Krzysztof Lisiecki

Foliated groupoids

In: Jarolím Bureš and Vladimír Souček (eds.): Proceedings of the Winter School "Geometry and Physics". Circolo Matematico di Palermo, Palermo, 1990. Rendiconti del Circolo Matematico di Palermo, Serie II, Supplemento No. 22. pp. [127]--149.

Persistent URL: http://dml.cz/dmlcz/702165

FOLIATED GROUPOIDS

Krzysztof Lisiecki

0. INTRODUCTION.

The notion of a groupoid comes from Brandt [1]. In the fifties Ch. Ehresmann [2],[3] introduced a development of the theory of Lie groupoids and, more generally, differential groupoids. This theory, from the formal point of view, is a natural extension of the theory of Lie groups. Next, the works by J. Pradines [12],[13] were the landmark in the theory of differential groupoids. The author associated, with every differential groupoid ϕ, some vector bundle consisting of all α-vertical vectors tangent to the units of the groupoid ϕ, equipped with the natural algebraic srtructure. This bundle was called by J. Pradines the Lie algebroid of the differential groupoid ϕ (an analogy to Lie algebra of Lie group).

In the theory of Lie groups and Lie algebras, one of important theorems is the theorem saying that, for a subalgebra h of a Lie algebra g of a Lie group G, there is exactly one Lie subgroup H of G whose Lie algebra is h. In paper [10] J. Kubarski proved the analogous theorem for Lie groupoids and Lie algebroids (see theorem 5.1 of our work).

The aim of this work is a generalization of this theorem for a more general class of groupoids than the class of Lie groupoids, namely, for foliated groupoids over a foliationwith singularities (which constitute a full subcategory of category of smooth groupoids introduced by J. Kubarski).

The present paper consists of five sections. The first and

This paper is in final form and no version of it will be submitted for publication elsewhere.

second contain the fundamental notions concerning differential spaces and groupoids. Especially, an attention was paid to the properties of a tangent bundle to a differential space, introduced by A. Kowalczyk [6], and to the notions of a smooth groupoid and a groupoid in the category of differential spaces, both coming from J. Kubarski [8].

In the third section, a construction of an algebroid of a smooth groupoid is described and the definition of an abstract algebroid is given.

Following the idea of definitions 1.6.7 and 1.7.1 [8], in the fourth section we introduce the notion of a foliated groupoid over Stefan's foliation; this term was proposed by J. Kubarski. While studying the structure of an algebroid of a foliated groupoid over a foliation, it has been found that after the pullback of an algebroid of a given groupoid to the leaf of the foliation we obtain a Lie algebroid.

In the last section we give the theorem on the inducing of an α-connected foliated subgroupoid (over the same Stefan's foliation) by a foliated subalgebroid.

1. PRELIMINARIES.

Let C be any non-empty family of real functions defined on a set M, and τ_M - the weakest topology under which all functions from C are continuous. We take

1) $C_A=\{\beta\colon A\longrightarrow R;\ \forall\ x\in A\ \exists\ x\in U\in\tau_C\ \exists\alpha\in C\ (\beta|A\cap U=\alpha|A\cap U)\}$ for $A\subset M$,

2) $scC=\{\phi\circ(\alpha_1(\cdot),\dots,\alpha_m(\cdot))\ ;\alpha_1,\dots,\alpha_m\in C,\ \phi\in C^{\infty}(R^m);\ m\in M\}$.

The family C is called a differential structure on M if $C_M=C$ and $scC=C$. The pair (M,C), sometimes briefly denoted by M, where M is a non-empty set and C is a differential structure on M, is called a differential space [15].

For example, $(R^m,C^{\infty}(R^m))$ is a differential space; more generally - for every manifold M, a pair $(M,C^{\infty}(M))$ is a differential space. If C_o is a family of real functions on M, then $C:=(scC_o)_M$ is the smallest differential structure on M containing C_o and we call it a differential structure generated by C_o [21]. If (M,C) is a differential space and $\emptyset\neq A\subset M$, then (A,C_A) is a differential space, too.

A differential space (N,D) will be called a differential subspace of a differential space (M,C) [9] if $N\subset M$ and if, for each point $y\in N$, there is a neighbourhood $U\in\tau_D$ of the point y such that $D_U=C_U$. Then we write $(N,D)\longrightarrow(M,C)$ (see [15], as well). If $D=C_N$,then (N,D) will be called a proper differential subspace of a differential space (M,C).

Let (M,C) and (N,D) be any differential spaces. The mapping $f:M\longrightarrow N$ is called :

1) smooth if $g\circ f\in C$ for $g\in D$, then we write $f:(M,C)\longrightarrow(N,D)$,
2) a diffeomorphism if it is a bijection and f and f^{-1} are smooth,
3) an embedding if $f:(M,C)\longrightarrow(f[M],D_{f[M]})$ is a diffeomorphism.

Let (M,C) and (N,D) be differential spaces. We denote by $C\times D$ the smallest differential structure generated by $\{\alpha\circ pr_1;\alpha\in C\}\cup\{\beta\circ pr_2;\beta\in D\}$ where $pr_1:M\times N\longrightarrow M$, $(x,y)\longmapsto x$, $pr_2:M\times N\longrightarrow N$, $(x,y)\longmapsto y$.

By a product of differential spaces (M,C) and (N,D) we mean a differential space $(M\times N,C\times D)$. One can show that $\tau_{C\times D}=\tau_C\times\tau_D$ [16]. The differential space (M,C) is called an n-dimensional differential manifold [15] if each point from M has a neighbourhood diffeomorphic to some open subset of the differential space $(R^n,C^\infty(R^n))$. The topology τ_C is then T2.

If (M,C) is an n-dimensional differential manifold, then there exists a uniquely determined C^∞-manifold M such that $C=C^\infty(M)$.

By a tangent vector to a differential space (M,C) at a point $x\in M$ [15] we mean a linear mapping $v:C\longrightarrow R$ such that $v(f\cdot g)=v(f)\cdot g(x)+v(g)\cdot f(x)$ for $f,g\in C$. The set of all tangent vectors to (M,C) at $x\in M$ form a vector space denoted by $T_x(M,C)$ and called a tangent space to (M,C) at x.

1.1. Remark. If (N,D) is a differential subspace of a space (M,C), and $i:N\longrightarrow M$ is an inclusion, then, for each $y\in N$, a mapping $i_{*y}:T_y(N,D)\longrightarrow T_y(M,C)$ is a monomorphism. We shall identify

(1.2) $$T_y(N,D)\cong i_*[T_y(N,D)]\subset T_y(M,C).$$

Let (M,C) be any differential space. We put

/i/ $T(M,C)=\bigsqcup_{x\in M} T_x(M,C)$ (disjoint union),

/ii/ $\pi:T(M,C)\longrightarrow M$, $\pi(v)=x \Leftrightarrow v\in T_x(M,C)$,

/iii/ $TC=(scC_o)_{T(M,C)}$ where $C_o=\{g\circ\pi;\ g\in C\}\cup\{dg;\ g\in C\}$ and $dg(v)=v(g)$ for $v\in T(M,C)$.

The differential space $(T(M,C),TC)$ is called (following A. Kowalczyk [6]) a tangent differential space to (M,C), and π - the canonical projection.

1.3. Proposition [6]. If a differential structure C is genereted by C_o, then a differential structure TC is generated by the set $\{g\circ\alpha;\ g\in C_o\}\cup\{dg;\ g\in C_o\}$.

1.4. Proposition [6]. $T(C\times D)=TC\times TD$.

1.5. Proposition [9]. Let $\lambda:(N,D)\longrightarrow(M,C)$. Then (cf. remark 1.1) $\lambda_*:T(N,D)\longrightarrow T(M,C)$. Moreover, if $U\in\tau_D$ is a set such that $C_U=D_U$, then $(TC)_{(\pi')^{-1}[U]}=(TD)_{(\pi')^{-1}[U]}$ ($\pi':T(N,D)\longrightarrow(N,D)$ is the canonical projection).

By a vector field on a differential space (M,C) [17] we mean a mapping $X:M\longrightarrow T(M,C)$ such that $\pi\circ X=id_M$.

A vector field is called smooth if $X:(M,C)\longrightarrow(T(M,C),TC)$.

A vector field is smooth if and only if $Xg\in C$ for $g\in C$, where $(Xg)(y)=X_y(g)$, $y\in M$, $g\in C$.

For any vector fields X,Y, we define their Lie bracket $[X,Y]$ as follows : it is a vector field which at a point $x\in M$ on a function α takes the value $X_x(Y\alpha)-Y_x(X\alpha)$. The set of all smooth vector fields on (M,C) is denoted by $\mathfrak{X}(M,C)$. This set form a module over the ring C; moreover, a pair $(\mathfrak{X}(M,C),[\cdot,\cdot])$ forms an R-Lie algebra .

