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ON THE SOLUTION OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS WITH AN UNBOUNDED DIRICHLET INTEGRAL

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A typical function-analytical method of solving linear elliptic partial differential equations is the variational method, which can be applied under comparatively weak conditions on domains and differential operators for elliptic equations and systems of arbitrary order. The solution of a boundary problem so obtained has the finite Dirichlet integral expressing the finiteness of energy. However, the solution of many very practical problems cannot have a finite Dirichlet integral, and it is therefore necessary to derive methods which can do without this assumption.

An efficient method of arriving at a solution with an unbounded Dirichlet integral is based on the study of the regularity of a solution of the dual problem and the dual transition to the original problem. This method, applied to a very broad range of problems, has been used in a number of papers as, for instance, by S. Agmon [1], G. Fichera [5], J. L. Lions, E. Magenes [13]–[18], E. Magenes, G. Stampacchia [19], M. I. Višik, S. L. Sobolev [30]. All the quoted papers are based on the fundamental assumption of a smooth boundary of the domain examined.

The regularity of a solution, which is important for the dual method, can be expressed by the so-called a priori estimates holding for weak solutions. Again it can be shown that these estimates are connected with the smoothness of the boundary. The weak solution of a Laplace equation in a plane domain with a single non-convex angular point generally does not satisfy these estimates. A priori estimates have been obtained for instance in a paper by S. Agmon, A. Douglis, L. Nirenberg [2].

This raises the question whether there is a further method which can be applied to the case of domains with Lipschitzian boundaries. This case comprises almost all domains which occur in practice. It may be pointed out that the class of domains with smooth boundaries is quite unsatisfactory for some problems of mathematical physics. It can be shown that one of the important means for solving these problems are Rellich's equalities, which were first used by F. Rellich for the Laplace operator in [29], then generalized by L. Hörmander in [8] for an elliptic second-order equation. The technique of these equalities was elaborated by L. E. Payne and H. F. Weinberger in [27], though for another purpose. The author studied these questions in the papers [20]–[25]. An important contribution in this direction is also due to the papers of J. Kadlec [9], P. Doktor [4], O. Horáček [7].

The results refer to elliptic second-order equations, to elliptic fourth-order equations which, with the exception of the third-order operator, are products of two second-order operators, and to strongly elliptic systems of second-order equations. We shall now explain the main results with shortened "proofs", which emphasise leading ideas.

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Let Ω be a bounded domain. The symbol $W_p^{(k)}(\Omega)$ denotes the space of all real functions such that the p-th powers of the absolute values of all their derivatives up to the k-th order are intergrable over Ω. Derivatives are supposed to be generalized derivatives. The norm will be introduced in the customary way,

$$|u|_{W_p^{(k)}(\Omega)} = \left(\sum_{0 \leqq i_1 + i_2 + \ldots + i_n \leqq k} \int_\Omega \left| \frac{\partial^{i_1 + i_2 + \ldots + i_n} u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}} \right|^p \mathrm{d}\Omega \right)^{1/p} .$$

$W_p^{(k)}(\Omega)$ is a separable Banach space, which is reflexive for $p > 1$. The symbol $\mathring{W}_p^{(k)}(\Omega)$ represents the closure of $\mathscr{D}(\Omega)$ in the norm of $W_p^{(k)}(\Omega)$. Here $\mathscr{D}(\Omega)$ is the space of infinitely differentiable functions with compact support. By $W_q^{(-k)}(\Omega)$ for $p > 1$ and $1/q + 1/p = 1$ we will understand the dual space to $\mathring{W}_p^{(k)}(\Omega)$.

We shall consider the second-order operator in the form

$$A = - \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right) + b . \tag{1}$$

Here we use the summation convention of summing over the indices which occur twice in the given expression. We assume further that the real coefficients a_{ij} satisfy the symmetry condition $a_{ij} = a_{ji}$ and that $a_{ij} \in C^{(1)}(\overline{\Omega})$, where $C^{(1)}(\overline{\Omega})$ is the space of continuous functions with continuous first partials on the closure of Ω and that $b \in C(\overline{\Omega})$ where $C(\overline{\Omega})$ is the space of functions continuous on Ω. We assume that the operator (1) is uniformly elliptic, i.e., that

$$[\xi_1, \xi_2, \ldots, \xi_n] \in E_n \Rightarrow a_{ij}\xi_i\xi_j \geqq \text{const} \sum_{i=1}^{n} \xi_i^2 , \quad \text{const} > 0$$

is valid. We further assume $b \geqq 0$. It may be pointed out that the paper [9] shows that one may omit the symmetry condition and replace the condition $a_{ij} \in C^{(1)}(\overline{\Omega})$ by the condition that the coefficients a_{ij} be Lipschitzian.

