ON A CERTAIN EXTENSION OF THE MAXIMUM PRINCIPLE

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In my paper I shall deal with two theorems connected with unicity theorems for the third boundary value problem for elliptic and parabolic equations. These theorems will guarantee the negativity of the normal derivative of a solution of a second order partial differential equation at a point at which the solution reaches its maximum value. Let me begin with some historical remarks. A theorem of this kind for elliptic equations was first proved by E. Hopf [1] and O. A. Oleĭnik [2]. The latter also showed, how the strong maximum principle and the continuous dependence of the solution on the coefficients can be deduced from this theorem. After the publication of these papers and after Nirenberg [3] proved the strong maximum principle for parabolic equations it was a simple matter to prove a theorem of this kind for parabolic equations. And Friedman [4] and I [5], [6] took this step independently. A general theorem, which included the theorems I have just mentioned was proved by Pucci [7]. In all these papers a certain assumption on the smoothness of the boundary was assumed, namely, that the maximum point can be touched from the interior of the region by a sphere which lies in the region and which has only one point in common with the boundary. The problem of weakening this assumption may be considered. A uniqueness theorem for the third boundary value problem, in which this assumption does not occur, is given by Krzyżański [8]. In this connection A. D. Alexandrov's [9] six papers on the maximum principale are extremely important. These papers are not only general concerning the boundary of the region but also with respect to the coefficients. In this short report I am unable to discuss Alexandrov's work. Let us now turn to the concrete facts. We shall assume once and for all, that for the region G, which is under consideration, the following conditions are fulfilled.

1) There is a function $\tau = \tau(x)$, which has the property, that $\tau = 0$ on $\dot{G}$ and $\tau > 0$ in G.

2) τ possesses in $\bar{G}$ continuous derivatives of the first order $\tau_i = \partial\tau/\partial x_i$, τ possesses continuous derivatives of the second order in G and the inequality

$$\beta > |\mathrm{grad}\, \tau| > \alpha > 0$$

holds.

Let us now consider the differential operator

$$L(u) = \sum_{i,j=1}^{n} a_{ij}(x)\, u_{ij}(x) + \sum_{i=1}^{n} b_i(x)\, u_i(x) + c(x)\, u(x)$$

where $x = (x_1, \ldots, x_n)$.

The indices i, j are used on a's and b's to indicate different functions, on u they indicate differentiation.

We shall assume, that the quadratic form

$$\sum a_{ij}(x)\,\lambda_i\lambda_j$$

is positive semidefinite for every $x \in G$. Further we shall assume, that the inequality

$$c(x) \leqq 0$$

holds in G.

Let us denote by z a point belonging to the boundary $\dot{G}$. Let the coefficients of L satisfy the condition

$$\text{(A)} \qquad \varliminf_{\substack{x\to z\\ x\in G}} \sum_{i,j=1}^{n} a_{ij}(x)\,\tau_i(x)\,\tau_j(x) > 0\,.$$

We denote by $B(t)$ a function, which is continuous and positive for $t \in (0, t_0\rangle$ and for which the integral

$$\int_0 B(z)\,\mathrm{d}z$$

is convergent.

Suppose now, that there is such a function B that the following conditions are satisfied.

$$\text{B 1)} \qquad \varlimsup_{\substack{x\to z\\ x\in G}} \frac{b_i(x)}{B(\tau(x))} < \infty\,, \qquad i = 1, \ldots, n$$

$$\text{B 2)} \qquad \varliminf_{\substack{x\to z\\ x\in G}} \frac{c(x)\,\tau(x)}{B(\tau(x))} > -\infty\,,$$

$$\text{B 3)} \qquad \varlimsup_{\substack{x\to z\\ x\in G}} \frac{|a_{ij}(x)\,\tau_{ij}(x)|}{B(\tau(x))} < \infty\,, \quad i, j = 1, \ldots, n\,.$$

If the functions b_i and c are bounded then obviously conditions B 1) and B 2) are satisfied. As for condition B 3) I only mention that if we weaken the condition 2)*) then B 3) is more general than the assumption of the existence of a sphere, which lies in G and which has only one point in common with the boundary.

We are now able to state the theorem

Theorem 1. *If u is a function, which possesses continuous derivatives of the second order in G and if the inequality*

$$L(u) \geqq 0$$

*) And such a generalization for nour theorems can be given (see [10]).

is satisfied in G and if for the point $z \in \dot{G}$ the following inequalities

$$u(x) < u(z) \quad \text{for} \quad x \in \bar{G},\ x \neq z\,,$$
$$0 < u(z)$$

hold, then

(1) $$\overline{\lim_{\substack{x \to z \\ x \in n}}} \frac{u(x) - u(z)}{|x - z|} < 0$$

provided that B 1)–B 3) *and the condition* (A) *are satisfied.*

In equation (1) $|x - z|$ denotes the distance between the points x and z, and n is the inner normal.

If the normal derivative exists, then its negativity is an immediate consequence of inequality (1).

If $c \equiv 0$, then the assumption $u(z) > 0$ is superfluous.

For a parabolic operator of the form

$$\hat{L}(u) = \sum_{i,j=1}^{n-1} a_{ij} u_{ij} + \sum_{i=1}^{n-1} b_i u_i - u_n + cu$$

there is no need to assume that $c \leqq 0$.

