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Non-Normality and Relative Normality of Niemytzki Plane

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A characterization of pairs of closed subsets of Niemytzki plane, which cannot be separated by open neighborhoods, is given. A few consequences about normality of Niemytzki plane on some subspaces are derived and an anwer to the problem 3.4 from Tkačenko, Tkachuk, Wilson, Yaschenko [TTWY] is given.

Notation

Let us recall the definition of Niemytzki plane and establish some notation. $\mathbb{R}$ will denote as usual real numbers, $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{N} = \{1,2,\dots\}$. Let $\mathbf{L} = \{(t,0) : t \in \mathbb{R}\}$, $\mathbf{E} = \{(r,s) : r \in \mathbb{R},\ s \in \mathbb{R}^+\}$, $\mathbf{N} = \mathbf{L} \cup \mathbf{E}$. For $x = (r,s) \in \mathbf{E}$ and $o < \varepsilon < s$ let

$$B_\varepsilon(x) = \{(r_1, s_1) \in \mathbf{E} : (r_1 - r)^2 + (s_1 - s)^2 < \varepsilon^2\}$$

and for $x = (t,0) \in \mathbf{L}$ and $\varepsilon \in \mathbb{R}^+$ let

$$B_\varepsilon(x) = B_\varepsilon(t,\varepsilon) \cup \{x\}.$$

The Niemytzki plane is the set $\mathbf{N}$ with topology generated by sets $B_\varepsilon(x)$ for $x \in \mathbf{N}$ and $\varepsilon \in \mathbb{R}^+$. On the set $\mathbf{L}$ we will also use the topology of the real line denoted by $\mathcal{R}$.

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1. Non-normality of Niemytzki plane

It is well known, that Niemytzki plane is an example of completely regular non-normal space [En]. In this section a general condition for closed subsets of $\mathbf{N}$, which can be separated by open neighborhoods, is described.

Theorem 1.1. Let G, H be disjoint closed subsets of $\mathbf{N}$. Then G and H can be separated by disjoint open sets if and only if there exist sets G_i and H_i for $i \in \mathbb{N}$ such that $G \cap \mathbf{L} = \bigcup_{i\in\mathbb{N}} G_i$, $H \cap \mathbf{L} = \bigcup_{i\in\mathbb{N}} H_i$ and

$$\overline{G_i}^{\mathcal{R}} \cap H = \emptyset = \overline{H_i}^{\mathcal{R}} \cap G$$

for every $i \in \mathbb{N}$.

We will use the following technical Lemma in the proof of Theorem 1.1.

Lemma 1.2. For each $x \in \mathbf{E}$ there exists some $\imath \in \mathbb{R}^+$ such that $x \notin B_\varepsilon(y)$ implies $B_{\varepsilon/2}(y) \cap B_\imath(x) = \emptyset$ for each $y \in \mathbf{L}$ and each $\varepsilon \in \mathbb{R}^+$, $\varepsilon \le 1$.

Proof of Lemma 1.2. Without loss of generality we may assume $x = (0, a)$. Take any $\imath$ such that $\imath + \imath^2 \le a^2/2$ and $\imath \le a/2$. We will prove that this $\imath$ works. Let $y = (b, 0) \in \mathbf{L}$ and $\varepsilon \in \mathbb{R}^+$, $\varepsilon \le 1$, be such that $x \notin B_\varepsilon(y)$ (and thus $\varepsilon^2 \le b^2 + (a - \varepsilon)^2$). We have to prove that $B_{\varepsilon/2}(y) \cap B_\imath(x) = \emptyset$. This fact can be reformulated as $(\imath + \varepsilon/2)^2 \le b^2 + (a - \varepsilon/2)^2$.

Now, note that

$$(\imath + \varepsilon/2)^2 = \varepsilon^2/4 + \varepsilon\imath + \imath^2 \le \varepsilon^2/4 + \imath + \imath^2 \le a^2/2 + \varepsilon^2/4.$$

If $a/2 \le \varepsilon$, then apply $0 \le b^2 + (a - \varepsilon)^2 - \varepsilon^2$ to get

$$a^2/2 + \varepsilon^2/4 \le a\varepsilon + \varepsilon^2/4 \le b^2 + (a - \varepsilon)^2 + a\varepsilon + \varepsilon^2/4 - \varepsilon^2 = b^2 + (a - \varepsilon/2)^2,$$

as required. Suppose that $0 < \varepsilon < a/2$. In particular, $0 \le a(a/2 - \varepsilon) = a^2/2 - a\varepsilon$, hence

$$a^2/2 + \varepsilon^2/4 \le a^2 - a\varepsilon + \varepsilon^2/4 \le b^2 + (a - \varepsilon/2)^2. \qquad \square$$

Proof of Theorem 1.1. We will put $G' = G \cap \mathbf{L}$, $H' = H \cap \mathbf{L}$.

