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NONLINEAR DIFFERENTIAL SYSTEMS OF THE THIRD AND FOURTH ORDER

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The theory of systems of nonlinear ordinary differential equations $\dot{x} = X(x, y, t)$, $\dot{y} = Y(x, y, t)$ $(\dot{x} = \mathrm{d}x/\mathrm{d}t)$ and in particular of the autonomous systems $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$ has been studied in modern treatises and monographies [1] to [6]; anyone interested in pure and applied mathematics may easily find complete expositions of the results obtained up till the present.

On the other hand, a systematic treatment of nonlinear systems of the third and fourth order has not appeared yet, and in this report there is an attempt at a classification of the results reached by Russian, American and other schools.

1 Linear Systems

The first classifications of the singular points for third-order linear systems are due to Poincaré [7], [8].

The system

$$\dot{x}_l = a_{l1}x_1 + a_{l2}x_2 + a_{l3}x_3 \quad (l = 1, 2, 3) \tag{1}$$

where the a_{lk} $(l, k = 1, 2, 3)$ are real constants, under the hypothesis $\det(a_{ik}) \neq 0$ has the unique singular point $x_1 = x_2 = x_3 = 0$.

By an affine transformation the system may be reduced to the canonical form

$$\dot{x} = \varrho_1 x, \quad \dot{y} = \varrho_2 y, \quad \dot{z} = \varrho_3 z \tag{2}$$

and therefore the characteristic lines have the equations

$$x = c_1 e^{\varrho_1 t}, \quad y = c_2 e^{\varrho_2 t}, \quad z = c_3 e^{\varrho_3 t} \quad (c_1, c_2, c_3 \text{ constants}). \tag{3}$$

From these equations, it is easy to deduce the behaviour of the characteristic lines around the origin O.

a) Let $\varrho_1 > 0$, $\varrho_2 > 0$, $\varrho_3 > 0$; it follows $\lim_{t\to-\infty} x = \lim_{t\to-\infty} y = \lim_{t\to-\infty} z = 0$; if $\varrho_1 < 0$, $\varrho_2 < 0$, $\varrho_3 < 0$, it follows $\lim_{t\to\infty} x = \lim_{t\to\infty} y = \lim_{t\to\infty} z = 0$ and the origin is called a *node* by Poincaré.

We remark that if $0 > \varrho_1 > \varrho_2 > \varrho_3$, then both semiaxes x $(c_1 \neq 0, c_2 = c_3 = 0)$ are characteristic lines (also the both semiaxes $y(z)$ are characteristic lines); all other characteristic lines are tangent to the x axis at the origin; we also remark that if $\varrho_1 = \varrho_2 = \varrho_3 < 0$, the characteristic lines tend to the origin in every direction and the node may be termed a *starred (proper) node.*

b) Let $\varrho_1, \varrho_2, \varrho_3$ be real, $\varrho_1 > 0$, $\varrho_2 > 0$, $\varrho_3 < 0$. Then all the semiaxes x, y, z are characteristic lines. If $c_3 = 0$ in (3), the corresponding characteristic lines lie on the plane $z = 0$ and tend to origin O as $t \to -\infty$; using the same terminology as in the plane case we may say that they have a node there.

Every other characteristic ($c_3 \neq 0$) remains in a neighbourhood of O for a finite time only and the point O is called a *saddle-point* by Poincaré.

c) If $\varrho_1 = r + is$, $\varrho_2 = r - is$, $r > 0$ $(s > 0)$, $\varrho_3 > 0$, the equation of the characteristic lines, in real form, are

$$x = c_1 e^{r(t-t_0)} \cos (t - t_0), \quad y = c_1 e^{r(t-t_0)} \sin (t - t_0), \quad z = c_3 e^{\varrho_3 t}$$
$$(c_1, c_3, t_0 \text{ real constant}).$$

Both semiaxes $z < 0$, $z > 0$ are characteristic lines; for $c_3 = 0$, $c_1 \neq 0$ the corresponding characteristics lie on the plane $z = 0$, and if $t \to -\infty$ they spiral around the origin and tend to it asymptotically.

The point O is called a *focus* by Poincaré (fig. 1).

The case $r < 0$, $\varrho_3 > 0$ corresponds (Poincaré) to a *saddle-focus* (fig. 2), the case $r = 0$, $\varrho_1 = is$, $\varrho_2 = - is$, $s > 0$, corresponds (Poincaré) to a *center* (fig. 3).

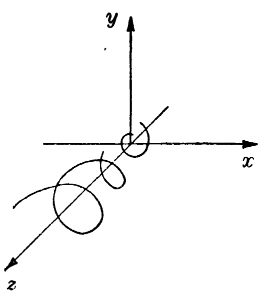

Fig. 1 — focus

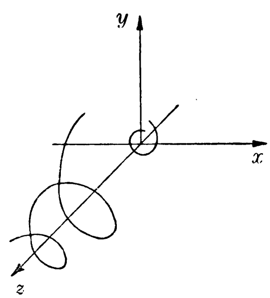

Fig. 2 — saddle-focus

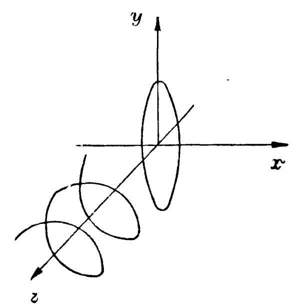

Fig. 3 — center

d) R. M. Minc in one of his papers [9] considers the qualitative behaviour at infinity of the trajectories of the real differential system

$$\dot{x} = P(x, y, z), \quad \dot{y} = Q(x, y, z), \quad \dot{z} = R(x, y, z)$$

where P, Q, R are polynomials in an Euclidean 3-space by means of the Poincaré transformation $x = 1/u$, $y = v/u$, $z = w/u$. In the case of system (1) under the hypothesis $\det (a_{ik}) \neq 0$, the author studies all the possible cases, also making use of appropriate figures.

2 Phenomena in the neighbourhood of a three-dimensional singularity and in three-dimensional space

In two-dimensional space, in addition to the elementary singular points there are more complicated singularities, and of course there is much greater complexity in the case of S_n with $n \geqq 3$.

S. Barocio [10] has given an example of three-dimensional vector field with a single singularity O. There exist two spheres Σ_1 and Σ_2 with center O, Σ_1 interior to Σ_2, with the following properties: Every trajectory of the field which intersects Σ_2 enters Σ_2. No trajectory which intersects Σ_2 ever intersects Σ_1, and hence every trajectory which intersects Σ_2 remains at a positive distance from O.

Also the Poincaré-Bendixson theorem is not true in a three-dimensional space.

This theorem states that solutions of second-order autonomous differential equations (and, more generally, systems of two first-order equations) which are bounded in the phase space and tend to no critical point must tend to a closed cycle as time increases to infinity.

R. N. d'Heende [11] shows that this result does not hold for third-order equations $\dddot{x} = Z(x, \dot{x}, x + \ddot{x}) - \dot{x}$ where Z is a proper function.