Let $f \in W_2^{(0)}(\Omega)$ or, if we wish, merely $\in W_2^{(-1)}(\Omega)$. We shall say that a function u from $W_2^{(1)}(\Omega)$ is a weak solution of the equation $Au = f$, if, for every $\varphi \in \mathscr{D}(\Omega)$,

$$\int_\Omega \left(a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial u}{\partial x_j} + b\varphi u \right) \mathrm{d}\Omega = \int_\Omega \varphi f \, \mathrm{d}\Omega = f(\varphi) . \tag{2}$$

We proceed to derive the so-called Rellich equality and the ensuing Rellich inequality for the homogeneous Dirichlet problem. These questions have been studied in detail by the author in the papers [20], [24].

Let Ω be a domain with a Lipschitzian boundary, and let $f \in W_2^{(0)}(\Omega)$. Further let u be a solution of the homogeneous Dirichlet problem $Au = f$ in Ω and $u = 0$ on Ω, i.e.

on the boundary of Ω. This has the following meaning. Let u be a weak solution of the equation $Au = f$ and further let $u \in \mathring{W}_2^{(1)}(\Omega)$. Let Ω_s be a sequence of subdomains of Ω with infinitely differentiable boundaries; let $\Omega_s \to \Omega$ topologically and, moreover, let the boundaries of the domains Ω_s, locally representable by functions, converge in local co-ordinates towards the boundary of the domain Ω in the spaces C, $W_2^{(1)}$ and let them be uniformly Lipschitzian. Such a sequence always exists, as shown in [23]. Let now u_s be the solution of the Dirichlet problem $Au_s = f$ in Ω_s with $u_s = 0$ on $\dot{\Omega}_s$. It follows from the a priori estimates that $u_s \in W_2^{(2)}(\Omega_s)$. Further let $H = [h_1, h_2, \ldots, h_n]$ be a vector with components in $C^{(1)}(\overline{\Omega})$. Then almost everywhere in Ω_s

$$\frac{\partial}{\partial x_k}\left[(h_k a_{ij} - 2h_i a_{kj}) \frac{\partial u_s}{\partial x_i} \frac{\partial u_s}{\partial x_j}\right] = 2h_i \frac{\partial u_s}{\partial x_i} Au_s - 2h_i \frac{\partial u_s}{\partial x_i} bu_s + \tag{3}$$

$$+ \left(\frac{\partial h_k}{\partial x_k} a_{ij} - 2 \frac{\partial h_i}{\partial x_k} a_{kj} + h_k \frac{\partial a_{ij}}{\partial x_k}\right) \frac{\partial u_s}{\partial x_i} \frac{\partial u_s}{\partial x_j}.$$

Using Green's theorem we obtain Rellich's equality

$$\int_{\dot{\Omega}_s} (h_k a_{ij} - 2h_i a_{kj}) \frac{\partial u_s}{\partial x_i} \frac{\partial u_s}{\partial x_j} \nu_k \, dS = \tag{4}$$

$$= \int_{\Omega_s} \left(2h_i \frac{\partial u_s}{\partial x_i} Au_s - 2h_i \frac{\partial u_s}{\partial x_i} bu_s + b_{ij} \frac{\partial u_s}{\partial x_i} \frac{\partial u_s}{\partial x_j}\right) d\Omega,$$

where $[\nu_1, \nu_2, \ldots, \nu_n]$ is the outer normal to the boundary of Ω_s and b_{ij} is the expression in the last bracket in (3). The first derivatives in the expression (4) are square-integrable on the boundary of Ω_s as implied by the Sobolev imbedding theorems. The vector $(h_i a_{kj} - h_k a_{ij}) \nu_k$ where i is the co-ordinate index while j is fixed, is perpendicular to the normal and therefore $(h_i a_{kj} - h_k a_{ij}) \nu_k \,.\, \partial u_s/\partial x_i = 0$ because this expression is the derivative of u_s in the tangent hyperplane of the boundary $\dot{\Omega}_s$. For the Dirichlet problem we can therefore replace the first expression in (4) by the expression $\int_{\dot{\Omega}} h_k \nu_k a_{ij} \,.\, \partial u_s/\partial x_i \,.\, \partial u_s/\partial x_j \,.\, dS$. The vector H can be chosen in such a way that for sufficiently large s the following relation holds: $h_k \nu_k \geqq \text{const} > 0$. For a weak solution of a homogeneous Dirichlet problem there holds

$$|u_s|_{W_2^{(1)}(\Omega_s)} \leqq \text{const}\, |f|_{W_2^{(0)}(\Omega_s)}, \tag{5}$$

where const is independent of s. Let $\partial u_s/\partial \nu = a_{ij} \,.\, \partial u_s/\partial x_j \,.\, \nu_i$. Combining (4) and (5) we finally obtain the basic Rellich inequality

$$\left|\frac{\partial u_s}{\partial \nu}\right|_{W_2^{(0)}(\dot{\Omega})} \leqq \text{const}\, |f|_{W_2^{(0)}(\Omega_s)}, \tag{6}$$

where const is independent of s.