Let $f:(M,C)\longrightarrow(N,D)$, $X\in\mathfrak{X}(M,C)$ and $Y\in\mathfrak{X}(N,D)$. The vector field X is called f-related to the field Y if, for each point $x\in M$, $f_{*x}(X_x)=Y_{f(x)}$. Then we write $X\underset{f}{\sim}Y$.

1.6. Proposition [22]. If $X,X'\in\mathfrak{X}(M,C)$, $Y,Y'\in\mathfrak{X}(N,D)$ and $X\underset{f}{\sim}Y$ and $X'\underset{f}{\sim}Y'$, then $[X,Y]\underset{f}{\sim}[X',Y']$.

1.7. Proposition [8]. The tangent differential space $(T(M,C),TC)$ has properties:

/1/ the mappings $+:T(M,C)\oplus T(M,C)\longrightarrow T(M,C)$, $(v,w)\longmapsto v+w$,

$$\cdot:R\times T(M,C)\longrightarrow T(M,C),\ (r,v)\longmapsto r\cdot v$$

are smooth, where $T(M,C)\oplus T(M,C)$ is a proper differential subspace of $T(M,C)\times T(M,C)$ containing only those pairs (v,w) for which $\pi(v)=\pi(w)$,

/2/ for any number $m\in N$, for any smooth vector fields

$X_1,\dots,X_m \in \mathcal{X}(M,C)$ and for a set $U \subset M$ (not necessarily open), such that, for each $x \in U$, the vectors $X_1(x),\dots,X_m(x)$ are linearly independent, the mapping
$\phi: (U\times R^m, C_U\times C^\infty(R^m)) \longrightarrow (T(M,C), TC)$, $(x,a) \longmapsto \sum_{i=1}^m a^i X_i(x)$
a diffeomorphism onto its image.

1.8. Proposition. Let $(N,D) \longrightarrow (M,C)$. Then:

/i/ if $X \in \mathcal{X}(M,C)$ is a vector field such that $X_x \in T_x(N,D)$ for each $x \in N$, then $X|N \in \mathcal{X}(N,D)$;

/ii/ if $X,Y \in \mathcal{X}(M,C)$ are vector fields such that $X_x \in T_x(N,D)$, $Y_x \in T_x(N,D)$ for each $x \in N$, then $[X,Y] \in T_x(N,D)$.

1.9. Definition. By a k-leaf of a differential space (M,C) we mean a subset $L \subset M$ if there exists a differential structure D on L such that:

/1/ (L,D) is a differential manifold of dimension k,

/2/ (L,D) is a differential subspace of (M,C),

/3/ for each locally arcwise connected topological space X and for a continuous mapping $f: X \longrightarrow (M,C)$, such that $f[X] \subset L$, the mapping $\bar{f}: X \longrightarrow (L,D)$ defined by the same formula is continuous.

A set L is called a leaf of the differential space (M,C) if it is a k-leaf of (M,C) for some $k \in N$.

1.10. Proposition [8]. Let (L,D) be a k-leaf of the differential space (M,C). Then

/i/ if (X,E) is any differential space whose topology τ_E is locally arcwise connected, then, for each smooth mapping $f:(X,E) \longrightarrow (M,C)$ such that $f[X] \subset L$, the mapping $f:(X,E) \longrightarrow (L,D)$ is also smooth,

/ii/ each connected component of the manifold (L,D) is equal to an arcwise connected component of L in the topological space (M,τ_C),

/iii/ if the set L is a leaf of a manifold (N,D) and (N,D) is a leaf of a differential space (M,C), then L is a leaf of (M,C).

2. GROUPOIDS - DEFINITIONS AND EXAMPLES.

Originally, the notion of a groupoid comes from the work by Brandt [1]. In the sequel, we shall use the notation given by

N. van Que [14].

2.1. Definition. By a groupoid we mean the system

(2.2) $(\Phi,\alpha,\beta,V,\cdot)$ consisting of sets Φ and V and mappings $\alpha,\beta:\Phi\longrightarrow V$, $\cdot:\Phi*\Phi\longrightarrow\Phi$, where $\Phi*\Phi:=\{(g,h)\in\Phi\times\Phi;\alpha g=\beta h\}$, fulfilling the axioms:

- $\alpha(GH)=\alpha h$ and $\beta(GH)=\beta g$ for $(g,h)\in\Phi*\Phi$,
- $f(gh)=(fg)h$ for (f,g), $(g,h)\in\Phi*\Phi$,
- for each point $x\in V$, there exists an element $u_x\in\Phi$ such that /i/ $\alpha u_x=\beta u_x=x$, /ii/ $h\cdot u_x=h$ when $\alpha h=x$, /iii/ $u_x\cdot g=g$ when $\beta g=x$ (for each $x\in V$, the element u_x is uniquely determined and called the unit over x,
- for each element $h\in\Phi$, there exists an element $h^{-1}\in\Phi$ such that /i/ $\alpha(h^{-1})=\beta h$, $\beta(h^{-1})=\alpha h$, /ii/ $h\cdot h^{-1}=u_{\beta h}$, /iii/ $h^{-1}\cdot h=u_{\alpha h}$ (for each $h\in\Phi$, the element h^{-1} is uniquely determined and called the element inverse to h).

Any equivalence relation $R\subset V\times V$, $V\neq\emptyset$, determines a groupoid

(2.3) $(R,\ pr_1|R,\ pr_2|R,\ V,\cdot)$

called a groupoid of the equivalence relation R, in which $(y\cdot z)\cdot(x,y)=(x,z)$ for (x,y), $(y,z)\in R$.

2.4. Definition. By a groupoid in the category of differential spaces we mean [8] groupoid (2.2) in which Φ and V are differential spaces and the mappings $\alpha,\beta:\Phi\longrightarrow V$, $u:V\longrightarrow\Phi$, $^{-1}:\Phi\longrightarrow\Phi$ and $\cdot:\Phi*\Phi\longrightarrow\Phi$ (where $\Phi*\Phi$ denotes the proper subspace of $\Phi\times\Phi$) are smooth .

Groupoid (2.2) will sometimes be denoted by Φ.

If R is any equivalence relation on a differential manifold V, then groupoid (2.3), in which R is taken as a proper differential subspace of the C^∞-manifold $V\times V$, is a groupoid in the category of differential spaces.

2.5. Example. Let Γ be any pseudogroup of smooth transformations on a differntial manifold V. Then, for each $k=1,2,\ldots$, the set of jets $\{j_x^kf;\ f\in\Gamma,\ x\in D_f\}\subset J^k(V,V)$, with the differential structure induced from $J^k(V,V)$, forms a groupoid in the category of differential spaces.

2.6. Definition. By a smooth groupoid on a differential manifold V [9] we mean a groupoid in the category of differential spaces (2.2) in which V is a differential manifold and, for each point $x\in V$, the set $\alpha^{-1}(x)$ is a leaf of the differential space Φ.

The set $\alpha^{-1}(x)$ equipped with a suitable differential manifold structure is called a leaf of the groupoid Φ over x and denoted by Φ_x.

For each $h\in\Phi$, the mapping

(2.7) $$D_h:\Phi_{\beta h}\longrightarrow\Phi_{\alpha h}\ ,\ \tau\longmapsto\tau\cdot h,$$

is a diffeomorphism; moreover, for $(g,h)\in\Phi*\Phi$, we have $D_h\circ D_g=D_{gh}$.

2.8. **Example.** Let R be any equivalence relation on a diiferential manifold V. The groupoid of the equivalence relation R, described above, is a smooth groupoid if and only if each abstract class of R is a leaf of V.

2.9. **Example [9].** Let (2.2) be any Lie groupoid (see definition below). Then, for an equivalence relation R for which (2.3) is a smooth groupoid, the subgrouipoid $\Phi^R=(\alpha,\beta)^{-1}[R]$ equipped with the differential structure of the proper differential subspace of Φ, turns out to be a smooth groupoid.

By a Lie groupoid [14] we shall mean a smooth groupoid over a differential manifold V in which:

1) Φ is a differential manifold,

2) the mappings α and β are submersions,

3) the mapping $(\alpha,\beta):\Phi\longrightarrow V\times V$, $h\longmapsto(\alpha h,\beta h)$, is a surjection (the transitivity condition).

Let $\Phi=(\Phi,\alpha,\beta,V,\cdot)$ and $\Phi'=(\Phi',\alpha',\beta',V,\cdot')$ be any groupoids. A mapping $F:\Phi\longrightarrow\Phi'$ is called a (strong) homomorphism of groupoids if : /i/ $\alpha'\circ F=\alpha$, /ii/ $\beta'\circ F=\beta$, /iii/ $F(g\cdot h)=F(g)\cdot'F(h)$ when $(g,h)\in\Phi*\Phi$.