3 Analytic case

Researches of R. E. Gomory, C. Coleman, F. Haas, L. È. Reĭzin', R. M. Minc

R. E. Gomory [12] has considered the equation

$$\dot{v} = F(v) = \sum_{s=m}^{\infty} F_s(v) \tag{4}$$

where $F(v)$ is real analytical vector function of the real 3-vector v, $v = (x, y, z)$, in some neighbourhood of the origin, and $F_s(v)$ is a vector whose components are terms of degree s in x, y, z, $F(0) = 0$.

If $v(t)$ is a solution of (4) tending to $P = (0, 0, 0)$ as $t \to \infty$ the behaviour of $v(t)$ is studied by projecting its motion onto the unit sphere D around P.

Let $v(t) = \sigma(t)\, u(t)$ where σ is the norm of $v(t)$ and $u(t)$ is a unit vector. Then as $v(t) \to P$, $u(t)$ traces out a path on D which may of course be self-crossing. A point u_0 on the unit sphere is a *limiting direction* of approach of the motion $v(t)$ if and only if there is a sequence $\{t_n\}$, $t_n \to \infty$, such that $u(t_n) \to u_0$.

To the set $L(v)$ of limiting directions the author lets correspond an autonomous differential system on S_2 which may be used to yield information about $L(v)$.

With the same method, C. Coleman [13] considers the case in which $F(v)$ is a vector whose components are homogeneous polynomials in x, y, z of degree $\geqq 1$.

Using the method of R. E. Gomory, R. E. Gomory and F. Haas have studied the system

$$\frac{dy_1}{dt} = \sum_{k=1}^{\infty} Y_{1k}, \quad \frac{dy_2}{dt} = \sum_{k=1}^{\infty} Y_{2k}, \quad \frac{d\vartheta}{dt} = 1 + \sum_{k=1}^{\infty} D_k \tag{5}$$

where Y_{ik}, D_k are polynomials in y_1 and y_2 of degree k, whose coefficients are periodic functions of ϑ wit period 1.

The existence of a limit cycle C is assumed and the behaviour of the solutions of (5) which approach C is studied.*)

L. È. Reĭzin' [16] investigates the system of differential equations

$$\dot{x}_i = X_i \quad (i = 1, 2, \ldots, n) \quad (\dot{x}_i = dx_i/dt) \tag{6}$$

where $X_i = X_i(x_1, \ldots, x_n)$ are homogeneous polynomials of degree m having only one common zero, at $x_1 = \ldots = x_n = 0$.

The theorems given by these authors make possible to determine the disposition of the integral curves in the neighbourhood of a singular point for certain special systems.

R. M. Minc [9] considers the qualitative behaviour at infinity of the trajectories of the real differential system

$$\dot{x} = P(x, y, z), \quad \dot{y} = Q(x, y, z), \quad \dot{z} = R(x, y, z) \tag{7}$$

where P, Q, R are polynomials in an Euclidean 3-space.

4 Analytic case – Proper node

Researches by L. È. Reĭzin', R. E. Zindler

L. È. Reĭzin' [17] has considered the system of equations

$$\frac{dx}{dt} = X(x, y, z) + \xi(x, y, z), \quad \frac{dy}{dt} = Y(x, y, z) + \eta(x, y, z), \tag{8}$$
$$\frac{dz}{dt} = Z(x, y, z) + \zeta(x, y, z)$$

where X, Y, Z are homogeneous polynomials of m-th degree in x, y, z with constant coefficients, and having only one common zero at the origin; the functions ξ, η, ζ are continuous, satisfying

*) The manner in which a trajectory can approach a limit cycle was studied by G. D. Birkhoff [15]. But in his classical paper on transformations of surfaces Birkhoff assumes the conservation of energy.

$$(9) \qquad \lim_{r\to 0} \xi(x, y, z)\, r^{-m} = \lim_{r\to 0} \eta(x, y, z)\, r^{-m} = \lim_{r\to 0} \zeta(x, y, z)\, r^{-m} = 0\,,$$
$$r = (x^2 + y^2 + z^2)^{1/2}$$

and such that the origin is an isolated singular point for $0 \leqq r \leqq r_0$, $r_0 > 0$.

By the introduction of polar coordinates r, φ, ψ

$$x = r \sin\varphi \cos\psi\,, \quad y = r\sin\varphi\sin\psi\,, \quad z = r\cos\varphi\,,$$
$$0 \leqq r \leqq r_0\,, \quad -\pi \leqq \psi \leqq \pi\,, \quad 0 \leqq \varphi \leqq \pi$$

into (8) we have

$$(10) \qquad \frac{\mathrm{d}r}{\mathrm{d}t} = r^m R(\varphi, \psi) + \varrho(r, \varphi, \psi)\,, \quad r\frac{\mathrm{d}\varphi}{\mathrm{d}t} = r^m \Phi(\varphi, \psi) + f(r, \varphi, \psi)\,,$$
$$r \sin\varphi \frac{\mathrm{d}\psi}{\mathrm{d}t} = r^m \Psi(\varphi, \psi) + g(r, \varphi, \psi)$$

with R, Φ, Ψ polynomials in $\sin\varphi$, $\cos\varphi$, $\sin\psi$, $\cos\psi$, and

$$\lim_{r\to 0} r^{-m} \varrho(r, \varphi, \psi) = \lim_{r\to 0} r^{-m} f(r, \varphi, \psi) = \lim_{r\to 0} r^{-m} g(r, \varphi, \psi) = 0\,.$$

The system (9) defines a certain field of directions which is obtained by associating with each point $P = (x, y, z)$ the direction of the tangent to the characteristic curve through P.

A direction defined by the coordinates (φ_0, ψ_0), for which there exists a sequence of points (r_1, φ_1, ψ_1), ..., (r_n, φ_n, ψ_n), ... such that for $n \to \infty$, i) $r_n \to 0$, $\varphi_n \to \varphi_0$, $\psi_n \to \psi_0$, ii) $\alpha_n \to 0$, where α_n is the tangent of the acute angle between the direction (φ_n, ψ_n) and the direction of the field at the point (r_m, φ_n, ψ_n), is called an *exceptional direction*.

The coordinates of the exceptional directions are completely determinated by the solutions of the system of equations

$$(11) \qquad \Phi(\varphi, \psi) = 0\,, \quad \Psi(\varphi, \psi) = 0\,,$$

and this system does not possess directions (φ^*, ψ^*) for which

$$\Phi(\varphi^*, \psi^*) = \Psi(\varphi^*, \psi^*) = R(\varphi^*, \psi^*) = 0\,.$$

If $\Phi \equiv 0$, $\Psi \equiv 0$, we have the following theorem of L. È. Reĭzin' which has been extended by R. E. Zindler [18], the analogy in three-dimensional space to one of Lonn's theorems [19] in two-dimensional space.