(6) is a necessary condition for the functions $\partial u_s/\partial \nu$ to converge weakly in the local co-ordinates in the space $W_2^{(0)}$. Using Green's theorem and the easily proved property

that $u_s \to u$ in $W_2^{(1)}(\Omega)$, we obtain finally that $\partial u_s/\partial \nu \rightharpoonup \partial u/\partial \nu$ (where $\rightharpoonup$ denotes weak convergence). We will call the function $\partial u/\partial \nu$ the generalized derivative in the direction of the outer co-normal. If now $v \in W_2^{(1)}(\Omega)$ then there holds Green's formula

$$\int_\Omega vf \, \mathrm{d}\Omega = -\int_{\dot{\Omega}} v \frac{\partial u}{\partial \nu} \, \mathrm{d}S + \int_\Omega \left(a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial u}{\partial x_j} + bvu \right) \mathrm{d}\Omega \,. \tag{7}$$

Naturally, for $\partial u/\partial \nu$ we have

$$\left| \frac{\partial u}{\partial \nu} \right|_{W_2^{(0)}(\dot{\Omega})} \leqq \mathrm{const} \, |f|_{W_2^{(0)}(\Omega)} \,. \tag{8}$$

Under the assumptions mentioned above there exists a continuous linear transformation of the space $W_2^{(0)}(\Omega)$ into $W_2^{(0)}(\dot{\Omega})$ which maps the right-hand side f of the homogeneous Dirichlet problem into the generalized derivative in the direction of the outer co-normal. Here the Green's formula also holds.

For the sake of simplicity in the sequel we shall ignore the limiting transition $\Omega_s \to \Omega$. If we have to deal with Dirichlet's problem and fourth-order equations, or a system of second-order equations, such a transition can be performed in exactly the same manner as described above. However, we meet new situations when studying other problems, e.g. Neumann's problem. Here one can perform the limiting transition, too, though the stability of Neumann's non-homogeneous problem must be proved, as was done in the case of the author's paper [24].

Let $g \in W_2^{(1)}(\dot{\Omega})$, $f \in W_2^{(0)}(\Omega)$ and let u be the solution of the Dirichlet problem $Au = f$ in Ω, $u = g$ on $\dot{\Omega}$. If we proceed in analogy with the method used for the derivation of Rellich's inequality in the homogeneous case, we obtain the generalized derivative in the direction of the outer co-normal, for which Green's theorem (7) and the inequality

$$\left| \frac{\partial u}{\partial \nu} \right|_{W_2^{(0)}(\dot{\Omega})} \leqq \mathrm{const} \, [|f|_{W_2^{(0)}(\Omega)} + |g|_{W_2^{(1)}(\dot{\Omega})}] \tag{9}$$

are valid.

We shall now turn our atention to the dual problem, represented by the non-homogeneous Dirichlet problem with a zero right-hand side. Let $h \in W_2^{(0)}(\dot{\Omega})$ and moreover let it be the trace of a function from $W_2^{(1)}(\Omega)$. In other words, let h be a function extensible to a function from $W_2^{(1)}(\Omega)$.

Let v be the solution of the non-homogeneous Dirichlet problem $Av = 0$ in Ω, $v = h$ on $\dot{\Omega}$. Then $v \in W_2^{(1)}(\Omega)$, and from Green's formula (7) there follows

$$\int_\Omega vf \, \mathrm{d}\Omega = \int_{\dot{\Omega}} h \frac{\partial u}{\partial \nu} \, \mathrm{d}S \,. \tag{10}$$

From the inequality (8) and from (10) there follows the dual inequality

$$|v|_{W_2^{(0)}(\Omega)} \leqq \mathrm{const} \, |h|_{W_2^{(0)}(\dot{\Omega})} \,. \tag{11}$$

Since the space of the traces is dense in $W_2^{(0)}(\dot{\Omega})$, the inequality (11) enables us to construct a linear and continuous transformation of the space $W_2^{(0)}(\dot{\Omega})$ into $W_2^{(0)}(\Omega)$ which is an extension of the operator which maps the space of traces into the space of solutions of the non-homogeneous Dirichlet problem. This extension is evidently unique. Thus we have obtained the solution of the non-homogeneous Dirichlet problem with a boundary condition, which is merely square-integrable on the boundary. It can be shown that the function so constructed has, in the domain Ω, second generalized derivatives integrable with an arbitrary power $p > 1$ and that it satisfies the equation $Av = 0$ almost everywhere.

The question concerning the behaviour of the solution in the neighbourhood of the boundary is more subtle. In the paper [22], the present author proved that in domains, which are convex in a certain sense, the inequality

$$(12) \qquad \int_\Omega \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2 \varrho \, d\Omega \leqq \text{const} \int_{\dot{\Omega}} h^2 \, dS$$

holds (here ϱ denotes the distance between the given point and the boundary) and that the function v assumes the boundary values in the mean. This is not a direct consequence of (12), because this inequality does not generally guarantee the existence of a trace.