If Φ and Φ' are groupoids in the category of differential spaces, then we say that Φ' is a subgroupoid of Φ if Φ' is a proper differential subspace of Φ and the inclusion $i:\Phi'\longrightarrow\Phi$ is a smooth homomorphismof groupoids.

All groupoids in the category of differential spaces, together with smooth homomorphism, form a category. Its full subcategory is the category of all smooth groupoids.

Proposition 1.4 implies

2.10. Proposition. Let (Φ',C') be any subgroupoid of a groupoid in the category of differential spaces (Φ,C). Then, for each $h\in\Phi$, there exists its neighbourhood $\Omega\subset\Phi$ such that

(2.11) $$(TC')_{(\pi')^{-1}[\Omega]}=(TC)_{\pi^{-1}[\Omega]}$$

where $\pi':T\Phi\longrightarrow\Phi$ and $\pi:T\Phi\longrightarrow\Phi$ denote the natural projections.

3. THE ALGEBROID OF SMOOTH GROUPOID.

3.1. A construction of the algebroid of smooth groupoid.

The notion of an algebroid appeared for the first time in connection with the investigation of differential groupoids. Namely, J.Pradines [12] constructed, for every differential groupoid, a vector bundle whose module of global cross-sections is isomorphic to the module of right-invariant vector fields on this groupoid. This bundle - called the Lie algebroid of a differential groupoid - plays analogous role as the Lie algebra of a Lie group. J.Kubarski [9] generalized this construction to the class of smooth groupoids. The basic elements of the construction are given below.

Let $\Phi=(\Phi,\alpha,\beta,V\cdot)$ be an arbitrary smooth groupoid and let C be a differential structure on Φ. We put

1) $A(\Phi):=\bigsqcup_{x\in V}T_{u_x}\Phi_x\subset T\Phi$,

2) $p:A(\Phi)\longrightarrow V$, $p(v)=x \Leftrightarrow v\in T_{u_x}\Phi_x$.

On the set $A(\Phi)$ we introduce the differential structure equalling $(TC)_{A(\Phi)}$, denoted later by T_AC, and obtain some proper differential subspace of $T\Phi$.

3.1.1. Proposition. The projection $p:A(\Phi)\longrightarrow V$ is smooth.

By an α-field on Φ we mean a vector field X on Φ if, for each $h\in\Phi$, $X(h)\in T_h\Phi_{\alpha h}$.

An α-field X is called right-invariant if $(D_h)_{*g}(X_g)=X_{gh}$, $g,h\in\Phi$ and $\alpha g=\beta h$. The set of all smooth right-invariant vector fields on Φ is denoted by $X^R(\Phi)$. It forms a module over the ring $C^\infty(V)$ with respect to the natural addition and to the multiplication $(f\cdot X=f\circ\beta\cdot X$, $f\in C^\infty(V)$. By $SecA(\Phi)$ we denote the set of all global smooth cross-sections of p.

3.1.2. Theorem [7]. If $X\in X^R(\Phi)$, then

(3.1.3) $X_o : V \longrightarrow A(\Phi)$, $x \longmapsto X(u_x)$ is a smooth cross-section of p. Conversely, for any smooth cross-section $\eta : V \longrightarrow A(\Phi)$ of p, there exists exactly one smooth right-invariant vector field on Φ, denoted by η', such that $(\eta')_o = \eta$. The mappings

$X^{R}(\Phi) \ni X \longmapsto X_o \in \mathrm{Sec}A(\Phi)$

fix an isomorphism of the $C^{\infty}(V)$-modules: $X^{R}(\Phi)$ and $\mathrm{Sec}A(\Phi)$.

3.1.4. Theorem. The Lie bracket of smooth right-invariant vector fields on Φ is smooth and the vector space $X^{R}(\Phi)$ is an R-Lie algebra with respect to this Lie bracket.

By 1.7 and 3.1.2, we obtain

3.1.5. Proposition [8]. The system $(A(\Phi), p, V)$ has the properties:

/1/ the mappings $+ : A(\Phi) \oplus A(\Phi) \longrightarrow A(\Phi)$, $(v, w) \longmapsto v + w$, and
$\cdot : R \times A(\Phi) \longrightarrow A(\Phi)$, $(r, v) \longmapsto r \cdot v$
are smooth,

/2/ for any number $m \in N$, any smooth cross-sections $\xi_1, \dots, \xi_m$ of p and any set $U \subset V$, such that the vectors $\xi_1(x), \dots, \xi_m(x)$ are linearly independent for $x \in U$, the mapping $\phi : U \times R^m \longrightarrow A(\Phi)$, $(x, (a^1, \dots, a^m)) \longmapsto \sum_{i=1}^{m} a^i \xi_i(x)$, is a diffeomorphism onto its image.

Now, in the module $\mathrm{Sec}A(\Phi)$ we introduce some structure of an R-Lie algebra.

3.1.6. Definition. For any cross-sections $\xi, \eta \in \mathrm{Sec}(\Phi)$, we define their Lie bracket in the following way:

(3.1.7) $$[\![\xi, \eta]\!] := ([\xi', \eta'])_o$$

3.1.8. Proposition [9]. The pair $(\mathrm{Sec}A(\Phi), [\![\cdot, \cdot]\!])$ forms a Lie algebra over R. Moreover, the canonical isomorphism of the $C^{\infty}(V)$-modules $X^{R}(\Phi)$ and $\mathrm{Sec}A(\Phi)$, described in 3.1.2, is an isomorphism of Lie algebras.

We define a mapping

(3.1.9) $$\tilde{\beta}_* : A(\Phi) \longrightarrow TV, \quad v \longmapsto \beta_*(v).$$

Notice that the diagram below is commutative:

$$\begin{array}{ccc} A(\Phi) & \xrightarrow{\tilde{\beta}_*} & TV \\ \downarrow & & \downarrow \\ V & \xrightarrow{=} & V \end{array}$$

3.1.10. Theorem [9]. Any vector field $X \in \mathfrak{X}^R(\Phi)$ is β-projective, i.e. there exists exactly one vector field $Y \in \mathfrak{X}(V)$ with which X is β-related. It is the field $Y := \tilde{\beta}_* \circ (X)_o$.
Further, the field $\tilde{\beta}_* \circ \xi$, $\xi \in SecA(\Phi)$, will be briefly denoted by $\beta_* \xi$. The following equality is true:

(3.1.11) $$\xi'(f \circ \beta) = (\beta_* \xi)(f) \circ \beta, \quad \xi \in SecA(\Phi), \quad f \in C^\infty(V).$$

3.1.12. Proposition [9]. The mapping $Sec\tilde{\beta}_* : SecA(\Phi) \longrightarrow V$, $\xi \longmapsto \tilde{\beta}_* \xi$, is a homomorphism of Lie algebras.

3.1.13. Proposition [9]. For right-invariant vector fields $X, Y \in \mathfrak{X}^R(\Phi)$ and a function $f \in C^\infty(V)$,

(3.1.14) $$[X, f \circ \beta \cdot Y] = f \circ \beta [X, Y] + (\beta_* X_o)(f) \circ \beta \cdot Y.$$

3.1.15. Corollary. The Lie algebra $SecA(\Phi)$ has the property

(3.1.16) $$[\![\xi, f \cdot \eta]\!] = f \cdot [\![\xi, \eta]\!] + (\beta_* \xi)(f) \cdot \eta$$

for $\xi, \eta \in SecA(\Phi)$ and $f \in C^\infty(V)$.

For a given smooth groupoid Φ, the system $(A(\Phi), p, V)$ is not, in general, a vector bundle. Proposition 3.1.5 asserts that it is a vector pseudobundle, according to the definition below

3.1.17. Definition [9]. By a vector pseudobundle (over a differential space V) we mean each system (A, p, V) containing differential spaces A and V and surjective smooth mapping $p: A \longrightarrow V$ in whose fibres some structure of vector spaces are defined and the following properties hold:

(i) $+: A \oplus A \longrightarrow A$, $(v, w) \longmapsto v + w$, $\cdot : R \times A \longrightarrow A$, $(r, v) \longmapsto r \cdot v$, are smooth mappings where $A \oplus A = \{(v, w) \in A \times A;\ p(v) = p(w)\}$ denotes a proper differential subspace of $A \times A$,

(ii) for any number $m \in M$, any smooth cross-sections $\xi_1, \ldots, \xi_m$ of p and any set $U \subset V$ (not necessarily open), such that the vectors $\xi_1(x), \ldots, \xi_m(x)$ are linearly independent for all $x \in U$, the mapping

$$\phi : V_{|U} \times R^m \longrightarrow A, \quad (x, a) \longmapsto \textstyle\sum_{i=1}^m a^i \xi_i(x),$$

is a diffeomorphism onto its image.