Given a system of differential equations (8) for which $\Phi \equiv \Psi \equiv 0$, if in a certain sufficiently small neighbourhood of the origin, $0 \leqq r \leqq r_0$, it is possible to find a continuous non-negative function $B(r)$ for which

$$\int_0^{r_0} B(r)\, r^{-m-1}\, \mathrm{d}r < \infty$$

and

$$|f(r, \varphi, \psi)| \leqq B(r), \; |g(r, \varphi, \psi)| \leqq B(r),$$

then all integral curves of the system (8) in a sufficiently small neighbourhood of the origin, $r < r_1$, approach the singular point with a definite tangent, and there exists in each direction one integral curve.

If the additional relations

$$|g(r, \varphi, \psi)/\sin \psi| \leqq B(r),$$
$$|\varrho(r, \varphi_1, \psi_1) - \varrho(r, \varphi_2, \psi_2)| \leqq C_\varrho \, B(r) \, [|\varphi_1 - \varphi_2| + |\psi_1 - \psi_2|],$$
$$|f(r, \varphi_1, \psi_1) - f(r, \varphi_2, \psi_2)| \leqq C_f \, B(r) \, [|\varphi_1 - \varphi_2| + |\psi_1 - \psi_2|],$$
$$\left| \frac{g(r, \varphi_1, \psi_1)}{\sin \varphi_1} - \frac{g(r, \varphi_2, \psi_2)}{\sin \varphi_2} \right| \leqq C_g \, B(r) \, [|\varphi_1 - \varphi_2| + |\psi_1 - \psi_2|]$$

are fulfilled, where C_ϱ, C_f, C_g are certain positive constants, then in each direction there exists only one integral curve. As in 1, a) we call the origin a *proper node.*

If (φ_0, ψ_0) is an exceptional direction, i.e. a solution of the system (11) and if there is a circular cone with vertex at the origin containing no other ray from the origin also corresponding to an exceptional direction, then the exceptional direction will be called *isolated.*

Let (φ_0, ψ_0) be an isolated exceptional direction. After a transformation of coordinates, we may suppose that in the new coordinates the z-axis coincides with the exceptional direction.

We shall restrict a part of the z-axis to the region S defined by the conditions

$$0 \leqq \varphi \leqq \delta, \quad 0 < r \leqq r_0,$$

where $K(\varphi, \psi) = \Phi(\varphi, \psi)/R(\varphi, \psi) \neq 0$.

L. È. Reĭzin' proves that if $K(\varphi, \psi) > 0$ in S, the integral curves which enter the region S through the lateral or rear surface of S tend to the origin as $t \to \infty$, and are tangent at the origin to z-axis (fig. 4).

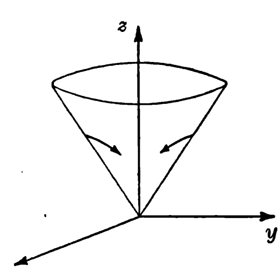

Fig. 4.

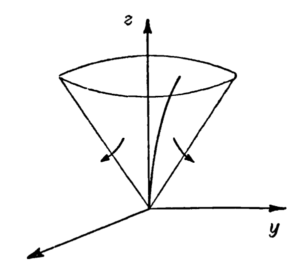

Fig. 5.

If $K < 0$ in S, then there is at least one integral curve which enters S through the rear surface of S, tangent at the origin to the z-axis; but an integral curve through the lateral surface of S, for t increasing, does not enter S (fig. 5).

5 Problem of M. A. Aĭzerman

Systematic researches on the stability of the systems which may be transformed to the canonical form

$$\dot{x}_\varrho = \lambda_\varrho x_\varrho + f(\sigma) \quad (\varrho = 1, \dots, n), \quad \sigma = \sum_{\varrho=1}^{n} \gamma_\varrho x_\varrho$$

(λ_ϱ, x_ϱ constants) were first performed by M. A. Aĭzerman [20], [21], [22],*) who used the direct (second) method of Lyapunov.

M. A. Aĭzerman's problem has been improved successively for its applications to nonlinear control system; two exhaustive treatments are in the monographs of A. I. Lur'e [24] and A. M. Letov [25].

In the following section 6 we shall notice particular researches of third-order systems of M. A. Aĭzerman.

We shall also give some other results on the nonlinear equations of third and fourth order.

6 Particular equations and systems of the third and fourth order

E. A. Barbašin [26] studied the stability of the solution $x = 0$ of the equation

(12) $$\dddot{x} + a_1\ddot{x} + \varphi(\dot{x}) + f(x) = 0$$

with the natural restrictions

$$a_1 < 0, \quad f(0) = 0, \quad f(x)/x > 0, \quad \varphi(0) = 0, \quad a_1\, \varphi(y)/y - f'(x) > 0$$

and supposing

$$2V = (z + a_1 y)^2 + 2\Phi(y) + 2f(x)\, y + 2a_1\, F(x) \to \infty$$

if

$$x^2 + y^2 + z^2 \to \infty, \quad \dot{x} = y, \quad \dot{y} = z,$$

$$\Phi(y) = \int_0^y \varphi(s)\, \mathrm{d}s, \quad F(x) = \int_0^x f(s)\, \mathrm{d}s;$$

it is proved that the origin is asymptotically stable (i.e., for arbitrary initial conditions every solution $x(t)$ of (12) tends to 0 as $t \to \infty$).

The asymptotical stability of the solution $x = 0$ of the equation

(13) $$\dddot{x} + f(x, \dot{x})\,\ddot{x} + b\dot{x} + cx = 0$$

where b, c are constants, with the restrictions

$$b > 0, \quad c > 0, \quad f(x, y) > c/b, \quad y\, f'(x) < 0$$

for all x, y has been proved by S. N. Šimanov [27].

*) See also [23] and [6].

N. N. Krasovskiĭ [28] gives some necessary and sufficient conditions for the asymptotic stability at the origin of the system

$$\dot{x}_i = f_i(x_1) + a_i x_2 + b_i x_3 \quad (i = 1, 2, 3)\,, \tag{14}$$

$|f_i(x)| \leqq M(x)$ for small $|x|$, assuming $a_1 b_1 \neq 0$.

A. P. Tusov [29], [30], [31] gives some conditions under which the system

$$\dot{x}_1 = \sum_{k=1}^{3} a_{1k} x_k + f(x_1)\,, \quad \dot{x}_i = \sum_{k=1}^{3} a_{ik} x_k \quad (i = 2, 3) \tag{15}$$

is stable in the large.

Here a_{ik} are real and f satisfies the standard existence and unicity conditions in the whole space, and furthemore $\alpha x_2^2 < x_2 f(x_2) < \beta x_2^2$ where α and β are the extreme values of constants a, such that on replacing f by $a x_2$ in (15), the characteristic roots of the corresponding linear system have negative real parts.

