THE THEORY OF SMALL CHANGES IN THE DOMAIN OF EXISTENCE IN THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

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In connection with basic problems of the theory of partial differential equations there are the so called conditions for the correct formulation of a problem. These conditions are cited in almost every basic textbook. In this sense we take a problem to be correctly formulated if the following conditions are satisfied:

1) the condition of solvability, i. e. the condition of the existence of the solution;
2) the condition of the uniqueness of the solution;
3) the condition of the continuous dependence of the solution on the boundary conditions or on the right-hand side.

The condition of the correct formulation of a problem is connected with mathematical applications. If a mathematical problem should describe a real physical problem, it must also fulfil its basic qualitative characteristics. The three conditions cited above, i. e. conditions of the existence, uniqueness and continuous dependence are fulfilled in principle by every physical problem.

Hadamard analysing the conditions of the continuous dependence on boundary conditions or on the right-hand side states that "the boundary conditions in almost every physical example are given empirically, i. e. only approximately and sometimes with a great deal of approximation. Therefore a solution, for which we are not able to determine its dependence on changes of boundary conditions, at least theoretically, has no practical sense".

Although it is clear that the question of approximation may be more subtle in practice than is apparent at first sight (e. g. it is a question of determining a measure for the approximation), Hadamard's analysis reveals the very kernel of the question of the applicability of a mathematical solution in problems of natural and technical sciences.

But for the same reasons which are cited by Hadamard, it is also clear that the correctly formulated problem must fulfil two more conditions in addition to the previous three, namely

4) the condition of the continuous dependence on small changes of the coefficients of the equation and

5) the condition of the continuous dependence on small changes of the domain of existence.

The coefficients of a differential equation usually characterise some physical qualities of the material or their fenomenological description.

The domain of existence, on which the problem is solved, is also an abstraction, which is fulfilled in practice only with various degrees of approximation. For example a building construction which is solved as a circular plate (i.e. the domain of existence is supposed to be a circle) in practice more closely resembles a polygon with a great number of sides (depending on the exact form of the shuttering used) than a circle etc. Therefore we must demand that a problem fulfils the fifth condition, i.e., the solution must depend continuously on small changes of the domain of existence.

In this paper we will deal with the mathematical problems connected with the question of dependence of the solution on small changes of the domain of existence, i.e. with the fifth condition for the correctness of the problem in the sense quoted above.

It has been shown that this problem has applications in many branches of mathematics. The mathematical significance of this condition is greater than it seems at first sight.

Due to lack of space, we must restrict ourselves to an illustrative choice of questions and mathematical assertions and – for the sake of simplicity – deal only with the simplest cases. We would like to clarify the basic problem and some interrelations but will not use precise formulation of definitions and the orems in their most general form nor give their proofs.

In the literature very little attention is paid to the problems of the dependence of the solution on small changes in the domain of existence, although, as we shall see it, there are very interesting and important mathematical problems originating here.

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Let us at first formulate this mathematical problem for a differential equation of the elliptic type, with a given domain of existence.

Definition 1. *Let us have a bounded domain $\Omega \subset E_2$ and a linear space $M(\Omega)$ of smooth functions defined on Ω with the scalar product*

$$[u, v]_{L_2{}^l(\Omega)} = \iint_\Omega \left(\sum_{i=0}^{l} \binom{l}{i} \frac{\partial^l u}{\partial x^i \, \partial y^{l-i}} \frac{\partial^l v}{\partial x^i \, \partial y^{l-i}} \right) \mathrm{d}x \, \mathrm{d}y \, .$$

Further let f be a function defined on Ω. The function $w(M(\Omega), f, l)$ defined on Ω will be the solution of the problem $\Delta^l w = f$ in relation to $M(\Omega)$ if

1) $w \in \overline{M(\Omega)}$, *where $\overline{M(\Omega)}$ is a completment of the space $M(\Omega)$ with respect to the norm introduced;*

2) $[w, u]_{L_2{}^l(\Omega)} = (-1)^l \iint_\Omega uf \, \mathrm{d}x \, \mathrm{d}y$ *for every* $u \in M(\Omega)$.