3.1.18. Definition [9]. By a homomorphism of vector pseudobundles (A, p, V) and (A', p', V) we mean a smooth mapping $H: A \longrightarrow A'$ such that $p' \circ H = p$ and, for each $x \in V$, $H_{|x} : A_{|x} \longrightarrow A'_{|x}$ is a linear homomorphism of vector spaces.

3.1.19. Definition [9]. By an algebroid on a differential manifold V we mean the system

(3.1.20) $\quad (A, [\![\cdot,\cdot]\!], \gamma)$,

in which:

1) $A=(A,p,V)$ is a vector pseudobundle,

2) $(\mathrm{Sec}A, [\![\cdot,\cdot]\!])$ is an R-Lie algebra,

3) $\gamma: A\longrightarrow TV$ is a strong homomorphism of vector pseudobundles called after K. Mackenzie [11], an anchor) such that the mapping $\mathrm{Sec}\gamma: \mathrm{Sec}A\longrightarrow X(V)$ is a homomorphism of Lie algebras,

4) $[\![\xi, f\cdot\eta]\!]=f\cdot[\![\xi,\eta]\!]+(\gamma\circ\xi)(f)\cdot\eta,\ \xi,\eta\in\mathrm{Sec}A, f\in C^{\infty}(V)$.

3.1.21. Proposition. For any smooth groupoid (2.2) the object

(3.1.22) $\quad A(\Phi)=(A(\Phi)\xrightarrow{p}V, [\![\cdot,\cdot]\!], \tilde{\beta}_{*})$

fulfils, by 3.1.8, 3.1.12, 3.1.13 and 3.1.15, the axioms of the definition of an algebroid.

3.2. Homomorphisms.

Let $(A', [\![\cdot,\cdot]\!]', \gamma')$ and $(A, [\![\cdot,\cdot]\!], \gamma)$ be any algebroids on a manifold V. A mapping $H: A'\longrightarrow A$ is called a homomorphism of algebroids [9] if:

(i) H is a homomorphism of vector pseudobundles,

(ii) $\mathrm{Sec}H: \mathrm{Sec}A'\longrightarrow \mathrm{Sec}A$ is a homomorphism of Lie algebras,

(iii) the diagram below commutes

$$\begin{array}{ccc} A & \xrightarrow{H} & A' \\ \gamma' & & \gamma \\ & TV & \end{array}$$

3.2.1. Proposition. Let $\Phi=((\Phi,C),\alpha_1,\beta_1,V,\cdot_1)$ and $\Psi=((\Psi,D),\alpha_2,\beta_2,V,\cdot_2)$ be any smooth groupoids and let $F:\Phi\longrightarrow\Psi$ be a homomorphism of groupoids. Then $\tilde{F}_{*}: A(\Phi)\longrightarrow A(\Psi),\ v\longmapsto F_{*}(v)$, is a homomorphism of algebroids.

Proof. Since $F_{|\alpha^{-1}(x)}: \Phi_x\longrightarrow\Psi_x$ is a smooth mapping between the leaves of groupoids, therefore $F_{*}(v)\in A(\Psi)$. By the evident smoothness of $\tilde{F}_{*}$ and linearity of $\tilde{F}_{|x}: A(\Phi)_x\longrightarrow A(\Psi)_x$, $x\in V$, it is sufficient to show that: 1) $\mathrm{Sec}\tilde{F}_{*}: \mathrm{Sec}A(\Phi)\longrightarrow \mathrm{Sec}A(\Psi)$ is a homomorphism of Lie algebras, 2) $\gamma_2\circ\tilde{F}_{*}=\gamma_1$ where $\gamma_1: A(\Phi)\longrightarrow TV$ and $\gamma_2: A(\Psi)\longrightarrow TV$ are the anchors in $A(\Phi)$ and $A(\Psi)$, respectively.

1) Let $\phi\in D$. For any $\mu\in\mathrm{Sec}A(\Phi)$ and $g\in\Phi$, we have

$\mu'(\phi\circ F)(g)=(F_{*})_g(\mu'(g))(\phi)=(F_{*})_g((D_g)_{*u_{\beta g}}(\mu_{\beta g}))=(F\circ D_g)_{*u_{\beta g}}(\mu_{\beta g})=$

$=(D_{F(g)}\circ F)_{*u_{\beta g}}(\mu_{\beta g})=(D_*)_{F(g)}\circ\tilde{F}_{*u_{\beta g}}(\mu_{\beta g})=D_{*F(g)}\circ(\tilde{F}_*\circ\mu)_{\beta g}=$
$=(\tilde{F}_*\circ\mu)'(\phi)\circ F(g)$. And so, for $\xi,\eta\in\mathrm{Sec}A(\Phi)$, we obtain

$$\tilde{F}_*(⟦\xi,\eta⟧)(x)(\phi)=⟦\xi,\eta⟧(x)(\phi\circ F)=[\xi',\eta'](u_x)(\phi\circ F)=$$
$$=\xi'(u_x)(\eta'(\phi\circ F))-\eta'(u_x)(\xi'(\phi\circ F))=\xi'(u_x)((\tilde{F}_*\circ\eta)'(\phi)\circ F)+$$
$$-\eta'(u_x)((\tilde{F}_*\circ\xi)'(\phi)\circ F)=(F_*)_{u_x}(\xi(x))((\tilde{F}_*\circ\eta)'\circ\phi)+$$
$$-(\tilde{F}_*)_{u_x}(\eta(x))((\tilde{F}_*\circ\xi)'\circ\phi)=(\tilde{F}_*\circ\xi)(x)((\tilde{F}_*\circ\eta)'\circ\phi)-$$
$$(\tilde{F}_*\circ\eta)(x)((\tilde{F}_*\circ\xi)'\circ\phi)=[(\tilde{F}_*\circ\xi)',(\tilde{F}_*\circ\eta)'](u_x)(\phi)=$$
$$=⟦\tilde{F}_*\circ\xi,\tilde{F}_*\circ\eta⟧(x)(\phi).$$

2) Let $v\in A(\Phi)$. Then $\gamma_2\circ\tilde{F}_*(v)=(\tilde{\beta}_2)_*(F_*(v))=(\beta_2\circ F)(v)=(\beta_1)_*(v)=$
$=(\tilde{\beta}_1)_*(v)=\gamma_1$.

3.2.2. Proposition. The assignment $\Phi\longmapsto A(\Phi)$, $F\longmapsto\tilde{F}_*$ is a covariant functor from the category of smooth groupoids to the category of algebroids.

3.2.3. Definition. By a subalgebroid of an algebroid $(A,⟦\cdot.\cdot⟧,\gamma)$ we mean an algebroid $(A',⟦\cdot,\cdot⟧',\gamma')$ such that A' is a proper differential subspace of A, and the inclusion $i:A'\longrightarrow A$ is a homomorphism of algebroids.

3.2.4. Proposition. Let $f:(M,C)\longrightarrow(N,D)$ be a smooth injective mapping between the differential spaces. If, for each $x\in M$, there exist a neighbourhood U of x in M and a neighbourhood W of the point $y=f(x)$ in N, such that $f|U:(U,C_U)\longrightarrow(W,D_W)$ is an embedding and $W\cap\mathrm{Im}f=f[U]$, then f is an embedding.

By this proposition and 2.4, we obtain

3.2.5. Corollary. If Φ' is a subgroupoid of a smooth groupoid Φ, then its algebroid $A(\Phi')$ is a subalgebroid of an algebroid $A(\Phi)$.

4. FOLIATED GROUPOIDS AND FOLIATED ALGEBROIDS.

4.1. Definitions and some properties.

In the sequel, we shall consider smooth groupoids which fulfil some additional axioms.

4.1.1. Definition. Let F be any partition of V into connected immersed submanifolds. By a foliated algebroid over F we mean an algebroid $(A,⟦\cdot,\cdot⟧,\gamma)$ in which:

(FA 1) for each $v\in A$ there exists a smooth cross-section $\xi\in\mathrm{Sec}A$

such that $\xi(p(v))=v$,

(FA 2) $\gamma[A]=TF$

(FA 3) the function $V\ni x\longmapsto \dim A_{|x}$ is constant on each $L\in F$.

4.1.2. Theorem. For a foliated algebroid over F we have: F is a foliation with singularities (in the sense of P. Stefan [18], [19]).