V. A. Pliss [32] has studied the system

$$\dot{x} = y - f(x)\,, \quad \dot{y} = z - x\,, \quad \dot{z} = - x\,; \tag{16}$$
$$f(0) = 0\,, \quad \eta f(\eta) > \eta^2 \quad \text{for} \quad \eta \neq 0$$

giving some sufficient conditions for its stability in the large.

Conditions for the existence of periodic solutions are also given.

Further [33], taking into account the system

$$\dot{x} = y - f(x)\,, \quad \dot{y} = z - x\,, \quad \dot{z} = - ax - bf(x) \tag{17}$$

where $f(x)$ is Lipschitzian, $f(0) = 0, f(x)/x > x + bf(x)/x$ for $x > 0$, V. A. Pliss has given sufficient conditions under which the system is stable in the large.

Furthermore V. A. Pliss proves also for the system (17) the existence of periodic solutions for $a > 0$, $0 \leqq b < 1$, $a + b \geqq 1$, $hx \leqq f(x) - ax/(1 - b)$ if $0 \leqq x \leqq x_1$, $0 < f(x) - ax/(1 - b) < \lambda$ for $x \geqq x_2$ where $h > (1 - b)/a$, λ, x_1, $x_2 - x_1$ are sufficiently small positive numbers.

Moreover V. A. Pliss [34] has given necessary and sufficient conditions for stability in the large for the system

$$\dot{x} = y - ax - f(x)\,, \quad \dot{y} = x - bf(x)\,, \quad \dot{z} = - cf(x) \tag{18}$$

with $ab > c$, $b > 0$, $c > 0$, $f(0) = 0$, $xf(x) > 0$ for $x \neq 0$,

$$\overline{\lim_{x \to -\infty}} \left[f(x) - \int_0^x f(s)\,\mathrm{d}s \right] = \infty, \quad \overline{\lim_{x \to -\infty}} \left[- f(x) - \int_0^x f(s)\,\mathrm{d}s \right] = \infty\,.$$

Finally V. A. Pliss [35] gives, without proof, many conditions under which all the solutions of a nonlinear system of the form

$$\dot{x} = Ax + f(x_1)\,b\,, \quad x(0) = c \tag{19}$$

approach zero as t tends to infinity. Here x is a 3-dimensional vector, A is a constant matrix, and $f(x_1)$ is a scalar function of one component of x, say x_1.

E. M. Vaĭsbord [36] has studied the system

$$(20) \qquad \dot{x} = f_{11}(x) + f_{12}(y)\,, \quad \dot{y} = f_{23}(z)\,, \quad \dot{z} = f_{31}(x) + f_{32}(y) + f_{33}(z)\,.$$

The existence and continuity of $f'_{ik}(x)$ is supposed and

i) $f_{ik}(0) = 0$;

ii) $f'_{i1}(x) < 0$, $\lim\limits_{|x|\to\infty} f_{i1}(x) = -\infty$ $(i = 1, 3)$;

iii) $f_{12}(y) < c$, $f'_{23}(z) > 0$, $|f_{23}(z)| < a|z|$, and when $\lim\limits_{y\to\infty} |f_{32}(y)| < \infty$, $\lim\limits_{y\to-\infty} |f_{32}(y)| < \infty$ then $\lim\limits_{y\to\infty} f'_{32}(y) = 0$, $\lim\limits_{y\to-\infty} f'_{32}(y) = 0$, respectively;

iv) on the curves satisfying the equation $f_{31}(x) + f_{32}(y) = 0$ we have $[f'_{21}(y) - f'_{11}(x)]/[f'_{31}(x) - f'_{32}(y)] > \eta > 0$, $0 < A < f'_{11}(y)/f'_{31}(x)$;

v) $ca/A\eta^2 < 1$;

vi) $f'_{23}(0)\,[f'_{33}(0)f'_{32}(0) + f'_{31}(0)f'_{12}(0)] - f'_{11}(0)f'_{33}(0)\,[f'_{11}(0) - f'_{33}(0)] < 0$.

Under these hypotheses system (20) has a periodic solution and all solutions for $t \to \infty$ are bounded in the large (i.e. every characteristic of (20) remains in a bounded domain independent of the characteristic).

First it is proved that the characteristic equation of the system associated with (20) in a neighbourhood of (0, 0, 0),

$$\dot{\xi} = f'_{11}(0)\,\xi + f'_{12}(0)\,\eta\,, \quad \dot{\eta} = f'_{23}(0)\,\zeta\,, \quad \dot{\zeta} = f'_{31}(0)\,\xi + f'_{32}(0)\,\eta + f'_{33}(0)\,\zeta\,,$$

has a root with negative real part and two roots with positive real parts. Then by the fixed point theorem the existence of a periodic solution is obtained.

From topological considerations there follows the boundedness of solutions.

E. M. Vaĭsbord [37] has also studied the existence of periodic solutions of the equation

$$(21) \qquad \frac{\mathrm{d}^3x}{\mathrm{d}t^3} + \frac{\mathrm{d}^2G_1(x)}{\mathrm{d}t^2} + \frac{\mathrm{d}G_2(x)}{\mathrm{d}t} + G_3(x) = 0\,.$$

Putting

$$g_i(x) = G_i(x)/x \quad \text{for} \quad x \neq 0\,, \quad g_i(0) = G'_i(0) \quad \text{for} \quad x = 0$$

we have the system

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 - g_1(x_1)\,x_1\,, \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = x_3 - g_2(x_1)\,x_1\,, \quad \frac{\mathrm{d}x_3}{\mathrm{d}t} = g_3(x_1)\,x_1\,;$$

existence and uniqueness is supposed.

Furthermore, assuming

i) $g_i(x) > 0$;

ii) $\lim\limits_{|x|\to\infty} g_i(x) = g_i(\infty) > 0$;

iii) $g_1(0)\, g_2(0) < g_3(0),\; g_1(\infty)\, g_2(\infty) > g_3(\infty)$

and under other hypotheses the existence of at least one periodic solution is proved.

Finally, E. M. Vaĭsbord has considered the following equation of the fourth order

$$\frac{d^4x}{dt^4} + \frac{d^3G_1(x)}{dt^3} + \frac{d^2G_2(x)}{dt^2} + \frac{dG_3(x)}{dt} + G_4(x) = 0\,. \tag{22}$$

M. L. Cartwright [38] has studied the system

$$\dot{x} = y\,, \quad \dot{y} = z\,, \quad \dot{z} = w\,, \quad \dot{w} = -a_1 w - a_2 z - a_3 y - f(x) \tag{23}$$

corresponding to the equation of the fourth order

$$x^{(4)} + a_1 \dddot{x} + a_2 \ddot{x} + a_3 \dot{x} + f(x) = 0\,,$$

which occurs in automatic control systems.