It can be shown that in the Lipschitzian domains the inequality (8) is generally not valid if we replace the space $W_2^{(0)}(\dot{\Omega})$ by the space $W_p^{(0)}(\dot{\Omega})$, where $p > 2$, i.e., in this sense this inequality and the corresponding dual inequality are the best. In his paper [7] O. Horáček showed the possibility of refining the space $W_2^{(0)}(\dot{\Omega})$ by introducing weights of logarithmical type and thus dually extending the space of boundary conditions for non-homogeneous Dirichlet problem. Roughly speaking, O. Horáček assumed a domain with piecewise regular boundary, i.e. with only "isolated" singularities.

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Following this line of argument the homogeneous Dirichlet problem and the corresponding dual non-homogeneous problem for a strongly elliptic system of equations was studied in the author's paper [25].

Let us consider the strongly elliptic system

$$(13) \qquad -\frac{\partial}{\partial x_i}\left(a_{ij}^{\alpha\beta}\frac{\partial}{\partial x_j}\right) + b^{\alpha\beta} .$$

Assume

$$a_{ij}^{\alpha\beta} \in C^{(1)}(\overline{\Omega}), \quad b^{\alpha\beta} \in C(\overline{\Omega}), \quad a_{ij}^{\alpha\beta} = a_{ji}^{\alpha\beta}, \quad a_{ji}^{\alpha\beta} = a_{ji}^{\beta\alpha},$$
$$i, j = 1, 2, \ldots, n, \quad \alpha, \beta = 1, 2, \ldots, N .$$

At first sight, the condition $a_{ij}^{\alpha\beta} = a_{ij}^{\beta\alpha}$ restricts the admissible systems. It is, however, satisfied for the general systems of classical elasticity, if we consider media both non-homogeneous and anisotropic. Assume that

$$[\xi_1, \xi_2, \ldots, \xi_n] \in E_n,\ [\eta_1, \eta_2, \ldots, \eta_N] \in E_N \Rightarrow$$

$$a_{ij}^{\alpha\beta} \xi_i \xi_j \eta_\alpha \eta_\beta \geqq \operatorname{const} \sum_{i=1}^{n} \xi_i^2 \sum_{\alpha=1}^{N} \eta_\alpha^2, \quad b^{\alpha\beta} \eta_\alpha \eta_\beta \geqq 0. \tag{14}$$

Further assume that for $\varphi_\alpha \in \mathscr{D}(\Omega)$, $\alpha = 1, 2, \ldots, N$ the following holds:

$$\int_\Omega \left(a_{ij}^{\alpha\beta} \frac{\partial \varphi^\alpha}{\partial x_i} \frac{\partial \varphi^\beta}{\partial x_j} + b^{\alpha\beta} \varphi^\alpha \varphi^\beta \right) \mathrm{d}\Omega \geqq \operatorname{const} \sum_{\alpha=1}^{N} |\varphi_\alpha|^2_{W_2^{(1)}(\Omega)}. \tag{15}$$

It may be pointed out that the last condition is the consequence of strong ellipticity, i.e., of the condition (14) for constant coefficients, a fact which can be proved by Fourier transforms. If the coefficients $a_{ij}^{\alpha\beta}$ are continuous, the relation (14) implies the so-called Garding inequality, which is equivalent to the statement that the system $- \partial/\partial x_i (a_{ij}^{\alpha\beta} . \partial/\partial x_j) + b^{\alpha\beta} + \lambda \delta^{\alpha\beta}$, where $\delta^{\alpha\beta}$ is Kronecker's delta and λ is a sufficiently large positive number, already satisfies condition (15).

Let us solve the homogeneous Dirichlet problem

$$- \frac{\partial}{\partial x_i} \left(a_{ij}^{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} \right) + b^{\alpha\beta} u^\beta = f_\alpha, \quad \alpha = 1, 2, \ldots, N,$$

$$f_\alpha \in W_2^{(0)}(\Omega), \quad u^\beta \in \mathring{W}_2^{(1)}(\Omega), \quad \alpha, \beta = 1, 2, \ldots, N.$$

Set

$$\left(\frac{\partial u}{\partial \nu} \right)_\alpha = a_{ij}^{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} \nu_i.$$

We shall proceed precisely as in the derivation of Rellich's equality for one equation, and finally arrive at the inequality

$$\int_{\dot{\Omega}} a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} \mathrm{d}S \leqq \operatorname{const} \sum_{\alpha=1}^{N} |f_\alpha|^2_{W_2^{(0)}(\Omega)}. \tag{16}$$

Let $c^{\alpha\beta}$ be the inverse to the matrix $a_{ij}^{\alpha\beta} \nu_i \nu_j$. This matrix is evidently uniformly positive definite for points of the boundary, and the following holds:

$$\int_{\dot{\Omega}} \left(c^{\alpha\beta} \left(\frac{\partial u}{\partial \nu} \right)_\alpha \left(\frac{\partial u}{\partial \nu} \right)_\beta - a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} \right) \mathrm{d}S = 0. \tag{17}$$