Remark. For the sake of a simple notation we assume that $\Omega \subset E_2$ and that the scalar product is given in the simplest form of the norm in L_2^l. However, an analogical definition can also be formulated for a general case.

This definition holds for a general domain of existence. It depends of course on the concrete choice of the space $M(\Omega)$; depending on this choice the solution exists, is just one, etc. The space also determines the homogeneous boundary conditions fulfilled by the solution on the boundary of the domain of existence.

To illustrate the different choice of the space $M(\Omega)$ let us examine the biharmonic problem $l = 2$.

I. Let us include in the space $M(\Omega)$ all functions having four continuous derivatives, the value of the functions and their first derivatives being equal to zero on the boundary of the domain of existence. Let us designate this space as $M_I(\Omega)$. In this case Definition 1 gives the solution of the classical biharmonic problem with the right-hand side, when a zero value and zero first derivative is prescribed for the function in the normal direction on the boundary of the domain of existence. Because this is a natural generalization of the classical Dirichlet's or Dirichlet-Poisson's problem for the Laplace equation we will call this problem, characterized by the space $M_I(\Omega)$ Dirichlet's problem for the biharmonic equation. It can be shown, that the solution of this problem fulfils the first three conditions of correctness, i.e. the common conditions for the existence, uniqueness and continuous dependence on the right-hand side. The corresponding physical interpretation is the problem of a built-in plate.

II. If we include in $M(\Omega)$ functions with four continuous derivatives and will demand, contrary to the first example, that only the values of the functions be equal to zero on the boundary of the domain, we obtain the space $M_{II}(\Omega)$. This space characterizes the so called intermediate problem for the biharmonic equation. In the classical formulation we are dealing again with the solution of the biharmonic problem. But here are fulfilled the special homogeneous boundary conditions, which, on a straight-line segment of the boundary, can be formulated as follows: we demand that the value of the desired function and its second derivative in the normal direction be equal to zero. These conditions are more complicated for a non-linear segment of the boundary, where the curvature of the boundary begins to play a role. Of course, Definition 1 does not include the assumption about the smooth boundary and existence of a normal and holds for every domain of existence, i.e. for an open connected set. The conditions for the normal derivative in the classical sense are of course fulfilled only in the case of a smooth domain. The physical interpretation of the intermediate problem for the biharmonic equation is the problem of a free supported plate.

It can be shown again, that the conditions of the existence, uniqueness and the continuous dependence on the right-hand side are fulfilled.

III. If we include in the space $M(\Omega)$ all smooth functions, with no boundary conditions to be fulfilled, we obtain the space $M_{III}(\Omega)$. This space characterizes the so called Neumann's problem for the biharmonic equation. We call it a Neumann's problem, because it represents a natural generalization of Neumann's or Neumann-Poisson's problem for the Laplace equation of second order and has similar properties concerning the existence and uniqueness of the solution.

In the classical case it is again a biharmonic problem with the following homogeneous conditions on a straight-line segment of the boundary: $\partial^2 w/\partial n^2 = \partial^3 w/\partial n^3 + 2 \, . \, \partial^3 w/(\partial s^2 \, \partial n) = 0$, s is an arc, n a normal.

The physical interpretation is the free plate problem. The basic characteristics of the solutions of the problems cited above follow from the functional analytic theory of elliptic differential equations. See for example [1], [2].

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Let us now formulate the problem of the stability of the solution.

Definition 2. *Let Ω be a domain and K a unit open circle, $\bar{\Omega} \subset K$. We will call the domain Ω stable from the interior with respect to the problem $\Delta^l w = f$ and the spaces $M(\Omega)$ and $\mathscr{K}$, if for every function $f \in \mathscr{K}$ defined on K and for every sequence Ω_n, $n = 1, 2, \ldots$ such that $\bar{\Omega}_n \subset \Omega$, $\Omega_n \to \Omega$ (i. e. for $\bar{F} \subset \Omega$ it holds $\bar{F} \subset \Omega_n$ for every $n > N(x)$) holds*

$$(1) \qquad w(M(\Omega_n), f, l) \to w(M(\Omega), f, l) \quad in \quad L_2^l(\Omega_n) \, .$$

If Ω is stable from the interior, we will call the function $w(M(\Omega), f, l)$ the interior solution of the given problem.