Proof. According to Sussmann's theorem [20] we need to obtain the smoothness of TF only, i.e. that

$$\forall\ v\in TV\ \exists\ X\in \mathfrak{X}(F)\ (X(\pi(v))=v)$$

($\mathfrak{X}(F)$ stands for the Lie subalgebra of $\mathfrak{X}(V)$ of all smooth vector fields with values at TF). To this purpose take arbitrary $v\in TF$. Let $\tilde{v}\in A$ be any element of A such that $\gamma(\tilde{v})=v$. From (FA 1) we choose some $\xi\in \mathrm{Sec}A$ such that $\xi(p(\tilde{v}))=v$. Of course $X:=\gamma\circ\xi$ has the property: $X(\pi(v))=v$ and $X\in\mathfrak{X}(F)$.

4.1.3. Definition. Let F be as in 4.1.1. By a foliated groupoid over F we mean a smooth groupoid Φ for which:

(FG 1) the family of abstract classes of the equivalence relation $R_\Phi=\{(x,y)\in V\times V;\ \exists\ h\in\Phi\ (\alpha h=x,\ \beta h=y)\}$ equals F,

(FG 2) for each $h\in\Phi$ and $v\in T_h(\Phi_{\alpha h})$ there exists a smooth right-invariant vector field X on Φ such that $X(h)=v$,

(FG 3) the mappings $\beta_x:\Phi_x\longrightarrow L_x$ ($x\in L\in F$) are submersions.

4.1.4. Proposition. The algebroid of a foliated groupoid over F is a foliated algebroid over F, in particular F is a foliation with singularities.

4.1.5. Example. Any Pradines-type groupoid is a foliated groupoid ([9]).

4.1.6. Example. Every groupoid of equivalence relation R which comes from a foliation with singularities F (in particular without singularities) is a foliated groupoid (over F).

4.2. The Lie algebroid $A(\Phi)_L$.

Let $\Phi=((\Phi,C),\alpha,\beta,V,\cdot)$ be any foliated groupoid over a foliation with singularities and let $A(\Phi)$ be its algebroid. By a construction analogous to the pullback of vector bundles, we pullback the algebroid $A(\Phi)$ via inclusion $i:L\longrightarrow V$ (L is any leaf of F).

Denote by $A(\Phi)_L$ the proper differential subspace of $L\times A(\Phi)$ with the set of points $\{(x,v)\in L\times A(\Phi)\ ;\ i(x)=p(v)\}$. Therefore the differential structure on $A(\Phi)_L$ is equal to $(C^{\infty}(L)\times(TC)_{A(\Phi)})_{A(\Phi)_L}$.

4.2.1. Theorem. The system $(A(\Phi)_L,\bar{p},L,R^k)$, where $\bar{p}\colon A(\Phi)_L\longrightarrow L$, $(x,v)\longmapsto v$, $k=\dim A(\Phi)_{|x}$, $x\in L$, is a vector bundle over the manifold L.

Proof. The mapping $\bar{p}$ is smooth because $\bar{p}=pr_1|A(\Phi)_L$ and the projection $pr_1\colon L\times A(\Phi)\longrightarrow L$ is smooth. Let $x\in L$. With the help of an isomorphism $i_{|x}\colon(A(\Phi)_L)_{|x}\longrightarrow(A(\Phi))_{|x}$, $(x,v)\longmapsto v$, we introduceon ther set $(A(\Phi)_L)_{|x}$ some structure of a vector space over R. Let the vectors $\langle v_1,\dots,v_k\rangle$ form a basis of $A(\Phi)_{|x}$; then the vectors $\langle(x,v_1),\dots,(x,v_k)\rangle$ form a basis of $(A(\Phi)_L)_{|x}$. There are cross-sections $\xi_1,\dots,\xi_k\in \mathrm{Sec}A(\Phi)$ such that $\xi_i(x)=v_i$, $i=1,\dots,k$. Thus, for each y from some neighbourhood U of x in V, the vectors $\xi_1(y),\dots,\xi_k(y)$ are linearly independent; moreover, by 3.1.5, the mapping

$$\phi\colon U\times R^k\longrightarrow \phi(U\times R^k)\subset A(\Phi),\quad (y,a)\longmapsto\sum_{i=1}^{k}a^i\xi_i(y)$$

is a diffeomorphism.

Now, by means of the cross-sections $\xi_1,\dots,\xi_k$ of p, we shall define some cross-sections of $\bar{p}$. Put $\bar{\xi}_i(x)=(x,\xi_i(x))$, $x\in L$, $i=1,\dots,k$. Then $\bar{\xi}_i\colon L\longrightarrow A(\Phi)_L\subset L\times A(\Phi)$, $x\longmapsto(x,\xi_i(x))$, are smooth mappings and the vectors $\bar{\xi}_i(y)$, $y\in U\cap L$, are lineary independent, $i=1,\dots,k$. We define a smooth mapping

$$\Psi\colon U\cap L\times R^k\longrightarrow A(\Phi)_L,\quad (y,a)\longmapsto(y,\phi(y,a)).$$

It is a local trivialization of the system $(A(\Phi)_L,\bar{p},L,R^k)$. Indeed, Ψ^{-1} is smooth because ϕ^{-1} is smooth and the diagram below commutes:

$$\begin{array}{ccc} L\times A(\Phi)\supset U\cap L\times\phi(U\times R^k)\supset\bar{p}^{-1}[U\cap L] & \xrightarrow{\ \Psi^{-1}\ } & U\cap L\times R^k \\ pr_2 & & \\ \phi(U\times R^k) & \xrightarrow{\ \phi^{-1}\ } & U\times R^k \end{array}$$

Now, we shall introduce a structure of a Lie algebroid in the vector bundle $(A(\Phi)_L,\bar{p},L,R^k)$ by using an auxiliary lemma whose proof, being very easy, will be omitted.

4.2.2. Lemma. If $i\colon L\longrightarrow V$ is an inclusion and L is a leaf of the differential manifold V (see definition 1.9), then $TL\longrightarrow TV$ is a leaf of the manifold TV.

We define $\gamma: A(\Phi)_L \longrightarrow TL$ by the formula $(x,v) \longmapsto \tilde{\beta}_*(v)$. It is smooth mapping. Indeed, $\tilde{\gamma}: A(\Phi)_L \longrightarrow TV$, $\tilde{\gamma}(x,v)=\gamma(x,v)$, is a smooth mapping from a locally arcwise connected topological space $A(\Phi)_L$ to the space TV with values in TL. By the above lemma and 1.10, is smooth.

4.2.3. Proposition. For each cross-section $\bar{\xi} \in \mathrm{Sec}A(\Phi)_L$ and each $x \in L$, there exist a neighbourhood U of the point x in L and a cross-section $\xi \in \mathrm{Sec}A(\Phi)$, such that $\bar{\xi}(y)=(y,\xi(y))$, $y \in U$.

Proof. Let $x \in L$. Take a neighbourhood W of x in L and cross-sections $\xi_1,\dots,\xi_k \in \mathrm{Sec}A(\Phi)$, such that $L_{|W} \longrightarrow V$ is an embedding and the vectors $\xi_1(y),\dots,\xi_k(y)$ form a basis of the space $A(\Phi)_{|y}$ for $y \in W$. Let now $\bar{\xi} \in \mathrm{Sec}A(\Phi)_L$ be an arbitrary cross-section. There are functions $a^i: W \longrightarrow R$, $i=1,\dots,k$, such that $\xi(y)=\sum_{i=1}^k a^i(y)\xi_i(y)$ for $y \in W$. From the equality $a^i=p^i \circ \phi^{-1} \circ \xi$ where $p^i: W \times R^k \longrightarrow R$, $(x,(a^1,\dots,a^k)) \longmapsto a^i$, we obtain the smoothness of a^i, $i=1,\dots,k$. Since $L_{|W} \longrightarrow V$ is a proper subspace of V, therefore $a^i \in C^\infty(V))_W$, $i=1,\dots,k$. Hence, there are a neighbourhood U of x in W and functions $\bar{a}^i \in C^\infty(V)$, such that $a^i|U=\bar{a}^i|U$, $i=1,\dots,k$. Define a cross-section $\xi: V \longrightarrow A(\Phi)$, $y \longmapsto \sum_{i=1}^k \bar{a}^i(y)\xi_i(y)$. Thus $\bar{\xi}|U=\xi|U$. By definition, $\xi \in \mathrm{Sec}A(\Phi)$.