This follows from

$$c^{\alpha\beta} \left(\frac{\partial u}{\partial \nu} \right)_\alpha \left(\frac{\partial u}{\partial \nu} \right)_\beta - a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} = A_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j},$$

where the vector $A_{ij}^{\alpha\beta}$ for fixed α, β, i is perpendicular to the boundary $\dot{\Omega}$. Hence we arrive at Rellich's inequality

$$\sum_{\alpha=1}^{N} \left| \left(\frac{\partial u}{\partial \nu} \right)_\alpha \right|_{W_2^{(0)}(\dot{\Omega})} \leqq \operatorname{const} \sum_{\alpha=1}^{N} |f_\alpha|_{W_2^{(0)}(\Omega)}. \tag{18}$$

By a dual process, using Green's formula, we obtain for weak solutions of the non-homogeneous Dirichlet problem with zero right-hand sides the inequality

$$\sum_{\alpha=1}^{N} |v^\alpha|_{W_2^{(0)}(\Omega)} \leqq \text{const} \sum_{\alpha=1}^{N} |h^\alpha|_{W_2^{(0)}(\dot{\Omega})}, \tag{19}$$

where $u^\alpha = h^\alpha$ at $\dot{\Omega}$. This leads again to the construction of a continuous linear operator from $[W_2^{(0)}(\dot{\Omega})]^N$ into $[W_2^{(0)}(\Omega)]^N$ (where $[\ ]^N$ denotes the Cartesian product), which is the unique continuous extension of the operator that maps boundary values into the corresponding solutions of the Dirichlet problem with a bounded Dirichlet integral.

Let us consider the fourth-order operator in the form

$$\frac{\partial^2}{\partial x_i\, \partial x_j}\left(a_{ijkl} \frac{\partial^2}{\partial x_k\, \partial x_l}\right) - \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial}{\partial x_j}\right) + b \tag{20}$$

(this problem was studied in detail in the author's paper [21]). Assume $a_{ijkl} \in C^{(2)}(\overline{\Omega})$, $a_{ij} \in C^{(1)}(\overline{\Omega})$, $b \in C(\overline{\Omega})$, $a_{ijkl} = a_{jikl} = a_{klij}$, $a_{ij} = a_{ji}$. Further assume that for every real, symmetric matrix ξ_{ij} the following relation holds: $a_{ijkl}\xi_{ij}\xi_{kl} \geqq \text{const} \sum_{i,j} \xi_{ij}^2$, for every vector $[\xi_1, \xi_2, \ldots, \xi_n] \in E_n \Rightarrow a_{ij}\xi_i\xi_j \geqq 0$ and $b \geqq 0$. We further assume there is a second decomposition of our operator in the form

$$\frac{\partial^2}{\partial x_i\, \partial x_j}\left(b_{ij}c_{kl} \frac{\partial^2}{\partial x_k\, \partial x_l}\right) + \frac{\partial}{\partial x_j}\left(a_{jkl} \frac{\partial^2}{\partial x_k\, \partial x_l}\right) + d_{ij} \frac{\partial^2}{\partial x_i\, \partial x_j} + \\ + e_i \frac{\partial}{\partial x_i} + b\,, \tag{21}$$

where

$$b_{ij}, c_{kl} \in C^{(2)}(\overline{\Omega})\,, \quad b_{ij} = b_{ji}\,, \quad c_{ki} = c_{ik}\,, \quad [\xi_1, \xi_2, \ldots, \xi_n] \in E_n \Rightarrow b_{ij}\xi_i\xi_j \geqq \\ \geqq \text{const} \sum_{i=1}^{n} \xi_i^2\,, \quad c_{kl}\xi_k\xi_l \geqq \text{const} \sum_{i=1}^{n} \xi_i^2\,, \quad a_{jkl} \in C^{(1)}(\overline{\Omega})\,, \quad d_{ij}, e_i \in C(\overline{\Omega})\,.$$

Such a situation obtains for constant coefficients and for the dimension $n = 2$ and usually under the same assumptions, also for non-constant coefficients.

Let u be the solution of the homogeneous Dirichlet problem $Au = f$ in Ω, $u = 0$, $\partial u/\partial n = 0$ at $\dot{\Omega}$ where A is the operator (20) and $\partial u/\partial n$ the derivative in the direction of the outer normal, $f \in W_2^{(0)}(\Omega)$. Here $u = 0$, $\partial u/\partial n = 0 \Leftrightarrow u \in \mathring{W}_2^{(2)}(\Omega)$ and the weak solution of the equation $Au = f$ is defined as in (2). Then we have the analogue to (3)

$$\frac{\partial}{\partial x_m}\left[(h_i a_{mjkl} + h_l a_{mkji} - h_m a_{lkji}) \frac{\partial^2 u}{\partial x_i\, \partial x_j} \frac{\partial^2 u}{\partial x_k\, \partial x_l}\right] = \\ = 2h_i a_{mjkl} \frac{\partial^2 u}{\partial x_i\, \partial x_j} \frac{\partial^3 u}{\partial x_m\, \partial x_k\, \partial x_l} + \frac{\partial}{\partial x_m}(h_i a_{mjkl} + h_l a_{mkji} - h_m a_{lkji}) \cdot \\ \cdot \frac{\partial^2 u}{\partial x_i\, \partial x_j} \frac{\partial^2 u}{\partial x_k\, \partial x_l}\,. \tag{22}$$

