Miloš Zlámal

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THE PARABOLIC EQUATIONS AS A LIMITING CASE OF HYPERBOLIC AND ELLIPTIC EQUATIONS

M. Zlámal, Brno

In this lecture I shall deal with some problems concerning partial differential equations with a small parameter at the highest derivatives. Contrary to ordinary differential equations these questions had not appeared in the journals until the course of the last twelve years. It was an article by N. Levinson [1] which drew attention of the mathematicians to these problems. With respect to its importance I will say a few words about it.

In his paper Levinson considers the Dirichlet problem for the equation

$$\varepsilon \, \Delta u + A(x, y) \, u_x + B(x, y) \, u_y + C(x, y) \, u = D(x, y) \tag{1}$$

in an open simply or multiply connected region R with a boundary ∂R consisting of a finite number of simple closed curves. ε is a small positive parameter and the question is whether the solution u of (1) satisfying the given boundary condition

$$u|_{\partial R} = \varphi \tag{2}$$

tends to some solution of the reduced equation

$$A(x, y) \, U_x + B(x, y) \, U_y + C(x, y) \, U = D(x, y) \tag{3}$$

which we get putting $\varepsilon = 0$ in (1). It is obvious that in general any solution of (3) cannot satisfy the boundary condition (2) as the solution of (1) does. That is why u cannot tend to U in the whole region $\bar{R}$. Levinson defines the so called regular quadrilateral. I can only say that it is a region bounded by two arcs S_1, S_2 of the boundary ∂R and two characteristics p_1, p_2 of a certain system of differential equations. Under assumptions concerning the smoothness of coefficients of (1) and of the boundary ∂R and under the assumption that there exists a function $\Gamma(x, y)$ which is twice continuously differentiable in a region $R_0 \supset \bar{R}$ and such that $A\Gamma_x + B\Gamma_y > 0$ Levinson proved the asymptotic formula

$$u = U + h(x, y) \, e^{-g(x,y)/\varepsilon} + 0(\varepsilon^{\frac{1}{2}}) \, . \tag{4}$$

In this formula U is the solution of (3) satisfying the initial condition $U|_{S_1} = u|_{S_1}$, $g(x, y)$ is a function which is positive outside of S_2 and equals zero on S_2 and $h = u - U$ on S_2. What follows from this formula? The third term on the right side converges to zero uniformly in the whole quadrilateral. The second term is of importance only in a very small neighbourhood of S_2. For if $(x, y) \notin S_2$, $g(x, y)$ is positive and the factor $\exp \left[-g(x, y)/\varepsilon\right]$ converges very rapidly to zero. Consequently we see that $u \to U$ uniformly in every closed subregion of the quadrilateral which does not contain S_2. The term $h(x, y) \exp[-g(x, y)/\varepsilon]$ is characteristic for the problems we speak about and Levinson calls it the boundary layer term.

Levinsons' paper had a rather great response in the literature. There appeared a number of papers generalizing his results and also many papers dealing with the same problem for elliptic equations of higher order. I only mention the important papers by Višik and Lyusternik [2], [3] who created a theory concerning elliptic equations of higher order. In all these papers as much as in Levinson's, the reduced equations are of lower order then the original ones and this accounts for the appearance of boundary layer terms in the asymptotic formula. The problems I am going to speak about now are such that the reduced equations are of the same order as the original ones, namely of the second order, but they are of a different type and this necessitates the appearance of boundary layer terms in the asymptotic formulas.