It is supposed that

i) a_1, a_2, a_3 are all positive;

ii) $f(0) = 0,\; f'(x) > 0,\; f''(x)$ continuous;

iii) $F(x) = \int_0^x f(s)\, ds \to \infty$ as $|x| \to \infty$;

iv) $A_4(x) = a_1 a_2 a_3 - a_3^2 - a_1^2 f'(x) \geqq \delta > 0$ for all x.

If

$$2V_B(x, y, z, w) = a_1^2 a_3 [w + a_1 z + (a_2 - a_3/a_1)\, y]^2 + a_3^2 a_1\,.$$
$$.\,[z + a_1 y + a_1 a_3^{-1} f(x)]^2 + a_3\, A_4(x)\, y^2 + 2a_1 a_3 (a_1 a_2 - a_3)\, F(x) - a_1^3 f^2(x)$$

for every $V_0 > 0$ there is a domain D_0 of the x, y, z, w space defined by $V_B(x, y, z, w) < V_0$; if $P_0 = (x_0, y_0, z_0, w_0)$ is any point of D_0, then the solution of (23) through P_0 tends to the trivial solution $x = y = z = 0$ of (23) as $t \to \infty$ provided that $|f'(x)\; y| < \delta/a$ for all x, y in D_0.

Another analogous theorem is established by the author.

J. O. C. Ezeilo [39] has studied the equation of the fourth order

$$x^{(4)} + a_1 \dddot{x} + a_2 \ddot{x} + g(\dot{x}) + a_4 x = 0\,. \tag{24}$$

Supposing that i) a_1, a_2, a_4 are all positive; ii) $g(0) = 0,\; g(y)/y \geqq a_3 > 0\; (y \neq 0)$; $g'(x)$ exists and is continuous and $g'(y) \leqq A_3$ for all y, where $\Delta_0 = (a_1 a_2 - A_3)\, a_3 - a_1^2 a_4 > 0$, then every solution $x = x(t)$ of (24) satisfies

$$x \to 0\,, \quad \dot{x} \to 0\,, \quad \ddot{x} \to 0\,, \quad \dddot{x} \to 0$$

as $t \to \infty$, provided that $g'(y) = g(y)/y \leqq \delta_1$ $(y \neq 0)$, where δ_1 is a constant such that

$$\delta_1 < 2a_4[(a_1 a_2 - a_3) a_3 - a_1^2 a_4]/(a_1 a_3^2).$$

The following criteria for asymptotic stability in the large are proved by A. I. Ogurcov [40] by means of Lyapunov's second method.

a) Let

$$\dddot{x} + \psi(x, \dot{x}) \ddot{x} + \varphi(\dot{x}) + f(x) = 0 \tag{25}$$

and write $F(x, y) = 2\alpha \int_0^x f(s)\, ds + f(x)\, y + \int_0^y \varphi(s)\, ds$, α a positive constant, $\varphi(0) = f(0) = 0$, $f(x)/x > 0$, $\psi(x, y) \geqq 2\alpha$, $2\alpha\, \varphi(y)/y - f'(x) > 0$, $x\, \psi_x(x, y) \leqq 0$, $F \to \infty$ as $(x, y) \to \infty$.

b) The same with condition $2\alpha\, \varphi(y)/y - f'(x) > 0$ replaced by $2\alpha\, \varphi(y)/y - f'(x) - \alpha^2[\psi(x, y) - 2\alpha] > 0$.

c) Let

$$\dddot{x} + a\ddot{x} + \varphi(x)\, \dot{x} + bx = 0, \tag{26}$$

$$a > 0, \quad b > 0, \quad \varphi(x) > b/a, \quad \int_0^x ds \int_0^s [\varphi(\tau) - b/a]\, d\tau \to \infty \quad \text{as} \quad x \to \infty.$$

d) For the equation

$$x^{(4)} + \psi(\dot{x}, \ddot{x}) \dddot{x} + c\ddot{x} + b\dot{x} + ax = 0, \tag{27}$$

c, b, a constants, $b > 0$, $a > 0$, $\psi(y, z) > 0$, $bc\, \psi(y, z) - b^2 - a^2\, \psi(y, z) > 0$, $z\, \psi_y(y, z) \leqq 0$

e) Let

$$x^{(4)} + d\dddot{x} + c\ddot{x} + \varphi(\dot{x}) + ax = 0, \tag{28}$$

$a > 0$, $d > 0$, $\varphi(0) = 0$, $\varphi(y)/y > 0$, $cd\, \varphi(y)/y - [\varphi(y)/y]^2 - ad^2 > 0$.

f) Let

$$x^{(4)} + d\dddot{x} + c\ddot{x} + \varphi(x)\, \dot{x} + ax = 0, \tag{29}$$

$a > 0$, $d > 0$, $\varphi(x) > 0$, $cd\, \varphi(x) - \varphi^2(x) - ad^2 > 0$.

V. A. Tabueva [41] has found conditions for stability and existence of a periodic solution, which is asymptotically stable, of the equation

$$\dddot{x} + \alpha\ddot{x} + \beta\dot{x} + \sin x = e(t).$$

J. O. C. Ezeilo in five further papers [42] – [46] has studied the nonautonomous equation of the thid order

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t), \tag{30}$$

$a > 0, b > 0$, and found sufficient conditions for boundedness of solutions. Supposing that $p(t)$ is a continuous periodic function of t, with a least period $\omega > 0$, sufficient conditions are given for the existence of periodic solutions with period $n\omega$, $n \geqq 1$, n integer and for the existence of at least one solution of (30) with a least period ω.

Remark. In No 10 (July 1962) of "Abstracts of papers accepted for publication in the Proceedings of the London Math. Society" a new paper of J. O. C. Ezeilo "A Boundedness Theorem for a Certain Third-Order Differential Equation $\dddot{x} + a\ddot{x} + f(x)\dot{x} + g(x) = p(t)$" is announced, in which a is a constant, $p(t)$ is continuous and $|\int_0^t p(\tau)\,d\tau|$ is bounded for all t considered.

J. O. C. Ezeilo [47] has given recently a new paper on the existence in the range $t > 0$ of a solution of

$$\dddot{x} + f(x, \dot{x})\,\ddot{x} + g(\dot{x}) + h(x) = p(t)$$

with

$$x(0) = x_0\,, \quad \dot{x}(0) = y_0\,, \quad \ddot{x}(0) = z_0\,.$$

I observe J. O. C. Ezeilo [48] has also given a theorem on the boundedness and the stability of solutions of differential equations

$$x^{(4)} + f(\ddot{x})\,\dddot{x} + \alpha_2\ddot{x} + g(\dot{x}) + \alpha_4 x = p(t)\,.$$

Furthermore, I will note that on page 57 of the "Abstracts of Communications" of the Conference EQUADIFF, J. O. C. Exeilo has announced a communication "Some Integrability Results for the Solutions of Nonlinear Differential Equation of the Third Order $\dddot{x} + f(x, \dot{x})\,\ddot{x} + g(\dot{x}) + h(x) = 0$".