Using Green's formula we obtain

$$\int_{\dot\Omega} (h_i a_{mjkl} + h_l a_{mkji} - h_m a_{lkji}) \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 u}{\partial x_k \partial x_l} v_m \, dS = \tag{23}$$

$$= 2 \int_{\dot\Omega} h_i \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_m} \left(a_{mjkl} \frac{\partial^2 u}{\partial x_k \partial x_l} \right) v_j \, dS - 2 \int_{\dot\Omega} \frac{\partial h_i}{\partial x_j} \frac{\partial u}{\partial x_i} a_{mjkl} \frac{\partial^2 u}{\partial x_k \partial x_l} v_m \, dS -$$

$$- 2 \int_\Omega h_i \frac{\partial u}{\partial x_i} \frac{\partial^2}{\partial x_m \partial x_j} \left(a_{mjkl} \frac{\partial^2 u}{\partial x_k \partial x_l} \right) d\Omega + \int_\Omega c_{ijkl} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 u}{\partial x_k \partial x_l} d\Omega +$$

$$+ 2 \int_\Omega a_{mjkl} \frac{\partial^2 h_i}{\partial x_j \partial x_m} \frac{\partial^2 u}{\partial x_k \partial x_l} \frac{\partial u}{\partial x_i} d\Omega .$$

Here

$$c_{ijkl} = \frac{\partial}{\partial x_m} (h_i a_{mjkl} + h_l a_{mkji} - h_m a_{lkji}) - 2 h_i \frac{\partial a_{mjkl}}{\partial x_m} + 2 \frac{\partial h_i}{\partial x_m} a_{mjkl} .$$

The vector $(h_i a_{mjkl} - h_m a_{ijkl}) v_m$ for fixed j, k, l is perpendicular to the outer normal and the same is true for the vector $(h_l a_{mkji} - h_m a_{lkji}) v_m$ for fixed k, j, i. Hence we can write the integral on the left-hand side of (23) in the form

$$\int_{\dot\Omega} h_m v_m a_{ijkl} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial^2 u}{\partial x_k \partial x_l} dS .$$

Since, moreover $\partial u / \partial x_i = 0$ on $\dot\Omega$ the integrals on the right-hand side of (23) over $\dot\Omega$ are zero. For weak solutions of the homogeneous Dirichlet problem the following inequality holds:

$$|u|_{W_2^{(2)}(\Omega)} \leqq \text{const} \, |f|_{W_2^{(0)}(\Omega)} . \tag{24}$$

This inequality, together with a form of (23), again implies Rellich's inequality

$$\sum_{i,j=1}^{n} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|_{W_2^{(0)}(\dot\Omega)} \leqq \text{const} \, |f|_{W_2^{(0)}(\Omega)} ; \tag{25}$$

its validity is independent of whether or not the decomposition (21) exists. However, the derivation of another inequality is based on this decomposition. Let $W_2^{(-1)}(\dot\Omega)$ denote the dual space to $W_2^{(1)}(\dot\Omega)$. Let v be the solution of the equation

$$- \frac{\partial}{\partial x_i} \left(b_{ij} \frac{\partial v}{\partial x_j} \right) = 0 \tag{26}$$

and let $v = g$ on $\dot\Omega$ where $g \in W_2^{(1)}(\dot\Omega)$. Green's formula then states that

$$\int_\Omega f v \, d\Omega = \int_{\dot\Omega} b_{ij} c_{kl} \frac{\partial^3 u}{\partial x_j \partial x_k \partial x_l} v_i v \, dS + \int \frac{\partial b_{ij}}{\partial x_j} c_{kl} \frac{\partial^2 u}{\partial x_k \partial x_l} v_i v \, dS + \tag{27}$$

$$+ \int_{\dot{\Omega}} a_{jkl} \frac{\partial^2 u}{\partial x_k \, \partial x_l} \nu_i v \, \mathrm{d}S - \int_{\dot{\Omega}} c_{kl} \frac{\partial^2 u}{\partial x_k \, \partial x_l} b_{ij} \frac{\partial v}{\partial x_i} \nu_j \, \mathrm{d}S +$$

$$+ \int_{\Omega} c_{kl} \frac{\partial^2 u}{\partial x_k \, \partial x_l} b_{ij} \frac{\partial^2 v}{\partial x_i \, \partial x_j} \mathrm{d}\Omega - \int_{\Omega} a_{jkl} \frac{\partial^2 u}{\partial x_k \, \partial x_l} \frac{\partial v}{\partial x_i} \mathrm{d}\Omega +$$

$$+ \int_{\Omega} \left(d_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + e_k \frac{\partial u}{\partial x_k} + bu \right) v \, \mathrm{d}\Omega .$$

