{aligned} &\operatorname{div} \mathfrak{S}[H] + \mathfrak{b}[H] + Bu - \mathfrak{b}_1 + \rho \ddot{u} = 0, \\ &\mathfrak{S}[H] n_o - \mathfrak{S}[H] - \Sigma u - \mathfrak{s}_1 = 0 , \end{aligned} \tag{4.3}$$

by v_i (where, as before, v_i form a basis for $\mathcal{S}$); integrate the result respectively over $\mathcal{B}_o$ and $\partial\mathcal{B}_o$ and sum. Finally try to solve the resulting global equation choosing for u the expression

$$u^{\star} = \sum_1^r {}_k \, \gamma_o^k (t) \, v_k . \tag{4.4}$$

It is sufficient for the functions $\gamma_o^k(t)$ to satisfy the ordinary differential system

$$\sum_1^r {}_k \, J_{ik} \ddot{\gamma}_o^k = m_i , \tag{4.5}$$

where the constants J_{ik} are generalized coefficients of inertia

$$J_{ik} = \int_{\mathcal{B}_o} \rho \, v_i \cdot v_k \, d(\mathrm{vol}) , \tag{4.6}$$

m_i are generalized resultant forces

$$m_i = \int_{\mathcal{B}_o} \mathfrak{b}_1 \cdot v_i \, d(\mathrm{vol}) + \int_{\partial\mathcal{B}_o} \mathfrak{s}_1 \cdot v_i \, d(\mathrm{surf}), \tag{4.7}$$

In Signorini's case, when $\mathcal{S} = \mathcal{T}$, eqns (4.4) provide linear approximations to a rigid body motion of $\mathcal{B}$; in general, they describe a much more complex movement of our body:

$$u^{\star} = \frac{1}{2} \left(\sum_1^r {}_{ks} \, v_k ((\overset{-1}{J})^{sk} \, m_s) \right) t^2 . \tag{4.8}$$

At this point it is possible to return to (4.3) and determine $u_1 - u^{\star}$ solving a compatible static boundary value problem where the loads $(\mathfrak{b}_1 - \rho \sum_1^r {}_k \, v_k ((\overset{-1}{J})^{sk} \, m_s), \mathfrak{s}_1)$ involve an apparent body force.

We have dealt in some detail with this almost trivial "case of incompatibility" because it is a simple model for slightly subtler cases of higher order. For instance, suppose that (2.8) applies, so that a solution of (2.4) for h = 1 exists, but (3.2) has no real solutions in γ_1^i; then one must return to the dynamic system

$$\text{(4.8)} \qquad \begin{aligned} &\operatorname{div} \mathfrak{S}[H] + \mathfrak{b}[H] + Bu - \bar{b}_2^{\star} + \rho \ddot{u} = 0, \\ &\mathfrak{S}[H] h_o - \mathfrak{G}[H] - \Sigma u - \bar{s}_2^{\star} = 0, \end{aligned}$$

and proceed as before to obtain a differential relation of the type (4.5), where now the generalized resultant forces are

$$\text{(4.9)} \qquad \begin{aligned} m_i = &\int_{\mathcal{B}_o} (\mathfrak{b}_2 \cdot v_i + \mathfrak{S}_2[H_1] \cdot \nabla v_i)\, d(\mathrm{vol}) + \\ &+ \int_{\partial\mathcal{B}_o} \mathfrak{s}_2 \cdot v_i\, d(\mathrm{surf}). \end{aligned}$$

An acceptable, necessarily dynamic, approximation to (4.8) is a new $u^{\star}$ of type (4.4) with γ_1^k, solution of (4.5), but with the specification (4.9) for m_i, plus a time-independent solution of a static problem involving the appropriate apparent body force.

5. Conclusion.

Signorini's perturbation method is a special case of a general technique, particularly adapted to a study of stability and branching phenomena in hyperelasticity. The interpretation already advanced [2] of the phenomena of incompatibility discovered by Signorini can be extended to apply to more general cases where the ground state is stressed and follower loads are present. In particular the well-known arbitrariness in the choice of amplitude of the buckled shapes within the first approximation can be interpreted either as a temporary freedom soon to be mitigated by conditions of compatibility of higher order systems or as a real scope in the choice of initial placements within the class of placements whence a motion begins where the acceleration is of higher order.

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