Now, we take any cross-sections $\bar{\xi},\bar{\eta} \in \mathrm{Sec}A(\Phi)_L$. By 4.2.3, for any point $x \in L$, there exist a neighbourhood U open in L and cross-sections $\xi,\eta \in \mathrm{Sec}A(\Phi)$, such that $\bar{\xi}(y)=(y,\xi(y))$ and $\bar{\eta}(y)=(y,\eta(y))$ for $y \in U$. We can define a bracket $[\![\cdot,\cdot]\!]_L: \mathrm{Sec}A(\Phi)_L \times \mathrm{Sec}A(\Phi)_L \longrightarrow \mathrm{Sec}A(\Phi)_L$ as follows:

4.2.4. Definition. Put

$$[\![\bar{\xi},\bar{\eta}]\!]_L(y):=(y,[\xi,\eta](y)),\quad y \in U. \tag{4.2.5}$$

We shall demonstrate thet the above definition is correct. Take any cross-sections $\xi,\xi_1,\eta,\eta_1 \in \mathrm{Sec}A(\Phi)$ such that $\bar{\xi}(y)=(y,\xi(y))=(y,\xi_1(y))$ and $\bar{\eta}(y)=(y,\eta(y))=(y,\eta_1(y))$ for $y \in U$. By a symmetry of the bracket, the above condition is equivalent to the following: $[\![\nu,\eta]\!]|U=0$ provided $\nu|U=0$. The lemma below proves that this condition is fulflled.

4.2.6. Lemma. Let $\nu,\eta \in \mathrm{Sec}A(\Phi)$ and $\nu|U=0$ for some set U open in L. Then $[\![\nu,\eta]\!]|U=0$.

Proof. Let $x_o \in L$. Take an element $h_o \in \Phi_{x_o}$ such that $y_o:=\beta h_o \in U$. $\nu'|\Phi_{x_o}$ and $\eta'|\Phi_{x_o}$ are smooth right-invariant vector fields on

Φ_{x_o}; moreover, /1/ $\nu'(h)=(D_h)_{*\beta h}(\nu_{\beta h})$, /2/ $\eta'(h)=(D_h)_{*\beta h}(\eta_{\beta h})$, /3/ $[\![\nu,\eta]\!]'(h)=(D_h)_{*\beta h}([\![\nu,\eta]\!](\beta h))$ for $h\in\Phi_{x_o}$. By /1/, $\nu'|\beta^{-1}_{x_o}[U]=0$. Since $(D_h)_{*\beta h}$ is an isomorphism, therefore, by /2/, to prove that $[\![\nu,\eta]\!]|U=0$, it is sufficient to see that $[\![\nu,\eta]\!]'(h)=0$ for $h\in\beta^{-1}_x[U]$. But $[\![\nu,\eta]\!]'(h)=[\nu',\eta'](h)$, so $[\nu',\eta']|\Phi_{x_o}=[\nu'|\Phi_{x_o},\eta'|\Phi_{x_o}]$, thus $[\![\nu,\eta]\!]'|\Phi_{x_o}=[\nu'|\Phi_{x_o},\eta'|\Phi_{x_o}]$ and, in the end, $[\nu'|\Phi_{x_o},\eta'|\Phi_{x_o}]|\beta^{-1}_{x_o}[U]=0$.

4.2.7. Proposition. The set $SecA(\Phi)_L$ with the natural operations is a vector space. Moreover, the pair $(SecA(\Phi)_L,[\![\cdot,\cdot]\!]_L)$ is an R-Lie algebra with the property

(4.2.8) $$[\![f\cdot\bar{\xi},\bar{\eta}]\!]_L=f\cdot[\![\bar{\xi},\bar{\eta}]\!]_L+(\gamma\circ\bar{\xi})(f)\cdot\bar{\eta}$$

for $\bar{\xi},\bar{\eta}\in SecA(\Phi)_L$ and $f\in C^\infty(L)$.

4.2.9. Proposition. The mapping

(4.2.10) $$Sec\gamma: SecA(\Phi)_L\longrightarrow X(L),\quad \xi\longmapsto\gamma(\xi),$$

is a homomorphism of Lie algebras.

An immediate consequence of 4.2.1, 4.2.3, 4.2.7 and 4.2.9 is

4.2.11. Theorem. The system $(A(\Phi)_L,[\![\cdot,\cdot]\!]_L,\gamma)$ is a Lie algebroid.

4.2.12. Remark. Let $(A',[\![\cdot,\cdot]\!]_L',\gamma')$ ba a subalgebroid of the algebroid $(A(\Phi)),[\![\cdot,\cdot]\!]_L,\gamma)$ of the foliated groupoid Φ over the same foliation F. If: 1/ the function $x\longmapsto \dim A'_{|x}$ is constant on each $L\in F$, 2/ $\gamma'[A']=\gamma[A(\Phi)]$, 3/ $\forall\, v\in A'\ \exists\ \xi\in SecA'\ (\xi(p'(v))=v\,)$ then A'_L is a Lie subalgebroid of the Lie algebroid $A(\Phi)_L$.

4.3. The Lie groupoid Φ_L.

Using the Lemma I [5, p.195], one can prove that if $\pi:P\longrightarrow B$ is a coregular mapping between differential manifolds and $\cdot:P\times G\longrightarrow P$ is a right free action of a Lie group G on P whose orbits coincide with the fibres of the projection π, then the system $(P,\pi,B,G,\cdot)$ is a principal bundle. The properties of a leaf and the above remark imply

4.3.1. Proposition. Let $(\Phi,\alpha,\beta,V,\cdot)$ be a foliated groupoid over a foliation with singularities F. Then, for each $x\in L$, $L\in F$, the set $G_x=\{h\in\Phi;\ \alpha h=\beta h=x\}=\beta^{-1}_x(x)$, together with the multiplication induced from Φ, is a Lie group, and the system

(4.3.2) $$\Phi_x = (\Phi_x, \beta_x, L, G_x, \cdot),$$

where $\cdot : \Phi_x \times G_x \to \Phi_x$, $(h,a) \mapsto h\cdot a$, is a principal bundle.

With the help of local cross-sections of principal bundle (4.3.2) we shall equip the groupoid $\Phi_L = (\alpha,\beta)^{-1}[L\times L]$ (with the operations induced from Φ) with the structure of a Lie groupoid. Next, we shall show that the Lie algebroids $A(\Phi)_L$ and $A(\Phi_L)$ are isomorphic.

Let $L\in F$ and $x\in L$. Take a family $\{(\phi_i : U_i \to \Phi_x)_{i\in I}\}$ of local cross-sections, of the principal bundle (4.3.2) such that $\bigcup_{i\in I} U_i = L$. Each cross-section ϕ_i determines a local coordinate representation of (4.3.2)

(4.3.3) $$\tilde{\phi}_i : U_i \times G_x \to \beta_x^{-1}[U_i], \quad (y,a) \mapsto \phi_i(y)\cdot a.$$

For $i,j\in I$, we put

$\phi_{i,j} : U_i \times G_x \times U_j \to (\alpha,\beta)^{-1}[U_i \times U_j] \subset \Phi_L$, $(y,a,z) \mapsto \phi_j(z)\cdot a\cdot \phi_i(y)^{-1}$.

Notice first that

1/ $\bigcup_{i,j\in I} \mathrm{Im}\,\phi_{i,j} = \Phi_L$,

2/ $\Phi_{i,j}$ is a bijection, $i,j\in I$,

3/ the set $\phi_{i,j}^{-1}[\mathrm{Im}\,\phi_{k,l}]$ is open in $U_i\times G_x\times U_j$ and the mapping $\phi_{k,l}^{-1}\circ\phi_{i,j}$ is a diffeomorphism for $i,j,k,l\in I$,

4/ for $g,h\in\Phi_L$, $g\neq h$, there exist disjoint sets Ω_1 and Ω_2 and $i,j,k,l\in I$ such that $g\in\Omega_1\subset(\alpha,\beta)^{-1}[U_i\times U_j]$, $h\in\Omega_2\subset(\alpha,\beta)^{-1}[U_k\times U_l]$ and the sets $\phi_{i,j}^{-1}[\Omega_1]$, $\phi_{k,l}^{-1}[\Omega_2]$ are open.

Therefore (see [4] p.29), on the set Φ_L there exists exac5tly one structure of differential manifold, such that the sets $(\alpha,\beta)^{-1}[U_i\times U_j]$ are open and the mappings $\phi_{i,j}$ are diffeomorphisms, $i,j\in I$. It is easy to see that it is a Hausdorff manifold with a countable basis.

4.3.5. Theorem. The groupoid $(\Phi_L, \alpha, \beta, L, \cdot)$ with the differential structure defined above is a Lie groupoid.