We will call the domain Ω stable from the exterior with respect to the problem $\Delta^l w = f$ and the spaces $M(\Omega)$ and $\mathscr{K}$, if for every function $f \in \mathscr{K}$ defined on K and for every sequence Ω_n, $n = 1, 2, \ldots$ such, that $\bar{\Omega} \subset \Omega_n$, $\bar{\Omega}_n \subset K$, $\Omega_n \to \Omega$ (i. e. for $\bar{F} \cap \Omega = \emptyset$ it holds $\bar{F} \cap \Omega_n = \emptyset$ for every $n > N(x)$, the following holds:

$$(2) \qquad w(M(\Omega_n), f, l) \to \widehat{w(M(\Omega)}, f, l) \quad in \quad L_2^l(\Omega) \, .$$

If Ω is stable from the exterior, we will call the function $\widehat{w(M(\Omega)}, f, l)$ the exterior solution of the given problem. A domain Ω without interior boundary points (i. e., a domain such that $\Omega = E_m - \overline{(E_m - \bar{\Omega})}$) we shall call a stable domain with respect to $\Delta^l w = f$ and the spaces $M(\Omega)$ and $\mathscr{K}$, if for every function $f \in \mathscr{K}$ defined on K and for every sequence Ω_n, $n = 1, 2, \ldots$ such that $\bar{\Omega}_n \subset K$, $\Omega_n \to \Omega$ (i. e. if $\bar{F} \subset \Omega$ then $\bar{F} \subset \Omega_n$ for every $n > N_1(x)$ and if $\bar{F} \cap \bar{\Omega} = \emptyset$, then $\bar{F} \cap \Omega_n \, \emptyset =$ for every $n > N_2(x)$) holds

$$(3) \qquad w(M(\Omega_n), f, l) \to w(M(\Omega), f, l) \quad in \quad L_2^l(\Omega \cap \Omega_n) \, .$$

Remark. In Definition 2 we demand the continuous dependence of the solution on Ω_n for every $f \in \mathscr{K}$. It is clear that $\mathscr{K}$ must be such as to guarantee the existence of the solution $w(M(\Omega_n), f, l)$.

In the case of Dirichlet's problem for the l-harmonic equation we write $\mathscr{K} \equiv L_2(K)$.

In the case of Neumann's problem for the l-harmonic equation $\mathscr{K}$ will include all functions $f \in L_2(K)$ having a compact support in Ω and moreover

$$(4) \qquad \iint_\Omega f x^\alpha y^\beta \, \mathrm{d}x \, \mathrm{d}y = 0 \quad \text{for} \quad \alpha + \beta \leqq l - 1 \, ,$$

where α, β are non-negative integers.

By substantially narrowing $\mathscr{K}$ we can achieve the stability of Ω. The Definition means that we should take $\mathscr{K}$ as large as possible so that $\mathscr{K}$ would be dense in that part of $L_2(\Omega)$, for which the solution in Ω, according to Definition 1, does exist.

In equation (2), contrary to the case of stability from the interior, we obtain the space $\widehat{M(\Omega)}$, which has generally no connection with $M(\Omega)$. This space must not of course depend on the choice of the sequence Ω_n, but only on Ω. Generally speaking $\widehat{M(\Omega)} \neq M(\Omega)$. A simple example of this situation is the case, where the domain of existence includes the so called interior boundary points; for example it may be the circle with an excluded segment. In this case we obtain a circle with excluded segment for Ω_n converging from the interior to Ω and a circle without a segment excluded for the convergence from the exterior. This is easy to understand, because in the limiting case we deal with a solution in two different domains of existence.