First, let us show that if the condition is not fulfilled, then the sets G and H cannot be separated. Suppose U and V are open set, such that $G \subset U$ and $H \subset V$. To each $x \in G'$ $(x \in H')$ assign $\varepsilon(x) \in \mathbb{R}^+$, for which $B_{\varepsilon(x)}(x) \subset U$ $(B_{\varepsilon(x)}(x) \subset V$, respectively). Now if $G_i = \{x \in G' : \varepsilon(x) > \frac{1}{i}\}$ and $H_i = \{x \in H' : \varepsilon(x) > \frac{1}{i}\}$ for $i \in \mathbb{N}$, then, without lost of generality, $(\exists j \in \mathbb{N})(\exists h \in \overline{G_j}^{\mathcal{R}})(h \in H')$. Otherwise G_i, H_i satisfy the given condition. This implies for such j and h,

$$\emptyset \ne \bigcup_{y\in G_j} B_{\varepsilon(y)}(y) \cap B_{\varepsilon(h)}(h) \subset U \cap V$$

and U and V are not disjoint.

Now let us fix sets G, H, G_i and H_i, $i \in \mathbb{N}$ which satisfy the condition stated in Theorem 1.1, and construct disjoint sets U and V separating G and H. In the first (and crucial) step we will separate G' and H'. For $x = (t,0) \in \mathbf{L}$ let $P_\varepsilon(x)$ be "the area between a horizontal line and a parabola":

$$P_\varepsilon(x) = \{(r,s) \in \mathbf{E} : \varepsilon > s > (t - r)^2\} \cup \{x\}.$$

Now for $x \in G_1$ take any $\varepsilon(x) \in (0,1)$. For each $x = (t,0) \in H_1$ fix an $\varepsilon(x) \in (0,1)$ such that $\{(t',0) \in \mathbf{L} : |t' - t| < 2\sqrt{\varepsilon(x)}\} \cap G_1 = \emptyset$. That is possible since $\overline{G_1}^{\mathcal{R}} \cap \cap H_1 = \emptyset$. Thus

$$P_{\varepsilon(x)}(x) \cap \bigcup_{y \in G_1} P_{\varepsilon(y)}(y) = \emptyset$$

for every $x \in H_1$.

Further, we may assume that sets G_i (H_i, respectively) are pairwise disjoint and we will continue inductively. To $x \in G_n$ (H_n, respectively) we assign $\varepsilon(x)$ in the same way: for $x = (t,0) \in G_n$ let $\varepsilon(x) \in (0,1)$ be such that

$$\{(t',0) \in \mathbf{L} : |t - t'| < 2\sqrt{\varepsilon(x)}\} \cap \bigcup_{i<n} H_i = \emptyset.$$

Such $\varepsilon(x)$ exists since $\overline{\bigcup_{i<n} H_i}^{\mathcal{R}} \cap G_n = \emptyset$. For x and $\varepsilon(x)$ chosen in this way

$$P_{\varepsilon(x)}(x) \cap \bigcup_{i<n} \bigcup_{y \in H_i} P_{\varepsilon(y)}(y) = \emptyset.$$

For $x \in H_n$ the construction (and also the resulting property) is similar. From the construction it follows that

$$\bigcup_{y \in G'} P_{\varepsilon(y)}(y) \cap \bigcup_{y \ H} P_{\varepsilon(y)}(y) = \emptyset.$$

Since $B_{\varepsilon\,2}(x) \subset P_\varepsilon(x)$ for $x \in \mathbf{L}$ and $\varepsilon \in (0,1)$,

$$U_1 = \bigcup_{x \in G'} B_{\varepsilon(x)\,2}(x)$$

and

$$V_1 = \bigcup_{x \in H'} B_{\varepsilon(x)\,2}(x)$$

are disjoint open sets in $\mathbf{N}$ and $G' \subset U_1$, $H' \subset V_1$.

In the second step we will separate G' from H. For each $x \in G'$ fix $\delta'(x) \in (0,1)$ such that $B_{\delta'(x)}(x) \cap H = \emptyset$. For $x \in G'$ let

$$\delta(x) = \min(\delta'(x)/2, \varepsilon(x)/2\}.$$

The set

$$U_2 = \bigcup_{x \ G} B_{\delta(x)}(x)$$

is open and covers G'. We will prove that $\overline{U}_2 \cap H = \emptyset$. Let us show that $h \in H \Rightarrow h \notin \overline{U}_2$.

If $h \in H'$, then $U_1 \cap V_1 = \emptyset$ and $U_2 \subset U_1$, V_1 is open and $H' \subset V_1$. Thus $h \notin \overline{U}_2$. If $h \in H \cap \mathbf{E}$, then $h \notin B_{\delta'(x)}(x)$ for each $x \in G'$. From this and Lemma 1.2 it follows that there exists $\iota \in \mathbb{R}$ such that $B_\iota(h) \cap B_{\delta(x)}(x) = \emptyset$ for all $x \in G'$, so $B_\iota(h) \cap U_2 = \emptyset$ and $h \notin \overline{U}_2$. Similarly we can construct an open set V_2 such that $H' \subset V_2$, $\overline{V}_2 \cap G = \emptyset$ and $V_2 \subset V_1$, which implies $U_2 \cap V_2 = \emptyset$.