From the inequalities (9), (24)–(27) there follows the inequality

$$(28) \qquad \left| b_{ij} c_{kl} \frac{\partial^3 u}{\partial x_j \, \partial x_k \, \partial x_l} \nu_i \right|_{W_2^{(-1)}(\dot{\Omega})} \leqq \text{const} \, |f|_{W_2^{(0)}(\Omega)} .$$

Let now v be a weak solution of the equation $Av = 0$ in Ω and let $v = h$, $\partial v / \partial n = g$ on $\dot{\Omega}$. We assume here that there is a function $v_0 \in W_2^{(2)}(\Omega)$ such that, in the sense of traces, $v_0 = h$, $\partial v_0 / \partial n = g$ on Ω. Since $u \in \mathring{W}_2^{(2)}(\Omega)$ we obtain

$$(29) \qquad \int_{\Omega} c_{kl} \frac{\partial^2 u}{\partial x_k \, \partial x_l} b_{ij} \frac{\partial^2 v}{\partial x_i \, \partial x_j} \mathrm{d}\Omega - \int_{\Omega} a_{jkl} \frac{\partial^2 u}{\partial x_k \, \partial x_l} \frac{\partial v}{\partial x_j} \mathrm{d}\Omega +$$

$$+ \int_{\Omega} \left(d_{ij} \frac{\partial^2 u}{\partial x_i \, \partial x_j} + e_k \frac{\partial u}{\partial x_k} + bu \right) v \, \mathrm{d}\Omega = 0 .$$

Thus we have derived Green's formula

$$(30) \qquad \int_{\Omega} fv \, \mathrm{d}\Omega = \int_{\dot{\Omega}} b_{ij} c_{kl} \frac{\partial^3 u}{\partial x_j \, \partial x_k \, \partial x_l} \nu_i v \, \mathrm{d}S + \int_{\dot{\Omega}} \frac{\partial b_{ij}}{\partial x_j} c_{kl} \frac{\partial^2 u}{\partial x_k \, \partial x_l} \nu_i v \, \mathrm{d}S +$$

$$+ \int_{\dot{\Omega}} a_{jkl} \frac{\partial^2 u}{\partial x_k \, \partial x_l} \nu_i v \, \mathrm{d}S - \int_{\dot{\Omega}} c_{kl} \frac{\partial^2 u}{\partial x_k \, \partial x_l} b_{ij} \frac{\partial v}{\partial x_i} \nu_j \, \mathrm{d}S .$$

From this formula and the inequalities (22), (25) there follows the dual inequality

$$(31) \qquad |v|_{W_2^{(0)}(\Omega)} \leqq \text{const} \left[|h|_{W_2^{(1)}(\dot{\Omega})} + |g|_{W_2^{(0)}(\dot{\Omega})} \right] .$$

Again we may extend continuously the operator which maps boundary values into solutions of the non-homogeneous Dirichlet problem with a bounded Dirichlet integral. This extension is unique and transforms $W_2^{(1)}(\dot{\Omega}) \times W_2^{(0)}(\dot{\Omega})$ into $W_2^{(0)}(\Omega)$. The corresponding theorem on density is valid again.

It may be pointed out that it is not necessary to require the decomposition (21), if in the boundary conditions there is $h \equiv 0$.

4

We will now again return to the second-order equation (cf. the author's paper [24]). For the sake of simplicity, let $b \not\equiv 0$. Let $f \in W_2^{(0)}(\Omega)$, $g \in W_2^{(0)}(\dot{\Omega})$. Let u be the solution of the Neumann problem $Au = f$ in Ω, $\partial u / \partial \nu = g$ on $\dot{\Omega}$. From (4) we obtain at once the inequality

$$(32) \qquad |u|_{W_2^{(1)}(\dot{\Omega})} \leqq \text{const}\,[|f|_{W_2^{(0)}(\Omega)} + |g|_{W_2^{(0)}(\dot{\Omega})}]\,.$$

Let now v be the solution of the Neumann problem $Av = 0$ in Ω, $\partial v/\partial \nu = h$ on $\dot{\Omega}$ where $h \in W_2^{(0)}(\dot{\Omega})$. Here Green's formula states $\int_\Omega vf\,\mathrm{d}\Omega = \int_{\dot{\Omega}} hu\,\mathrm{d}S$ and hence from (32) there follows the dual inequality

$$(33) \qquad |v|_{W_2^{(0)}(\Omega)} \leqq \text{const}\,|h|_{W_2^{(-1)}(\dot{\Omega})}\,.$$

The space $W_2^{(0)}(\dot{\Omega})$ is dense in $W_2^{(-1)}(\dot{\Omega})$ and thus from (33) there follows the existence of a solution of the Neumann problem for the boundary condition from $W_2^{(-1)}(\dot{\Omega})$. It may be mentioned that for $n = 2$, we can consider h as an arbitrary measure, e.g. the Dirac's one.