There also exist, of course, nontrivial examples, i. e., examples of domains without internal boundary points and hence with the same limiting domain of existence for both the internal and external convergence, for which nevertheless $M(\Omega) \neq \widehat{M(\Omega)}$.

For the domains without internal boundary points it is reasonable to introduce the conception of a stable domain according to Definition 2.

4

In the preceding paragraphs we have introduced three characteristic problems for the l-harmonic equation. We will now study the stability of the domains in these cases.

Let us begin with Dirichlet's problem for the l-harmonic equation, which is characterized by the space $M_l(\Omega)$. The classical formulation leads to the l-harmonic equation with the boundary conditions

$$w = \frac{\partial w}{\partial n} = \ldots = \frac{\partial^{l-1} w}{\partial n^{l-1}} = 0 .$$

The following Theorem can be proved:

Theorem 1. *Let us consider Dirichlet's problem for the l-harmonic equation and let* $\mathscr{K} = L_2(K)$. *Then*

1) *every domain is stable from the interior and from the exterior,*

2) *if $m = 2l$, where m is the dimension of the space of the domain, then there exists a domain which is not stable.*

3) *A sphere is a stable domain.*

4) *If $l = 1$, i. e. for the harmonic equation, then in the plane every Caratheodory domain with zero boundary measure is stable.*

For proof see [3].

For illustration let us now introduce some open problems. Does there exist an unstable domain for $m < 2l$? In the plane every Caratheodory domain is stable for $l = 1$. Is this domain also stable for $l > 1$?

Let us proceed now to the second problem cited above, i.e. to the intermediate problem. We will restrict ourselves to the case $l = 2$, i.e. to the case of the biharmonic problem with physical meaning – a free supported plate. Here we obtain a surprising result.

Theorem 2. *For the intermediate problem of the biharmonic equation, with $\mathscr{K}$ including all functions $f \in L_2(\Omega)$ with a compact support on Ω, there is no domain stable either from the exterior, or from the interior.*

For proof see [3]. This property is very interesting because this is a problem often met in practice and we will study it in more detail later.

Let us now examine the third problem cited, namely Neumann's problem for the polyharmonic equation. Here we have an analogous situation to the case of Dirichlet's problem. The following theorem holds:

Theorem 3. *Let us have Neumann's problem for the l-harmonic equation and let $\mathscr{K}$ include all functions $f \in L_2(\Omega)$ with a compact support in Ω and fulfilling equation* (4). *Then*

1) *all domains are stable from the interior and from the exterior,*
2) *there exists an unstable domain,*
3) *a sphere is a stable domain,*
4) *In the plane every Caratheodory domain with zero boundary measure is stable for $l = 1, 2$.*

For proof see [3] and [4].

Let us mention now at least one open problem. Is the Caratheodory domain stable in the plane for $l > 2$?

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We will now examine in more detail the question of instability of the intermediate problem for the biharmonic equation, the physical meaning of which is the free supported plate in the classical formulation of Sophie Germain. This is the basic problem in the theory of elasticity with wide applications in technical practice. Nevertheless there are interesting instability phenomena also in this case. One concrete example of this instability follows.

Let Ω be a circle with radius R, the coordinates of its centre being $(0, 0)$ and let Ω_n be a regular n-gon with the same center. Let us suppose, that these n-gons converge to the circle with ascending n. Let $f \equiv 1$, i. e. let us solve the problem of a plate homogeneously loaded. Now it can be proved that

$$w(M_{II}(\Omega_n),\ f \equiv 1, 2)\,(0, 0) \rightarrow \frac{32R^2}{64} \quad \text{for} \quad n \rightarrow \infty\,,$$

$$w(M_{II}(\Omega),\ f \equiv 1, 2)\,(0, 0) = \frac{52R^2}{64}\,.$$

So we see that the value of the solution $w(M_{II}(\Omega_n), f, 2)$ in the coordinate beginning, i.e., in the center of the n-gon, converges to a certain value, but this value differs from the value for the circle. This difference cannot be neglected.