Five years ago the following very simple problem arose from a consultation with an engineer: to study the dependence of the solutions and its first derivatives on the parameter ε in the mixed problem for the equation

$$\varepsilon u_{tt} + u_t - u_{xx} = 0 \tag{5}$$

where ε is a small positive parameter. It seems natural to take as the first approximation for this solution the corresponding solution U of the equation of heat flow $U_t = U_{xx}$. But in general this solution cannot satisfy both initial conditions $U(x, 0) = f(x)$ and $U_t(x, 0) = g(x)$. On the other hand to solve the mixed problem for (5) is an elementary matter. We seek the solution by means of the Fourier method in the form of a series

$$u = \sum_{n=1}^{\infty} u_n(t) \sin \frac{\pi n}{l} x .$$

Now one can expect that having calculated $u_n(t)$ we get from this series an asymptotic formula for the solution u in which the first term is the solution U. But if we calculate $u_n(t)$ we get a linear second order differential equation and the discriminant of the corresponding characteristic equation is equal to $1 - 4\pi^2 n^2 \varepsilon / l^2$. If $n \to \infty$ and $\varepsilon \to 0$ this expression is not of the same sign and this greatly complicates the situation. In short the problem mentioned above is not so simple as it seems to be and it is not possible to obtain the asymptotic formula directly from the Fourier series.

Dealing with this question I found from [4] the following equation:

$$\Delta p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} + \alpha \frac{\partial p}{\partial t} . \tag{6}$$

In this equation α is a constant and c is the velocity of sound so that $1/c^2$ is very small. The author who needed to know $\partial p / \partial t$ neglected the first term of the right-hand side of (6) so that he got the equation of heat flow instead of the wave equation. The justification of this approximation is missing, of course.

Generalizing both problems I put the following one (see [8]): to find the asymptotic formulas for the solution and its first derivatives in the mixed problem for the equation

(7) $$\varepsilon u_{tt} + \beta(t)\, u_t - Lu = F(x, t)$$

where

(8) $$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - a(x)\, u\,.$$

In this equation $x = (x_1, \ldots, x_n)$ denotes a point in E_n, the coefficients $a_{ij}(x)$, $a(x)$ are defined in the closure of a bounded domain $\bar{\Omega} \subset E_n$ and $F(x, t)$ in the cylinder $\bar{Q} = \bar{\Omega} \times \langle 0, \infty)$, $\beta(t)$ is positive for all $t \geqq 0$, $a(x)$ is nonnegative and the operator Lu is uniformly elliptic in $\bar{\Omega}$:

(9) $$a_{ij}(x) = a_{ji}(x)\,, \quad \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geqq \alpha \sum_{i=1}^{n} \xi_i^2\,, \quad \alpha = \text{const} > 0\,.$$

Equation (7) is hyperbolic in $\bar{Q}$ and we consider the mixed problem with the boundary condition of the first kind $u|_S = 0$ (S is the boundary of Ω) though the case with boundary conditions of the second and third kind can be treated in the same way. The reduced equation is of the form

(10) $$\beta(t)\, U_t - LU = F(x, t)$$

which is an equation of parabolic type. There is only one initial condition: $U(x, 0) = f(x)$. The boundary condition remains the same: $U|_S = 0$.

First I sought the solution u in a form which was similar to that used by Levinson. I succeeded in finding asymptotic formulas but for the boundary values on S of $f(x)$, $g(x)$ and $F(x, t)$ I had to assume more than is necessary and sufficient to the existence of the solution u. This is, of course, unsatisfactory. The reason is that, however, smooth the boundary S of Ω may be, the actual boundary is not smooth since the domain in which (7) is considered is the cylinder $\Omega \times \langle 0, T\rangle$ or $\Omega \times \langle 0, \infty)$ and it is necessary to seek the solution u in a substantialy more complicated form.

I overcame this difficulty and proved the asymptotic formulas under assumptions which, for the boundary values of the functions f, g, F, are necessary and sufficient to the existence of the solution u. In the homogeneous case when the formulas hold in the whole interval $\langle 0, \infty)$ they read

a) $[\frac{1}{2}n] + 3$ is even:

(11)
$$u = U + 0(\varepsilon)\,, \quad u_t = U_t + \frac{\beta(t)}{\beta(0)}\, k(x)\, e^{-\nu(t)/\varepsilon} + 0(\varepsilon^{1/4})\,, \quad u_{x_i} = U_{x_i} + 0(\varepsilon^{3/4})\,.$$

b) $[\frac{1}{2}n] + 3$ is odd:

(12)

$$u = U + 0(\varepsilon), \quad u_t = U_t + \frac{\beta(t)}{\beta(0)} k(x)\, e^{-\nu(t)/\varepsilon} + 0(\varepsilon^{3/4}), \quad u_{x_i} = U_{x_i} + 0(\varepsilon).$$

Here

$$\nu(t) = \int_0^t \beta(s)\, ds, \quad k(x) = g(x) - U_t(x, 0).$$

The boundary layer term appears in the formula for u_t only. This is natural since $u(x, 0) = U(x, 0)$ but in general $u_t(x, 0) \neq U_t(x, 0)$. We also see that this term is of importance only for small positive t. In addition I must say that the equation (7) was considered ten years ago by Krzyżański [5] in the special case $u_t(x, 0) = U_t(x, 0)$ which meant that the boundary layer term did not appear.

In proving formulas (11) and (12) I needed rather strong assumptions concerning the smoothness of the boundary S. In case of the telegraphic equation

(13) $$\varepsilon u_{tt} + \beta(t)\, u_t = \Delta u$$

I derived the formulas

(14) $$u = U + 0(\varepsilon), \quad \left\| u_t - U_t - \frac{\beta(t)}{\beta(0)} k(x)\, e^{-\nu(t)/\varepsilon} \right\|_{L_2(\Omega)} = 0(\varepsilon),$$

$$\| u_{x_i} - U_{x_i} \|_{L_2(\Omega)} = 0(\varepsilon).$$

These are not so precise as the preceding ones; but as to the smoothness of the boundary S it is sufficient that it is a Lyapunov surface.

It would be possible to prove formulas (14) for equation (7), too, under the mere assumption that the domain Ω is normal (a domain is normal if the Dirichlet problem for the Laplace equation is solvable whatever continuous values are prescribed on its boundary) if one would use the results of V. A. Il'in published two years ago [6].

All estimates that were necessary for proving the asymptotic formulas (11) and (12) were carried through by means of the Fourier method. It is an obvious idea to use the Fourier integrals and to prove in a similar way the asymptotic formulas for the solution of the Cauchy problem for equation (7). This means, of course, to limit oneself to the equation with constant coefficients. I dealt with this problem in [9] and proved the asymptotic formulas similar to that mentioned above under one restrictive assumption, namely that the initial values have a compact support.

In all questions I have spoken about the original equation was always hyperbolic and the reduced one parabolic. It is an obvious idea to take as the original equation an elliptic one such that the reduced equation is again parabolic. In [10] I formulated the following problem: to investigate the solution of the Dirichlet problem for the elliptic equation

(15) $$u_{xx} + \varepsilon u_{yy} + A(x, y)\, u_x - B(y)\, u_y + C(x, y)\, u = F(x, y)$$

in a region R whose boundary ∂R consists of a segment of the line $y = y_1$, of two continuous curves $x = v_1(y)$, $x = v_2(y)$ with $v_1(y) < v_2(y)$ and of a segment of the line $y = y_2$. Under usual assumptions of the smoothness of the coefficients and of the boundary ∂R and under the assumption $B(y) > 0$ I derived the following formula

$$u = U + h(x, y, \varepsilon)\, e^{-\alpha(y)/\varepsilon} + 0(\varepsilon^{\frac{1}{2}}) \tag{16}$$

where $h = 0(1)$ in $\bar{R}$ and $\alpha(y) = \int_y^{y_2} B(s)\, ds$. In this formula U is the solution of the reduced parabolic equation

$$U_{xx} + A(x, y)\, U_x - B(y)\, U_y + C(x, y)\, U = F(x, y) \tag{17}$$

which assumes the same boundary value as the solution u on the segment of the line $y = y_1$ and on the curves $x = v_1(y)$, $x = v_2(y)$. The second term on the right-hand side is the boundary layer term since it becomes important only near the line $y = y_2$ where it equalises different values of u and U.

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