V. A. Pliss [49] gives a generalization of the results of J. O. C. Ezeilo for the equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = P(x, \dot{x}, \ddot{x}, t)\,;$$

analogous results are stated for the equations

$$\dddot{z} + a\ddot{z} + \varphi(\dot{z}) + z = G(z, \dot{z}, \ddot{z}, t)\,,$$
$$\dddot{\zeta} + g(\ddot{\zeta}) + \dot{\zeta} + a\zeta = Q(\zeta, \dot{\zeta}, \ddot{\zeta}, t)\,.$$

In the case that $P(x, u, v, t)$, $G(z, u, v, t)$, $Q(\zeta, \eta, \vartheta, t)$ are periodic with respect to t, the standard application of Brouwer's fixed point theorem gives the existence of periodic solution.

The existence of forced oscillations for the system

$$\dot{x}_i = r_i x_i^k + e_i(x, t) \quad (i = 1, 2, 3)$$

where the r_i are real and non-zero, $k = 2s + 1$, s positive integer, and the $e_i(x, t)$ are polynomials in x of degree no greater than $k - 1$ with continuously differentiable coefficients which have period 1 in t, has been studied by C. Coleman [50].

The study of the behaviour of the integrals of the system

$$\dot{v} = Cv \tag{31}$$

where C is a constant $n \times n$ matrix and v is an n-vector, and of the associated system

$$\dot{v} = Cv + g(t, v) \tag{32}$$

where $g(t, v)$ is an n-vector of perturbations has been developed considerably.*)

The system (32) with C a singular matrix, i.e. $\det C = 0$, has been first studied by A. Lyapunov [51].

The case $n = 2$ has been studied by K. A. Keil (1955), S. Barocio (1956), S. Lefschetz (1957), P. Santoro (1957), H. Shintani (1959), but we wish to give notices on some results of B. N. Skačkov and R. M. Minc in the case $n = 3$.

B. N. Skačkov [52] has considered the behaviour of the integral curves and stability in the large of the system

$$\frac{dx}{dt} = \alpha x + \beta y + X, \quad \frac{dy}{dt} = \gamma x + \delta y + Y, \quad \frac{dz}{dt} = Z, \tag{33}$$

$\alpha, \beta, \gamma, \delta$ constants, $\alpha\delta - \beta\gamma \neq 0$, X, Y, Z are power series with real coefficients beginning with terms of degree two at least.

The author also supposed that the roots ϱ_1, ϱ_2 of $(\alpha - \varrho)(\delta - \varrho) - \beta\gamma = 0$ have real parts of the same sign and that if $x = u(z)$, $y = v(z)$ is a solution of the system

$$\alpha x + \beta y + X(x, y, z) = 0, \quad \gamma x + \delta y + Y(x, y, z) = 0,$$

then $Z(u(z), v(z), z) = 0$, i.e. the point $(u(z), v(z), z)$ will be singular for the system (33).

For a theorem of Lyapunov the system (33) has a first integral

$$z = f(x, y, c) + c \tag{34}$$

where $f(x, y, c)$ is holomorphic in the neighbourhood of x, y, c, $f(0, 0, 0) = 0$ and on every surface (34) there is an equilibrium point, i.e. the singular point

$$x = u(c), \quad y = v(c), \quad z = c. \tag{35}$$

The author proves that the characteristic curves trace out on the surfaces (34) in the neighbourhood of the point (35):

i) a *focus* for sufficiently small c if ϱ_1 and ϱ_2 are complex;

ii) *a node* for sufficiently small c if ϱ_1 and ϱ_2 are real and different;

*) See e.g. [1] to [6].

iii) if $\varrho_1 = \varrho_2$ we have a *focus* over the surfaces $c > 0$, a *node* over the surfaces $c < 0$.

For the system (33), supposing $\varrho_1 \neq 0$, $\varrho_2 \neq 0$, R. M. Minc [56] describes in three theorems, with the indications of proofs, the local phase-portraits near the origin.

8 Invariant surfaces — Invariant domains

The singular points of the system

$$(36) \qquad \dot{x} = yz\,, \quad \dot{y} = -xz\,, \quad \dot{z} = -K^2xy \quad (0 < K < 1)\,,$$

of the Jacobian elliptic functions are all the points of the axes x, y, z. This system has the integral

$$(38) \qquad x^2 + y^2 = \text{const}\,,$$

$$(39) \qquad Kx^2 + z^2 = \text{const}\,,$$

and consequently all the characteristic curves are quartic curves, intersections of cylinders (38), (39).

Cylinders (38) and cylinders (39) are called invariant surfaces; more generally an *invariant surface* M, of dimension r, for

$$(40) \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = f(x, t)\,,$$

x, f n-vectors, $f \in C^1$, is a C^r manifold in S_n with the property that if $U(t)$ is a solution of (40), and $U(t_0) \in M$, then $U(t) \in M$ for all t.*)

M. D. Marcus [57]**) has investigated the existence of a family of invariant tori for

*) For the applications of the theory of invariant surfaces see Yu. A. Mitropol'skiĭ, The Method of Integral Manifolds in the Theory of Nonlinear Differential Equation. Paper presented to the IVth International Mathematical Congress in Stockholm, 1962.

**) For the autonomous system $\dot{x} = f(x)$, $x = (x_1, \dots, x_n)$ an *integral invariant*, according to Poincaré, is an expression of the form

$$\int \dots \int_D M(x_1, \dots, x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n$$

where the integration is extended over any domain D with the property

$$\int \dots \int_D M(x_1, \dots, x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n = \int \dots \int_{Dt} M(x_1, \dots, x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n$$

where D_t is the domain occupied at the instant t by the points which for $t = 0$ occupy the domain D.

The so-called ergodic theorems may be linked to the theory of the integral invariant. See V. V. Nemyckiĭ-V. V. Stepanov [2], pp. 425—519.

the holonomic three-dimensional system:

$$\frac{d\vartheta}{dt} = 1 + z\,H(\vartheta, \varphi, z) + \lambda\,P(\vartheta, \varphi, z)\,, \quad \frac{d\varphi}{dt} = 1 + z\,K(\vartheta, \varphi, z) + \lambda\,Q(\vartheta, \varphi, z)\,,$$

$$\frac{dz}{dt} = A(\vartheta, \varphi)\,z + z^2\,L(\vartheta, \varphi, z) + \lambda\,R(\vartheta, \varphi, z)\,.$$

It is assumed that $H, K, L, P, Q, R \in C^3$, and they are doubly periodic in ϑ and φ of period ω_1 and ω_2, respectively.

Moreover, $H(\vartheta, \varphi, 0) = 0$, $K(\vartheta, \varphi, 0) = 0$, $|\lambda| \leqq \lambda^*$; $\int_0^{\omega_2} A(t + a, t)\,dt < 0$ for $0 \leqq a \leqq \omega_2$; $\int_0^{\omega_1} A(t, t + a)\,dt < 0$ for $0 \leqq a \leqq \omega_1$.