The significance of this result can be compared with the fact, mentioned in the Introduction, that the circular plate made from concrete can often be treated as an n-gon rather than a circle. The question is, which of these solutions is physically correct, and whether this instability is a real fact or only a property of the mathematical abstraction. From the physical point of view it is clear that in reality the solution must be stable and that the instability found by mathematical methods is a result of unsuitable formulation of the problem of the plate. The difference between the limiting solution and the solution for a circle depends on the manner of convergence of the domains Ω_n to the circle Ω. If for example the Ω_n^* are smooth domains converging to the circle including the curvature of the boundary, then there is no instability, because

$$w(M_{II}(\Omega_n^*),\ f \equiv 1, 2) \to w(M_{II}(\Omega),\ f \equiv 1, 2) \quad \text{for} \quad n \to \infty\,.$$

From this we see that the instability depends strongly on the manner in which the domain converges. From the physical point of view as well as from the point of view of the mathematical formulation of the problem of a plate we can ask if it is suitable to assume the solution only on smooth domains, so as to exclude that paradox. To support this standpoint we can argue, that the angle-points do not exist in reality. But it is also clear, that the answer to a question so formulated depends on the personal opinion of the person asked.

Let us examine this question in more detail from another point of view. The plate, the stress of which we would like to ascertain, is in fact a three-dimensional body. So we shall formulate this problem as a three-dimensional problem of the mathematical theory of elasticity. But in practice we simplify the solution and solve this problem as a two-dimensional one and we are convinced that this approximation is precise enough for technical applications. But, as we have pointed out earlier, we might solve the problem of a plate as a three-dimensional one, which mathematically leads to the solution of a strongly elliptical system of partial differential equations, the so called equations of Lamé

$$\begin{aligned}
&\frac{\partial \Theta}{\partial x} + \Delta u = 0\,,\\
&\frac{\partial \Theta}{\partial y} + \Delta v = 0\,,\\
&\frac{\partial \Theta}{\partial z} + \Delta w = 0
\end{aligned} \tag{5}$$

with the corresponding boundary conditions on the cylindrical-shaped domain, the height of which is $2h$ and the base is Ω.

The solution, used in practice for more than 100 years is the two-dimensional approximation, leading to the Sophie Germain equation. The relation between the solution obtained in this way and the correct solution of the three-dimensional problem gives the Bernoulli hypothesis of plane sections; we can express it in the form

$$
\begin{aligned}
(6) \qquad u(x, y, z) &= -z\frac{\partial w}{\partial x}, \\
v(x, y, z) &= -z\frac{\partial w}{\partial y}, \\
w(x, y, z) &= w(x, y).
\end{aligned}
$$

If we know the function w, which is the solution of the Sophie Germain equation, we suppose that the functions u, v, w given by the equations (6) are a good approximation to the solution of the Lamé equations. Therefore it must be stressed that the solution in the classical formulation of Sophie Germain is to be taken as an approximation of the three-dimensional problem into the two-dimensional one. Therefore let us examine the stability of the problem in the original three-dimensional formulation.

In the three dimensional case it can be proved that the solution is stable not only for a circle, but also for every Caratheodory domain. Thus we have for n-gons a continuous transition (see [5]).

In the two-dimensional formulation the solution is unstable for every domain.

We can therefore see that the instability arises from the transition of the three-dimensional problem to the two-dimensional one with the help of the Bernoulli hypothesis. This is also a sufficient reason to contradict the opinion that for the exclusion of the instability paradox we must restrict ourselves to smooth domains.

Now we can ask if there exists a transformation of the three-dimensional problem to the two-dimensional one, which preserves the original stability and describes the three-dimensional solution with sufficient degree of precision.