Proof. Let $h\in\Phi_L$. Take $i,j\in I$ such that $h\in(\alpha,\beta)^{-1}[U_i\times U_j]$. Since $\phi_{i,j}^{-1}$ is a diffeomorphism and the projections $pr_1 : U_i\times G_x\times U_j \to U_i$ and $pr_3 : U_i\times G_x\times U_j \to U_j$ are submersions, therefore, from the commutativity of the diagrams below it follows that α,β are submersions.

$$\begin{array}{ccc} (\alpha,\beta)^{-1}[U_i\times U_j] & \xrightarrow{\ \alpha\ } & U_i \\ \phi_{i,j}^{-1} & & pr_1 \\ U_i\times G_x\times U_j & & \end{array} \qquad \begin{array}{ccc} (\alpha,\beta)^{-1}[U_i, U_j] & \xrightarrow{\ \beta\ } & U_j \\ \phi_{i,j}^{-1} & & pr_3 \\ U_i\times G_x\times U_j & & \end{array}$$

Now, we notice that if $h\in(\alpha,\beta)^{-1}[U_i\times U_j]$ for some $i,j\in I$, then $h^{-1}\in(\alpha,\beta)^{-1}[U_j\times U_i]$. Thus, the smoothness of the mapping ${}^{-1}:\Phi_L\longrightarrow\Phi_L$ follows from the equality
$\phi_{j,i}^{-1}\circ\phi_{i,j}=(U_i\times G_x\times U_j\ni(y,g,z)\longmapsto(z,g^{-1},y)\in U_j\times G_x\times U_i$.
To show that the partial multiplication $\cdot:\Phi_L\longrightarrow\Phi_L$ is smooth, take $i,j\in I$ and a mapping
$(\phi_{k,i}\times\phi_{j,k}):(U_k\times G_x\times U_i)\times(U_j\times G_x\times U_k)\longrightarrow\Phi_L\times\Phi_L$. Note that $T'=(\phi_{j,k},\phi_{k,i})[U_k\times G_x\times U_i\times U_j\times G_x\times U_k]\cap\Phi_L*\Phi_L=(\phi_{j,k},\phi_{k,i})[T]$ where $T=\{(x,g_1,y_1,y_2,g_2,x);\ x\in U_k,\ g_1,g_2\in G_x,\ y_1\in U_i,\ y_2\in U_j\}$ (T is open in $\Phi_L*\Phi_L$). It defines the diffeomorphism $\theta=(T\ni\tau\longmapsto(\phi_{j,k},\phi_{k,i})(\tau))\in\Phi_L*\Phi_L|T'$. Then the mapping $\phi_{j,k}^{-1}\circ\cdot\circ\theta=((s,a,z,t,a_1,s)\longmapsto(t,a\cdot a_1,z))$ is smooth, so the multiplication is smooth. Of course, Φ_L is a transitive groupoid.

4.3.6. Theorem. For any $L\in\Phi$ and $x\in L$, the following equality of differential manifolds holds: $(\Phi_L)_x=\Phi_x$.

Proof. Clearly, the sets of points $(\Phi_L)_x$ and Φ_x are equal. To show the equality of their differential structures, we put $h\in(\Phi_L)_x$. Take cross-sections $\phi:U\longrightarrow\Phi_x$ and $\psi:W\longrightarrow\Phi_x$, where U and W are open in L, such that $x\in U$, $\phi(x)=u_x$ and $\psi(x)=h$. Since the mapping $\varkappa:U\times G_x\times W\longrightarrow(\alpha,\beta)^{-1}[U\times W]$, $(y,g,z)\longmapsto\psi(z)\cdot g\cdot\phi(y)^{-1}$, is a diffeomorphism, therefore $\varkappa(x,\cdot,\cdot):G_x\times W\longrightarrow(\Phi_L)_x\cap\mathrm{Im}\varkappa$ is a diffeomorphism; moreover, $h\in\mathrm{Im}\varkappa(x,\cdot,\cdot)$. On the over hand, $\tilde{\psi}:G_x\times W\longrightarrow\pi^{-1}[W]$, $(g,z)\longmapsto\psi(z)\cdot g$, as a coordinate representation, is a diffeomorphism. So, by the equality $\varkappa(x,\cdot,\cdot)=\tilde{\psi}$, we obtain commutative diagram.

$$\begin{array}{ccc}(\Phi_L)_x\cap\mathrm{Im}\varkappa\subset(\Phi_L)_x & \xrightarrow{\ \ id\ \ } & \Phi_x\subset\pi^{-1}[U]\\ \approx\ \varkappa(x,\cdot,\cdot) & & \approx\ \tilde{\psi}\\ G_x\times W & \xrightarrow{\ \ id\ \ } & G_x\times W\end{array}$$

Thus the mapping $id:(\Phi_L)_x\longrightarrow\Phi_x$ is a diffeomorphism.

4.3.7. Theorem. For $L\in F$, the mapping $\varkappa:A(\Phi_L)\longrightarrow A(\Phi)_L$, $(x,v)\longmapsto v$, is an isomorphism of Lie algebroids.

Proof. Of course, $\varkappa$ is an isomorphism on the fibres of vector bundles. Thus it is sufficient to show that:

1/ the diagram below commutes, where $\beta_L:=\beta|\Phi_L$,

$$\begin{array}{ccc}A(\Phi)_L & \xrightarrow{\ \ \varkappa\ \ } & A(\Phi_L)\\ \gamma & & (\tilde{\beta}_L)_*\\ & TL & \end{array}$$

2/ $\varkappa^{-1}$ is a smooth mapping,

3/ if a commutator in the Lie algebra $\mathrm{Sec}A(\Phi)_L$ is denoted by $[\![\cdot,\cdot]\!]^L$, then -for any $\bar{\xi},\bar{\eta}\in\mathrm{Sec}A(\Phi)_L$, the equality $\varkappa\circ[\![\bar{\xi},\bar{\eta}]\!]_L=[\![\varkappa\circ\bar{\xi},\varkappa\circ\bar{\eta}]\!]^L$ holds.

1/ is clear.

2/ Note that $\varkappa^{-1}=(A(\Phi_L)\ni v\longmapsto(\pi(v),v)\in A(\Phi)_L)$ where $\pi\colon A(\Phi_L)\longrightarrow L$ is the canonical projection. Since π is smooth, it is sufficient to show that the inclusion $i\colon A(\Phi_L)\longrightarrow A(\Phi)$ is smooth. To prove this, take a smooth inclusion $i\colon\Phi_L\longrightarrow\Phi$. Thus the mapping $i_*\colon T(\Phi_L)\longrightarrow T\Phi$ is smooth, and so, $i\colon A(\Phi_L)\longrightarrow A(\Phi)$ is smooth.

3/ Let $\bar{\xi},\bar{\eta}\in\mathrm{Sec}A(\Phi)_L$ and $x\in L$. Then there exist a neighbourhood U of x open in L and cross-sections $\xi,\eta\in\mathrm{Sec}A(\Phi)$, such that $\bar{\xi}(y)=(y,\xi(y))$, $\bar{\eta}(y)=(y,\eta(y))$ and $[\![\bar{\xi},\bar{\eta}]\!]_L(y)=(y,[\![\xi,\eta]\!](y))$ for $y\in U$. Thus $\varkappa\circ([\![\bar{\xi},\bar{\eta}]\!]_L(y))=\varkappa(y,[\![\xi,\eta]\!]y=[\![\xi,\eta]\!](y)$ for $y\in U$ it means $(*)$ $\varkappa([\![\bar{\xi},\bar{\eta}]\!]_L)|U=[\![\xi,\eta]\!]|U$. At the same time, $\varkappa\circ\bar{\xi}\in\mathrm{Sec}A(\Phi_L)$ and $(\varkappa\circ\bar{\xi})(y)=\varkappa(y,\xi(y))=\xi(y)$. Hence $(\varkappa\circ\bar{\xi})|U=\xi|U$ and
$(**)$ $[\![(\varkappa\circ\bar{\xi})|U,(\varkappa\circ\bar{\eta})|U]\!]^L=[\![\xi|U,\eta|U]\!]^L$. By $(*)$ and $(**)$, the equality $\varkappa\circ[\![\bar{\xi},\bar{\eta}]\!]_L=[\![\varkappa\circ\bar{\xi},\varkappa\circ\bar{\eta}]\!]^L$ follows from the following lemma:

4.3.8. Lemma. For any $x\in L$ and $\xi,\eta\in\mathrm{Sec}A(\Phi)$, we have $\xi|L$, $\eta|L\in\mathrm{Sec}A(\Phi_L)$ and $[\![\xi,\eta]\!](x)=[\![\xi|L,\eta|L]\!]^L(x)$.