Proof of Theorem 1. There is a positive constant η and a neighborhood U of the point z such that the following inequalities are satisfied

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \tau_i(x)\, \tau_j(x) > \eta\,,$$
$$|b_i(x)| < B(\tau(x))\,,$$
$$|a_{ij}(x)\, \tau_{ij}(x)| < B(\tau(x))\,,$$
$$c(x)\, \tau(x) > -\, B(\tau(x))$$

for $x \in U \cap G$. If necessary, we may choose U so small, that

$$\int_0^\tau B(t)\, \mathrm{d}t < \alpha = \frac{\eta}{2(n^2 + n\beta + 1)}$$

for $x \in U \cap G$. Let us define the function h by the equation

$$h(x) = \int_0^\tau B(t)\, (\tau - t)\, \mathrm{d}t + \alpha\tau\,.$$

It can be proved by calculation that

$$L(h) > 0$$

for $x \in U \cap G$. For the function $v = u + \mu h$ is evidently

$$L(v) \geqq \mu\, L(h) > 0$$

for $x \in G \cap U$ and therefore the function v reaches its maximum value on the boundary of $G \cap U$. But for $x \neq z$, $x \in U \cap \dot{G}$ the inequality

$$v(x) < u(z) \tag{2}$$

holds and this inequality is valid for $x \in \dot{U} \cap G$, too, if μ is sufficiently small. From inequality (2) it follows that

$$\frac{u(x) - u(z)}{|x - z|} < -\mu \frac{h(x) - h(z)}{|x - z|}.$$

If we make use of the fact, that

$$\frac{\partial h}{\partial n} > 0$$

we obtain the assertion of the theorem by letting $x \to z$. The proof is complete.

Let us now consider a quasilinear operator of the form

$$E(u) = \sum_{i,j=1}^{n} a_{ij}(x, u, p)\, u_{ij} - a(x, u, p),$$

where p denotes the vector $p = (u_1, \ldots, u_n)$. For an operator of this kind the maximum principle was investigated by R. M. Redheffer [11].

Let g be a continuous, positive and increasing function on the interval $(0, \infty)$ such that

$$\int_0 \frac{d\xi}{g(\xi)} = +\infty.$$

Theorem 2. *If the following estimates*

$$|a_{ij}(x, u, p) - a_{ij}(x, u, 0)| \leqq g(u^*)\, B(\tau(x)),$$
$$|a(x, u, p) - a(x, u, 0)| \leqq g(u^*)\, B(\tau(x)),$$
$$|a_{ij}(x, u, 0)\, \tau_{ij}(x)| < B(\tau(x))$$

where

$$u^* = \sqrt{(u_1^2 + u_2^2 + \ldots + u_n^2)}$$

are valid and if

$$a(x, u, 0) \geqq 0$$

and if

$$\lim_{\overline{x \to z}} \sum_{i,j=1}^{n} a_{ij}(x, u, 0)\, \tau_i(x)\, \tau_j(x) > 0$$

and if the quadratic form

$$\sum_{i,j=1}^{n} a_{ij}(x, u, 0)\, \lambda_i \lambda_j$$

is positive semidefinite, and if

$$E(u) \geqq 0\,,$$
$$u(x) < u(z) \quad \textit{for} \quad x \in \bar{G},\ x \neq z$$

then

$$\overline{\lim_{\substack{x \to z \\ x \in n}}} \frac{u(x) - u(z)}{|x - z|} < 0 \tag{1}$$

provided, that u possesses continuous derivatives of the second oder.

Let us now investigate the assumptions of the theorem in more detail. The assumptions that

$$\int_0 \frac{d\xi}{g(\xi)} = +\infty\,,$$
$$\int_0 B(t)\,dt < +\infty$$

are essential. An example of an operator E or L and of a function u can be given in such a way that inequality (1) does not hold, but all assumptions are satisfied with only one exception, that the integral

$$\int_0 B(t)\,dt$$

is divergent. A similar remark can be made about the convergence of the integral

$$\int_0 \frac{d\xi}{g(\xi)}\,.$$

Let us now outline the proof of theorem 2. First we shall show that the function

$$h(t) = C_1 t + C_2\, g(t)$$

(where C_1 and C_2 are arbitrary positive constants) satisfies the same conditions as the function g. We choose suitable constants C_1 and C_2. Then we solve the initial value problem

$$\begin{aligned} v'' &= B(t)\, h(v')\,, \\ v(0) &= 0\,, \\ v'(0) &= a > 0 \end{aligned}$$

and denote its solution by $v_a(t)$.

Now we define the function

$$w(x) = u(x) + v_a(\tau(x))$$

and investigate this function in a sufficiently small neighborhood of the point z. It can be proved, that this function reaches its maximum value in the set $\overline{U \cap G}$ on its boundary. But on $\dot{G}$ $v_a = 0$, hence

$$w(x) < u(z) ;$$

on the other part of the boundary we have,

$$u(x) < u(z) - \eta$$

where η is a positive constant.

If we choose a sufficiently small, then

$$v_a(\tau(x)) < \eta .$$

And together we get

$$w(x) < u(z)$$

on the boundary of U and hence in U. From this inequality the assertion of the theorem can be derived in the usual manner.

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