It can be shown that there exists a transformation, which in certain natural sense can be assumed as an optimal one, which gives very good results and preserves the stability. This transformation leads to the following system of three partial differential equations of the second order and of strongly elliptical type for the unknown quantities p, q, w

$$
\begin{aligned}
(7) \qquad \Delta p + \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 q}{\partial x\,\partial y} + \frac{3}{h^2}\left(p - \frac{\partial w}{\partial x}\right) &= 0, \\
\Delta q + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 p}{\partial x\,\partial y} + \frac{3}{h^2}\left(q - \frac{\partial w}{\partial y}\right) &= 0, \\
\Delta w + \frac{\partial p}{\partial x} - \frac{\partial q}{\partial y} &= \frac{1}{h}\,g.
\end{aligned}
$$

The connection between the original three-dimensional problem and this two-dimensional one is defined by the following approximate relations:

(8) $u(x, y, z) = - z\, p(x, y)\,, \quad v(x, y, z) = - z\, q(x, y)\,, \quad w(x, y, z) = w(x, y)\,.$

By noting the difference between the classical problem of Sophie Germain and the approximation in this formulation we see that both approximations are very similar. In the latter case we abandon the assumption of the perpendicular section with respect to the deformed central surface, or in other words, we take into account in a suitable way the influence of the displacing forces.

The question of how to transform the multi-dimensional problems to problems with a lower number of dimensions is very extensive (see [6], [7]). Due to lack of space, we have only touched some of the aspects which are closely connected with the problem of stability.

6

On the basis of the detailed mathematical examination we have explained in a physical way the reason for the instability of the intermediate problem.

But we have proved that there exists an unstable domain with respect to Dirichlet's problem for the Laplace equation. This problem also has a physical meaning and we can again ask why such a domain exists. It can be shown that the unstable domains are characterized by some porosity, i. e. by very small holes. These holes (i. e. porosity) are in certain sense in contradiction to the assumption of a continuous medium, which is the basic assumption in the derivation of Dirichlet's problem for the Laplace equation, for example when we are studying heat conduction.

If this interpretation is right, then there must exist a relation between the stability of the domain of existence in the case of a differential equation of the elliptic type and the stability of cylindrical-shaped domains for equations of the parabolic and hyperbolic type. In this case the time variable, which creates the cylindrical domain, can have no influence on the stability, and it is the space form of the domain which is decisive.

It can be shown, that this relation really exists. Let us examine the stability of the domain $D = \Omega \times \langle 0, T \rangle$ for the equation of parabolic and hyperbolic type, $\partial u/\partial t = \Delta u + f$, $\partial^2 u/\partial t^2 = \Delta u + f$ respectively, with homogeneous Dirichlet's conditions on the boundary. We will assume small changes of the domain Ω and study their influence on the solution in the same way, as we have studied the stability for the equation of elliptical type.

Now, what we have expected on the basis of physical intuition can be proved, namely Theorem 4.

Theorem 4. *The domain $D = \Omega \times \langle 0, T \rangle$, $\bar{\Omega} \subset K$ for an equation of parabolic type $\partial u/\partial t = \Delta u + f$ or hyperbolic type $\partial^2 u/\partial t^2 = \Delta u + f$ with homogeneous Dirichlet's conditions and for a function f, which is continuous and smooth on*

$K \times \langle 0, T \rangle$, *is stable if and only if Ω is stable with respect to Dirichlet's problem for the harmonic equation and* $\mathscr{K} = L_2(K)$.

Proof. See [8] and some unpublished results.

Remark. The solution of the equation of parabolic and hyperbolic type is understood in the usual sense.

For the same physical reasons, which we have mentioned above, we can expect a relation between the stability of the domain and the stability of eigen-values, representing physically the eigen-frequencies.

Let us examine the eigen-value problems

$$\Delta^l u = \lambda u\,, \quad (x, y) \in \Omega\,, \quad \frac{\partial^k u}{\partial u^k} = 0 \quad \text{for} \quad k = 0, 1, \ldots, l - 1$$

with homogeneous Dirichlet's boundary conditions. In quite the same way as we have introduced the idea of the interior and exterior solution for the problem $\Delta^l u = f$ as limiting solutions for the domain converging from the interior or the exterior, we can introduce the idea of the interior and exterior eigen-value or eigen-function.