N. Levinson [58] and L. Amerio [59] have studied the system (40) supposing $f(x, t)$ periodic in t of period ω, for all t holomorphic in x.

Let $\bar{t}$ be a fixed value of t, and $x_i = x_i(t)$ $(i = 1, \ldots, n)$ the solution of the system (40) satisfying the initial conditions $x_i(\bar{t}) = \bar{x}_i$. Consider the points $P = (\bar{x}_1, \ldots, \bar{x}_n)$ and $P' = (x_1(\bar{t} + \omega), \ldots, x_n(\bar{t} + \omega))$ and let $\mathcal{T}$ be the transformation $P \to P'$. A closed domain D is called an *invariant domain* if $P' \in D$ whenever $P \in D$.

L. Amerio has proved the existence of analytic invariant domains D (representable by holomorphic functions depending of some parameters) such that the functions $x_1(t), \ldots, x_n(t)$ which transform D in itself are almost periodic.

9 On centers of higher dimension

M. Urabe and Y. Sibuya [60] transform the system into a form analogous to that deduced by M. Hukuhara [61] for the system of two differential equations.

Assuming that the center is at the origin and that the characteristics passing through any point lying in the neighbourhood of the origin are all closed and the period $\omega(x_1, \ldots, x_n)$ of the characteristic passing through any point $(x_1, \ldots, x_n)$ is bounded in the neighbourhood, the system is:

$$\text{(41)} \qquad \frac{dx_k}{dt} = -\vartheta_k x_{\bar{k}} + [x_1, \ldots, x_n]_2\,, \quad \frac{dx_{\bar{k}}}{dt} = \vartheta_k x_k + [x_1, \ldots, x_n]_2$$

$$(k, \bar{k} = 1, \ldots, m)\,,$$

$$\text{(42)} \qquad \frac{dx_p}{dt} = [x_1, \ldots, x_n]_2 \quad (p = 2m + 1, \ldots, n)\,,$$

and the authors give a procedure for a series expansion of the solutions.

For the solutions $\varphi_\nu(x_1, \ldots, x_n, t)$ of the system (41), (42) such that $\varphi_\nu(x_1, \ldots, x_n, 0) = x_\nu$ $(\nu = 1, \ldots, n)$ the authors give:

$$\varphi_k(x_1, \ldots, x_n, t) = x_k \cos \vartheta_k t - x_{\bar{k}} \sin \vartheta_k t + \sum_{p_1 + \ldots + p_n \geqq 2} \varphi_{k, p_1, \ldots, p_n}(t)\, x_1^{p_1} \ldots x_n^{p_n}\,,$$

$$\varphi_{\bar{k}}(x_1, \dots, x_n, t) = x_k \sin \vartheta_k t + x_{\bar{k}} \cos \vartheta_k t + \sum_{p_1+\dots+p_n \geqq 2} \varphi_{\bar{k},p_1,\dots,p_n}(t)\, x_1^{p_1} \dots x_n ,$$

$$\varphi_p(x_1, \dots, x_n, t) = x_p + \sum_{p_1+\dots+p_n \geqq 2} \varphi_{p,p_1,\dots,p_n}(t)\, x_1^{p_1} \dots x_n^{p_n} .$$

The analytical representation of the solutions of the nonlinear systems in the neighbourhood of a singular point in the case that it is not a center does not seem to have been treated in the literature.

10 Systems in connection with mechanics and mathematical physics

E. Kasner [62] has studied the solutions of Einstein equation involving functions of only one variable, i.e.

$$\text{(43)} \qquad \dot{x} = yz - x^2 , \quad \dot{y} = zx - y^2 , \quad \dot{z} = xy - z^2 ,$$

and has integrated this system completely.

The points of the line $x = y = z$ are all singular points of the system (43), and there are no others.

We have $\mathrm{d}(xy + yz + zx)/\mathrm{d}t = 0$, and consequently the invariant surfaces

$$\text{(44)} \qquad xy + yz + zx = c \quad (c = \text{constant}) .$$

For $c > 0$ the characteristic lines are on a hyperboloid of revolution of one sheet, for $c < 0$ are on a hyperboloid of revolution of two sheets, for $c = 0$ on a circular cone.

The generating lines of the cone have the equation

$$x = \lambda/(t - t_0) , \quad y = \mu/(t - t_0) , \quad z = \nu/(t - t_0) ,$$
$$\lambda + \mu + \nu = 1 , \quad \lambda\mu + \mu\nu + \nu\lambda = 0 , \quad t_0 \text{ constant} ,$$

and $x \to 0$, $y \to 0$, $z \to 0$ for $t \to \infty$; every other characteristic (having $c \neq 0$) does not tend to the origin as $t \to \infty$; therefore the origin is a saddle point.

The system (43) and the system of the Jacobi elliptic functions $\dot{x} = yz$, $\dot{y} = -zx$, $\dot{z} = -K^2xy$ $(0 < K < 1)$ are particular cases of the quadratic differential system of L. Markus [63]

$$\text{(45)} \qquad \frac{\mathrm{d}x^i}{\mathrm{d}t} = \sum_{j,k}^{1\dots n} a^i_{jk} x^j x^k , \quad a^i_{jk} = a^i_{kj} .$$

To this system L. Markus makes correspond a certain non associative commutative real linear algebra, and the problem of affine equivalence of the differential system is shown to be the same as the isomorphism problem for the corresponding algebra.

H. Weyl [64] gives a complete solution of the problem of viscous fluids which consists in finding a function $x(t)$ of a real variable $t \geqq 0$, which is a solution of the boundary-value problem

$$(46)\qquad \dddot{x} + 2x\ddot{x} + 2\lambda^2(K^2 - \dot{x}^2) = 0 \quad \text{for} \quad t \geqq 0\,;$$

$$(47)\qquad x(0) = \dot{x}(0) = 0\,, \quad \dot{x}(t) \to K \quad \text{as} \quad t \to \infty\,.$$

H. Weyl gives a complete solution of the problem, first for two special values $\lambda = 0$*) and $\lambda = \frac{1}{2}$**) by a process of alternating approximations, rapidly converging and thus suitable for numerical computations, and then he treats the general case by the method of G. D. Birkhoff, O. Kellogg and J. Leray, J. Schauder of fixed points of transformations in a functional space.

K. O. Friedrichs [68] ***) studies a particular electric circuit involving a triode by means of the system

$$(48)\qquad j'(s)\frac{\mathrm{d}s}{\mathrm{d}\vartheta} = -\,s + u\,, \quad \frac{\mathrm{d}z}{\mathrm{d}\vartheta} = -\,s + (1 - \varrho)\,u\,, \quad \alpha\,\frac{\mathrm{d}u}{\mathrm{d}\vartheta} = \varrho z - \varrho u\,,$$

$$0 < \varrho < 1\,, \quad \alpha > 0\,,$$

and demonstrates the existence of a periodic solution.