Proof. First, we notice that, for $\xi\in\mathrm{Sec}A(\Phi)$, the equality $(***)$ $(\xi|L)'=\xi'|\Phi_L$ holds. Indeed, for $h\in\Phi_L$, $(\xi|L)'(h)=$ $(D_h)_{\beta h}(\xi|L)_{\beta h}=(D_h)_{*\beta h}(\xi_{\beta h})=\xi'(h)=(\xi'|\Phi_L)(h)$. From $(***)$ it follows, in particular, that $\xi|L\colon L\longrightarrow A(\Phi_L)$ is smooth. Take $\xi,\eta\in\mathrm{Sec}A(\Phi)$. From $(***)$, 4.3.6 and 1.8 we obtain
$[\![\xi|L,\eta|L]\!]^L(x)=[(\xi|L)',(\eta|L)'](u_x)=[\xi'|\Phi_L,\eta'|\Phi_L](u_x)=$
$=[\xi'|(\Phi_L)_x,\eta'|(\Phi_L)_x](u_x)=[\xi'|\Phi_x,\eta'|\Phi_x](u_x)=[\xi',\eta']|\Phi_x(u_x)=$
$=[\![\xi'\eta]\!](x)$.

5. On the inducing of a foliated subgroupoid by a foliated subalgebroid.

To end with our paper we give a generalization of the following theorem:

5.1. Theorem [10]. Let $\Phi=(\Phi,\alpha,\beta,V,\cdot)$ be a Lie groupoid and A - its algebroid. For each Lie subalgebroid $A'=(A',[\![\cdot,\cdot]\!]',\tilde{\beta}_*|A')$ of the Lie algebroid A, there exists exactly one connected Lie subgroupoid of Φ whose algebroid is equal to A'.

5.2. Theorem. Let F be a foliation with singularities of the manifold V, $((\Phi,C),\alpha,\beta,V,\cdot)$ - a foliated groupoid over F, and $A=(A(\Phi),[\![\cdot,\cdot]\!],\gamma)$ - its algebroid. Then, for any subalgebroid $A'=(A',[\![\cdot,\cdot]\!]',\gamma|A')$ of the algebroid A, the following conditions are equivalent:

1) there exists an α-connected foliated subgroupoid over the foliation F whose algebroid is equal to A', (α-connectedness of an arbitrary smooth groupoid means that all leaves of it are connected),

2) A' is a foliated algebroid over F.

For all such subgroupoids, their leaves over the same point are identical. Among these groupoids, there exists a subgroupoids with the topology induced from Φ. It is uniquely determined.

Proof. 1) $\Rightarrow$ 2) is clear.

2) $\Rightarrow$ 1). Let A' be a foliated subalgebroid over the foliation F and let $L\in F$. By theorem 4.3.5, the groupoid $\Phi_L=(\alpha,\beta)^{-1}[L\times L]$ has a natural differential structure of a Lie groupoid, and, by theorem 4.3.7, the algebroidA (Φ_L) is isomorphic to $A(\Phi)_L$. By remark 4.2.12, A'_L is a Lie algebroid. Moreover, it is a Lie subalgebroid of the Lie algebroid $A(\Phi)_L$. Denote by j an inclusion $A'_L\longrightarrow A(\Phi)_L$. Since $\varkappa: A(\Phi)_L\longrightarrow A(\Phi)_L$, $(x,v)\longmapsto v$, is an isomorphism, therefore $\varkappa\circ j$ is a monomorphism of vector bundles. Then ,obviously, the image of A'_L under $\varkappa\circ j$ is a Lie subalgebroid of the Lie algebroid $A(\Phi_L)$. By theorem 5.1, there is exactly one connected subgroupoid Φ'_L of the groupoid Φ_L whose algebroid is equal to $(\varkappa\circ j)[A(\Phi)_L]$. Put $\Phi':=\bigcup\Phi'_L$. That the system $(\Phi',\alpha|\Phi',\beta|\Phi',V,\cdot|\Phi'*\Phi')$ is, of course, a groupoid, and, Φ' together with a differential structure induced from the groupoid Φ, is a groupoid in the category of differential spaces. We shall show that it is a smooth groupoid. Let $x\in V$. The differential manifold Φ'_x, being a leaf of the groupoid Φ_L when $x\in L$, is a leaf of the manifold Φ_x (see [10]). Since Φ_x is a leaf of the differential space Φ, so is Φ'_x. Thus Φ'_x is a leaf of the differential space Φ' because the differential structure on Φ' is induced from Φ. Next, we notice that the

equality $T_{u_x}(\Phi'_L)_x = A'_{|x}$ implies the equality of sets $A(\Phi') = A'$. Since the inclusion $i: A' \longrightarrow A$ is an embedding, the differential structure of the algebroid A' is induced from A, and so, it is the same as the differential structure of the algebroid $A(\Phi')$. Hence A' is an algebroid of the groupoid Φ'. Let Φ', Φ'' be two α-connected subgroupoid whose algebroids are equal to A'. Since, for $x \in V$, the manifolds Φ'_x, and Φ''_x are the maximal integral manifolds of the same distribution on the manifold Φ_x, the leaves Φ'_x and Φ''_x are equal.

Now, let Φ' and Φ'' have the topologies induced from Φ. Since Φ'_x and Φ''_x are connected sets for $x \in V$, therefore Φ'_L and Φ''_L are connected Lie groupoids whose Lie algebroids are equal A_L where L is the leaf through x. Thus, by theorem 5.1, $\Phi'_L = \Phi''_L$, hence the sets Φ' and Φ'' are equal. And since both have the differential structures induced from Φ, the groupoids Φ' and Φ'' are identical.

Acknowledgments.

I wish express my warmest thanks to Doctor J.Kubarsli for his precious remarks concerning the topics of this paper. I would also like to thank for their careful analysis of the text as well as suggested simplifications.

References:

[1] BRANDT H. "Uber eine Verallgemeinnerung des Gruppenbegriffes", Math. Ann. 96 /1927/, p.360-366,

[2] EHRESMANN CH. "Les connexions infinitesimales dans un espace fibre differentiable", Colloq. Topologie, Bruxelles, 1950,

[3] EHRESMANN CH. "Categories topologiques et categories differentiables", Geom. Differ. Globale, Bruxelles 1959,

[4] GREUB W.,HALPERIN S.,VANSTONE R. "Connections, Curvature, and Cohomology" Vol.I, Academic Press New York and London, 1972,

[5] GREUB W., HALPERIN S., VANSTONE R. "Connections, curvature, and Cohomology" Vol.II, Academic Press New York and London, 1973,

[6] KOWALCZYK A. "Tangent differential spaces and smooth forms", Dem. Math. 13, 1980, No 4, p.893-905,

[7] KUBARSKI J. "Exponential mapping for Lie groupoids", Colloq. Math. Vol.XLVII, No 2, 1982, p.267-282,

[8] KUBARSKI J. "Smooth groupoids over foliations and their algebroids. Part I." Institute of Mathematics, Technical University of Lodz, Preprint No 1, May 1986,

[9] KUBARSKI J. "Pradines-type groupoids", Proceedings of the Winter School of Geometry and Physics, Srni, 10-17 January, 1987,

[10] KUBARSKI J. "Exponential mapping for Lie groupoids. Applications." Colloq. Math. Vol.LIV, No 2,1987, p.39-48,

[11] MACKENZIE K. "Lie groupoids and Lie algebroids in Differential Geometry", Cambridge University Press 1987, London Mathematical Society, Lecture Note Series 124,

[12] PRADINES J. "Theorie de Lie pour les groupoides differentiables. Relations entre proprietes locales et globales", C.R.A.S., Paris, t.263, p.907-910, 1966,

[13] PRADINES J. "Theorie de Lie pour les groupoides differentiables. Calcul differential dans la categorie des groupoides infinitesimaux", C.R.A.S., Paris, t.264, p.245-247, 1967.

[14] QUE V. "Du prolongement des espaces fibres et des structures infinitesimales", Ann. Inst. Fourier, Grenoble 17, 1., 1967, p.157-223,

[15] SIKORSKI R. "Abstract covariant derivative", Coll. Math. 18, 1967, p.251-272,

[16] SIKORSKI R. "Wstep do geometrii rozniczkowej", PWN, Warszawa 1972, /in Polish/,

[17] SIKORSKI R. "Differential modules", Coll. Math. 24, 1971, p.45-79,

[18] STEFAN P. "Accessibility and foliations with singularities", Bull. Amer. Math. Soc., 80, 1974, p.1142-1145,

[19] STEFAN P. "Accessible sets, orbits, and foliations with

singularities", Proc. London Math. Soc., 29, 1974, p.699-713,

[20] SUSSMANN H.J. "Orbits of families of vector fields and integrability of distributions", Trans. Amer. Math. Soc. 180, 1973, p.171-188,

[21] WALISZEWSKI W. "Regular and coregular mappings of differential spaces", Ann. Pol. Math. XXX, 1975, p.263-281,

[22] WALISZEWSKI W. "On vector fields coinduced on a differential space and their Lie product", Ann. Pol. Math., XXXII, 1976, p.129-133.

INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF LODZ,
AL.POLITECHNIKI 11, PL-90-924 LODZ, POLAND.