Theorem 5. *The domain Ω is stable with respect to Dirichlet's problem for an l-harmonic equation and $\mathscr{K} = L_2(K)$ if and only if the corresponding exterior and interior eigen-values for the problem $\Delta^l u = \lambda u$ with Dirichlet's boundary conditions are the same. In the stable case the exterior and interior eigen-functions will also be identical.*

For proof see [9].

We have shown in an illustrative manner some relations concerning the problem of stability of the domain of existence for different kinds of problems and their physical interpretations. The problem is of course substantially wider.

7

We will mention now the relation of the problem of stability to the problem of the theory of approximations.

The following theorem holds:

Theorem 6. *Let Ω be a domain, K a unit circle, $\overline{\Omega} \subset K$,* mes $\mathscr{F}(\Omega) = 0$. *Then Ω is stable with respect to Dirichlet's problem for the l-harmonic equation for $\mathscr{K} = L_2(K)$ if and only if for every function u, $u \in L_2^l(K)$ which is harmonic on Ω and for every $\varepsilon > 0$ there exists a domain Ω_ε, $\overline{\Omega} \subset \Omega_\varepsilon$ and a function v_ε l-harmonic on Ω_ε so that $\|u - v_\varepsilon\|_{L_2{}^l(K)} < \varepsilon$.*

For proof see [3].

In the special case where $l = 2$, Theorem 7 follows directly from Theorem 6.

Theorem 7. *In the space of functions which are harmonic and square-integrable in Ω, functions which are harmonic in open sets containing $\bar{\Omega}$ form a dense subset if and only if Ω is stable with respect to Dirichlet's problem for the biharmonic equation and $\mathscr{K} \equiv L_2(K)$.*

For proof see [3].

Let us compare Theorem 7 with the well known assertion from the theory of a complex variable about the density of polynoms in the space of square-integrable holomorphic functions. In this case the classical theorem holds: for a Caratheodory domain the polynoms form a dense subset in the space of all square-integrable holomorphic functions. Our Theorem 7 resembles this classic assertion in that we are dealing with the density assuming only the real and imaginary parts of the corresponding holomorphic functions.

If for every harmonic and square-integrable function on Ω there would also exist the conjugated square-integrable function, then our assertion would be a simple consequence of the theorem cited from the theory of a complex variable. But this holds only for domains with a smooth boundary, and it is possible to construct simple examples of domains with angle points for which the conjugated square-integrable function does not exist.

A relation analogous to the problem of stability with respect to Dirichlet's problem also holds for stability with respect to Neumann's problem.

Theorem 8. *The domain Ω is stable with respect to Neumann's problem for the l-harmonic equation and for $\mathscr{K}$ including all functions with a compact support in Ω, $f \in L_2(K)$ and fulfilling (4) if and only if the set of functions infinitely differentiable on some neighbourhood $\bar{\Omega}$ forms a dense subset in $L_2^l(\Omega)$.*

For proof see [3].

In special case $l = 1, 2$ in the plane, Theorem 3 gives us the density of smooth functions in $L_2^2(\Omega)$ if Ω is a Caratheodory domain.

There are unsolved problems for $l > 2$.

8

We have dealt so far with various characteristic aspects of the problem of the theory of stability, which studies the problem of the continuous dependence of the solution on small changes of the domain of existence. If we compare this conception with the elementary idea of continuity for a function of a real variable, we can note some analogy. Therefore we will point out what in this analogy corresponds to the conception of the derivative or of the differential in the case of a function of a real variable.

Let a domain Ω_λ be slightly different from the unit circle, defined by the expression

$$\Omega_\lambda = E[(x, y); x^2 + y^2 < (1 - \lambda f(\varphi))^2, 0 \leqq \varphi \leqq 2\pi].$$

The domain Ω_λ is characterized by the parameter λ and the problem of stability is to show how the solution behaves for $\lambda \to 0$.