It remains an open question whether or not this periodic solution is unique and stable.

The balance of the proof is the following.

A closed three-dimensional region which is topologically equivalent to a solid torus is defined in the phase space.

The vector field is shown to point inward at all points of the boundary of the region, and the paths of vector field circulate around inside the torus.

A surface of section of the torus is a simply connected two-dimensional closed region. A continuous mapping of any point back into the region is defined by following the corresponding path around the torus until it intersects the surface of section again. An application of Brouwer's fixed point theorem establishes the existence of a fixed point of the mapping. Therefore one of the paths is closed after one evolution around the torus. This corresponds to a periodic solution.

With the same method L. Lee Rauch [70] in a paper on the "*closed-loop control system*" proves that the differential equation

$$(48)\qquad K_1\dddot{x} + [K_2 + K_3\,g(x)]\,\ddot{x} + K_3\,g'(x)\,\dot{x}^2 + g(x)\,\dot{x} + x = 0$$

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*) The case $\lambda = 0$ was the first boundary-layer problem to be numerically integrated by H. Blasius [65].

**) The case $\lambda = \frac{1}{2}$ from the point of view of the numerical integration was considered by F. Homann [66]. The solution is obtained by adjoining a solution of the differential equation in ascending powers of t to an asymptotic solution for large t. J. Siekermann in [67] makes use of the ideas of Weyl for the numerical evaluation of the solution.

***) For a systematic treatment of the nonlinear oscillations see [69], in particular pp. 120–132.

where K_1, K_2, K_3 are constants,

$$g(0) < -\frac{K_2}{2K_3} + \sqrt{\left(\frac{K_2^2}{4K_3^2} + \frac{K_1}{K_2}\right)},$$

$$m_1 > 0, \quad f(x) = m_1 \left[\frac{(K_1 - K_2K_3)^2}{K_1K_3^2} + m_1 + 1\right]^2 x - m_1 \frac{K_1 - K_2K_3}{K_1K_3} \int_0^x g(s)\,\mathrm{d}s,$$

$$f^2(x) \leqq C < \infty,$$

$$1 > 4{,}6 \frac{K_1 - K_2K_3}{K_1} \sup_{-\infty < x < \infty} \frac{f(x)}{x} + 9{,}7 \frac{K_1 - K_2K_3}{K_1} +$$

$$+ \frac{5{,}0}{m_1} \frac{K_3^3}{K_1 - K_2K_3} + 8{,}4 \frac{K_3^3}{K_1 - K_2K_3},$$

has a periodic solution.

D. Graffi [71]*) in his research on two nonlinear circuits studied the system

$$(50) \qquad y_1 = L_1 \frac{\mathrm{d}x_1}{\mathrm{d}t} + M \frac{\mathrm{d}x_2}{\mathrm{d}t} + g_1(x_1), \quad y_2 = M \frac{\mathrm{d}x_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}x_2}{\mathrm{d}t} + g_2(x_2),$$

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = e_1(t) - \frac{x_1}{C_1}, \quad \frac{\mathrm{d}y_2}{\mathrm{d}t} = e_2(t) - \frac{x_2}{C_2}$$

with C_1, C_2, L_1, L_2 positive constants, M constant, $M^2 < L_1L_2$,

$$e_1(t + T) = e_1(t), \quad e_2(t + T) = e_2(t) \quad (T > 0);$$

$g_1'(x_1)$, $g_2'(x_2)$ continuous,

$$\lim_{|x_1| \to \infty} g_1(x_1)/x_1 = R_1 > 0, \quad \lim_{|x_2| \to \infty} g_2(x_2)/x_2 = R_2 > 0,$$

$$M^2/L_1L_2 < 4R_1C_1R_2C_2/(R_1C_1 + R_2C_2)^2.$$

D. Graffi considers a suitable positive definite quadratic form U in the variables x_1, x_2, y_1, y_2 and having $\mathrm{d}U/\mathrm{d}t < 0$ along a solution of system (50), by application of Brouwer's theorem he proves the existence of a periodic solution of system (50) with period T.

G. Colombo [75]**) in a problem of coupling of two Froude's penduli***) considers the system

$$(51) \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = u, \quad \frac{\mathrm{d}u}{\mathrm{d}t} = K_1(1 - x^2)\,u - x - M_1 y,$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = K_2 v - \omega^2 y - M_2 x,$$

K_1, K_2, ω positive constants, M_1, M_2 constants.

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*) See also [72], [73], [74].

**) See also [76] to [82].

***) For Froude's pendulum see [83], pp. 21–22, 178–180.

This system is transformed into itself by changing x, u, y, v in $-x, -u, -y, -v$. Supposing $M_1 M_2 \neq \omega^2$, system (51) has only one singular point (0, 0, 0, 0).

Colombo finds a region C_3 of the $(x, 0, y, v)$-space such that every solution $x(t)$, $u(t)$, $y(t)$, $v(t)$ of system (51) starting at the time $t = 0$ from any point of this region has a point $\bar{P}$ of intersection with the region C_3' symmetric to C_3 with respect to the origin.

If P' is the symmetric of $\bar{P}$ with respect to the origin, Colombo proves that transformation of $\bar{P}$ to P' maps C_3 into a part of itself and therefore by application of the fixed point theorem there is an almost periodic solution.

J. Béthenod [84] has discovered a phenomenon known under the name of the *phenomenon of Béthenod.*

Suppose an iron pendulum near to an electromagnet carrying an alternating current.

The magnet exerts a force on the pendulum, and the pendulum affects the self-inductance of the magnet. Under suitable conditions a selfsustained oscillation of the pendulum will develop, governed by the nonlinear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[L(\vartheta)\, i\right] + Zi = e(t)\,, \tag{52}$$

$$J\ddot{\vartheta} + D\dot{\vartheta} + C\vartheta = \frac{\mathrm{d}}{\mathrm{d}\vartheta}\left[\tfrac{1}{2}L(\vartheta)\, i^2\right],$$

J, D, C, Z real constants.

This system has been investigated by the stroboscopic method by N. Minorsky [85], [86] and recently by R. Faure [87].

V. S. Serebryakova and E. A. Barbašin [88] have studied the motion of two points interacting on a circle, by the reduction of the problem to the system

$$\begin{aligned} \dot{x} &= u\,, \quad \dot{u} = -\, R_1(x, u) - f(x) - K_1\, \psi(y - x)\,, \\ \dot{y} &= v\,, \quad \dot{v} = -\, R_2(y, v) - f(y) - K_2\, \psi(y - x)\,. \end{aligned} \tag{53}$$

In particular the authors study the form of the region of stability around the equilibrium point (0, 0, 0, 0) and give also the phase portrait under different assumptions on f, R_1, R_2.

As stated in the beginning of this report there is no doubt that a monograph on nonlinear differential equations and systems of the third and fourth orders would have an intrinsic interest for mechanics, for mathematical physics and for applied science: this report perhaps sketches its outline.

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