Finally, let us separate whole sets. Since $\mathbf{E}$ is an open normal subspace of $\mathbf{N}$, $G \cap \mathbf{E}$ and $H \cap \mathbf{E}$ are disjoint closed subsets of $\mathbf{E}$, there exist disjoint open subsets U_3, V_3 of $\mathbf{E}$ (and thus open in $\mathbf{N}$) such that $G \cap \mathbf{E} \subset U_3$, $H \cap \mathbf{E} \subset V_3$. Hence $U = (U_2 \cup U_3)\backslash\overline{V}_2$ and $V = (V_2 \cup V_3)\backslash\overline{U}_2$ are the desired disjoint open sets separating G and H.

2. Normality of Niemytzki plane on its Euclidean part

The notion of normality on a subspace was introduced by Arhangel'skii in his survey on relative topological properties [Ar]. A space X is called *normal on a subspace* Y if any pair of disjoint closed sets G and H of X with $\overline{G \cap Y} = G$ and $\overline{H \cap Y} = H$ can be separated by open subsets of X. This definition can be equivalently reformulated: X is normal on Y if for each pair $G, H \subset Y$, such that $\overline{G} \cap \overline{H} = \emptyset$, $\overline{G}$ and $\overline{H}$ can be separated by open sets.

It is known [Ar] that every countable (moreover, every Lindelöf) space is strongly normal in any larger regular space. A space Y is *strongly normal in X*, if for each par G, H of closed in Y disjoint subsets of Y there are open disjoint subsets U and V in X, such that $G \subset U$ and $H \subset V$. Here a question raises when a regular space is normal on its (dense) countable subspace. This is studied in [TTWY].

Example 2.1 ([TTWY]). In this example a countable dense subset C of $\mathbf{N}$, such that $\mathbf{N}$ is not normal on C, was constructed. Let

$$A = \{(x,y) \in \mathbf{E} : x, y \in \mathbb{Q}\} \text{ and } Q = \{(x,0) : x \in \mathbb{Q}\}.$$

Then $\mathbf{N}$ is not normal on $C = A \cup Q$. Details can be found in the original article.

Example 2.2 ([TTWY]). There is a separable Tychonoff space which is not normal on any countable dense subspace. This space is constructed by a modification of the Niemytzki plane. It is again a kind of a "bubble" space but this space is not first countable.

In the light of previous examples, the authors of [TTWY] asked the following question ([TTWY, Problem 3.4]): It is true that the Niemytzki plane is not normal on any of its countable dense subspaces? However, as a corollary of Lemma 2.3 this appears not to be true.

Lemma 2.3. **N** is normal on **E**.

Proof. Consider G, H subsets of **E**, $\overline{G} \cap \overline{H} = \emptyset$. We will show, that $\overline{G}$ and $\overline{H}$ fulfill the condition of Theorem 1.1 and thus they can be separated. Put

$$G_i = \{x \in \overline{G} \cap \mathbf{L} : B_{1\,i}(x) \cap \overline{H} = \emptyset\}$$

and

$$H_i = \{x \in \overline{H} \cap \mathbf{L} : B_{1\,i}(x) \cap \overline{G} = \emptyset\}$$

for $i \in \mathbb{N}$. It is obvious that $\overline{G} \cap \mathbf{L} = \bigcup_{i \in \mathbb{N}} G_i$ and $\overline{H} \cap \mathbf{L} = \bigcup_{i \in \mathbb{N}} H_i$, so it remains to show that $\overline{G_i^{\mathcal{R}}} \cap \overline{H} = \emptyset$ ($\overline{H_i^{\mathcal{R}}} \cap \overline{G} = \emptyset$, respectively).

For contradiction assume that there is some $n \in \mathbb{N}$ and $h \in \overline{G_n^{\mathcal{R}}}$ such that $h \in \overline{H}$. Since $h \in \overline{H}$, we can fix $h' \in H \cap B_{1\,n}(h)$. Now $h \in \overline{G_n^{\mathcal{R}}}$,

$$B_{1\,n}(h) \subset \bigcup_{x \in G_n} B_{1\,n}(x)$$

and this implies that $h' \in B_{1\,n}(g)$ for some $g \in G_n$– a contradiction. The case $(\exists n \in \mathbb{N})(\exists g \in \overline{H_n^{\mathcal{R}}})(h \in \overline{G})$ is similar. □

Corollary 2.4. **N** is normal on each subset of **E**. □

So each dense countable subset of **E** (and such clearly exists) gives us an example of countable dense subspace of **N** on which **N** is normal.

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