Let g be a function given in the neighbourhood $\bar{\Omega}_0$. Further let u_λ be a function defined on Ω_λ so that $\Delta u_\lambda = 0$ on Ω_λ, $u_\lambda = g$ on $\mathscr{F}(\Omega_\lambda)$.

Because Ω_0 is a unit circle which is a stable domain, it follows from the theory cited above that $\lim_{\lambda \to 0} u_\lambda = u_0$. So far the procedure is analogous to the conception of continuity. The analogy with the differential or derivative leads to the question when, for small λ, we can write the function asymptotically in the form $u_\lambda = u_0 + \lambda v + O(\lambda^2)$.

In this simple case it can be proved, that the function u_λ can be written in the asymptotic form if the second derivatives for the function g and the first derivatives for the function f fulfil Hölder's condition. The v is the solution of Dirichlet's problem on the unit circle with the boundary condition $v = f\, \partial(u_0 - g)/\partial r$ on $\mathscr{F}(\Omega_0)$ (see [10]).

We will mention now some applications of this theory to the problem of notch-stress.

One of the basic problems of the theory of elasticity, which has outstanding importance in practice, is the problem of stress concentration around a notch. Let us have a stressed half-plane which is disturbed by a notch, characterized by the function $\mu(x)$, i.e. $\Omega = E[(x, y); y < \mu(x)]$. This notch is the cause for the concentration of stress, this fact being of great technical importance. The coefficient describing the ascent of the stress caused by the notch will be called the coefficient of the concentration on the stress.

If we use the aforementioned method, we obtain for shallow notches the following asymptotic expression for the coefficient of the concentration of the stress K (see [11]):

$$K = 1 + \frac{2}{\pi} V.P \int_{-\infty}^{\infty} \frac{1}{x} \mu'(x)\, dx\,.$$

We take the integral in the sense of Cauchy's main value. In the special case of a circular notch we obtain the expression

$$K = 1 + \frac{4}{\pi} \arcsin \sqrt{\left(2\frac{t}{\varrho} - \frac{t^2}{\varrho^2}\right)},$$

where t is depth and ϱ radices of the notch.

Because of its great practical importance the theory of the notch-stress is widely elaborated. In practice there is the classical method of Neuber, who derived the asymptotic expressions for this coefficient for some special cases such as the circular notch etc. In the simple case of the half-plane with the circular notch it is possible to determine the coefficient of the concentration of the stress precisely. From a comparison of the results of the precise theory, Neuber's approximation, and the theory mentioned above we see that for shallow notches our results are better than the Neuber's ones and that for deep notches the results are the same. The advantage of the method, mentioned as an application of the theory of the dependence of the solution on small changes of the domain is that here we can express the influence of the form of the notch more clearly than in the Neuber's method.

We can use this advantage by solving the problem of the optimal form of the notch. The problem is to find the function $\mu(x)$,

$$\mu(x) = 0 \quad \text{for} \quad x < 0\,, \quad \mu(x) = \lambda \quad \text{for} \quad x > a\,,$$

which expresses the form of the notch in such a way that no coefficient of the stress concentration reaches its minimum value. It can be shown that the asymptotic solution of the optimal form of the notch for small λ is given by the expression

$$\mu(x) = \lambda \frac{2}{\pi}\left[\operatorname{arctg} \sqrt{\left(\frac{x/a}{1 - x/a}\right)} - \sqrt{\left(\frac{x}{a}\left(1 - \frac{x}{a}\right)\right)}\right].$$

The minimum coefficient of the concentration corresponding to the optimal notch is $K_{\min} = 1 + 4/\pi \,.\, \lambda/a$.

For comparison the coefficient of the concentration in the case of the elliptical-shaped notch is

$$K_{\mathrm{el}} = 1 + \frac{4}{\pi}\frac{\lambda}{a}\, 1.2854\,.$$

By comparing these two expressions we can see the influence of the optimum form of the notch on the coefficient of the concentration.

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