In a similar way we may generalize and solve the Newton problem or the mixed problem. In the latter case it is necessary to prescribe the same type of boundary condition for every component of $\dot{\Omega}$.

Before we approach the inequalities, describing the regularity of the solutions of second-order equations by using the spaces $W_p^{(1)}(\Omega)$, $p \neq 2$, we will mention that Rellich's equality can be used as a proof of the numerical method proposed by M. Picone, as is done in P. Doctors' paper [4]. This paper also generalizes Riesz' theorem on conjugate harmonic functions in a domain with Lipschitzian boundary.

5

Several authors – among them S. Agmon in [1] and J. L. Lions in [12] – proved, almost simultaneously, the following statement:

Let K be a domain with a sufficiently smooth boundary. Let $p > 1$. Then the operator (1) is an isomorphism of the space $\mathring{W}_p^{(1)}(K)$ onto $W^{(-1)}_p(K)$. Let us assume that the closure of the examined domain Ω, with a Lipschitzian boundary, is contained in K and that the operator is defined on K. Let u be the solution of the Dirichlet problem $Au = f$ in Ω, $u = g$ on $\dot{\Omega}$ where $f \in W_{2n/(n+1)}^{(0)}(\Omega)$, $g \in W_2^{(1)}(\dot{\Omega})$. By using the theorem mentioned above and Rellich's equality (4) we arrive at Rellich's inequality in the following form:

$$(34) \qquad \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|_{W_2^{(0)}(\dot{\Omega})} \leqq \text{const}\,[|f|_{W_{2n/(n+1)}^{(0)}(\Omega)} + |g|_{W_2^{(1)}(\dot{\Omega})}]\,.$$

(A detailed derivation may be found in [24].) It is even more interesting that we obtain the inequality

$$(35) \qquad |u|_{W_{2n/(n-1)}^{(1)}(\Omega)} \leqq \text{const}\,[|f|_{W_{2n/(n+1)}^{(0)}(\Omega)} + |g|_{W_2^{(1)}(\dot{\Omega})}]\,.$$

Analogous inequalities for the other problems may also be obtained.

An interesting consequence of the inequality (35) and of the im bedding theorems is that the examined solution of the Dirichlet problem for $n = 2$ is of Hölderian type with exponent $\frac{1}{2}$.

Let v be the solution of the Dirichlet problem $Av = f$ in Ω, $v = h$ on $\dot{\Omega}$ where $f \in W_2^{(0)}(\Omega)$, h is a trace. From Green's formula (35) there follows

$$|v|_{W_{2n/(n-1)}^{(0)}(\Omega)} \leqq \text{const}\left[|f|_{W_{2n/(n+1)}^{(-1)}(\Omega)} + |h|_{W_2^{(0)}(\dot{\Omega})}\right]. \tag{36}$$

The space $W_2^{(0)}(\Omega)$ is dense in $W_{2n/(n+1)}^{(-1)}(\Omega)$ and thus using (36) we may extend the corresponding operator continuously. In other words: There is a solution of the Dirichlet problem for any right-hand side in $W_{2n/(n+1)}^{(-1)}(\Omega)$ and with a square-integrable boundary condition.

In conclusion we should like to point out that within the frame of these results there is still a great number of open problems. There are, first of all, questions of attaining boundary values, other problems of second-order systems and equations of the fourth order, the question whether the above-mentioned method is applicable to general fourth-order operators which are not products of second-order operators, and other problems. For instance, a question of special interest is whether, for $h = 0$ the inequality (36) is valid in the form

$$|v|_{W_{2n/(n+1)}^{(1)}(\Omega)} \leqq \text{const}\,|f|_{W_{2n/(n+1)}^{(-1)}(\Omega)}. \tag{37}$$

We did not always emphasize that the considered solutions with an unbounded Dirichlet integral are sufficiently smooth inside the examined domain. They always satisfy weakly the differential equation and, for those right-hand sides which are functions, they satisfy the equation almost everywhere. With stronger assumptions on the coefficients and right-hand sides, the solutions satisfy the differential equations in the classical sense.

A study of these problems may also take another direction, using weighed Sobolev's spaces, a special case of which has already been mentioned. These spaces were used in the author's papers [22] and [26]. This method appears to be useful also in domains with smooth boundaries. It is possible to solve several singular cases occuring in the papers by J. L. Lions, E. Magenes [13]–[17], mentioned above, in which the authors use Sobolev's spaces with a fractional derivative. It may well become that some of their results can be applied to domains with a non-smooth boundary as, for instance, for convex domains. As a matter of fact, J. Kadlec in his paper [10] has already proved the validity of some of the a priori estimates for such a domain.

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