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Choosing an Attacker by a Local Derivation

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Within the ω-limit set of a point, a uniform method is given for such problems as choosing for each point in its kernel a recurrent point that attacks it. In the light of recent counter-examples, the method appears to be optimal. It is inspired by a lemma of Kunugui and forms a local derivation.

1. Introduction

Let $\mathscr{X}$ be a Polish space with metric d and $f : \mathscr{X} \to \mathscr{X}$ a continuous function. We introduce the attacking relation $x \curvearrowright_f y$ as in [5, 6, 7]:

$$x \curvearrowright_f y \Leftrightarrow_{\mathrm{df}} \forall \varepsilon > 0 \forall m :\in \mathbb{N}\ \exists \ell [\ell \geqslant m \,\&\, d(f^{\ell}(x), y) < \varepsilon].$$

One may reduce the number of variables by saying

$$\forall m > 0\ \exists \ell \geqslant m\ d(f^{\ell}(x), y) < \tfrac{1}{m}.$$

We write $\mathscr{N}$ for *Baire space*, the space of infinite sequences of natural numbers: for each finite such sequence r we have the basic open set $\{\alpha \mid \alpha \upharpoonright \ell h(r) = r\}$. The (backward) *shift function* $\mathfrak{s} : \mathscr{N} \to \mathscr{N}$ is given by $\mathfrak{s}(\alpha)(n) = \alpha(n + 1)$.

In [7] two points a, b in Baire space $\mathscr{N}$ were constructed such that for $f = \mathfrak{s}$, both the set $\{\delta \mid \exists \beta :\in \mathscr{N}\ a \curvearrowright_{\mathfrak{s}} \beta \curvearrowright_{\mathfrak{s}} \delta\}$ and the set $\{\delta \mid \exists \varrho :\in \mathscr{N}\ b \curvearrowright_{\mathfrak{s}} \varrho \curvearrowright_{\mathfrak{s}} \varrho \curvearrowright_{\beta} \delta\}$ are complete analytic and therefore not Borel. These constructions formed counterexamples to conjectures made in [5], [6]. The second one in particular was contrary to the expectations of the author and of other researchers.

The purpose of this note is to describe, in the general setting of an arbitrary Polish space $\mathscr{X}$, continuous function $f : \mathscr{X} \to \mathscr{X}$ and point c in $\mathscr{X}$, a uniform

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method for choosing for each y such that $\exists x\; c \curvearrowright_f x \curvearrowright_f y$ a point x in $\omega_f(c)$ that attacks it, and for each y such that $\exists \varrho\; c \curvearrowright_f \varrho \curvearrowright_f \varrho \curvearrowright_f y$ a recurrent point ϱ in $\omega_f(c)$ that attacks it. The method was found in the course of attempting to prove the conjectures refuted by the examples of [7], and may still be of interest as providing a possible mechanism for proving of particular functions f and initial points c that the sets in question will, in their particular case, be indeed Borel.

1·0 HISTORICAL NOTE The method we shall describe was inspired by reading the discussion in [8] pp 254 – 259 of Kunugui's Lemma, which we now state, and its applications.

1·1 KUNUGUI'S LEMMA *If A is closed and π is continuous and $\pi[A] \subseteq \bigcup_k K_k$ where each K_k is closed, then there is a basic open set O_r and a k such that $O_r \cap A$ is non-empty and $\pi[O_r \cap A] \subseteq K_k$.*

2. Choosing an attacker

We define closed sets $K_{y,m}$ for each $y \in X$ and positive integer m:

2·0 DEFINITION $K_{y,m} =_{\mathrm{df}} \bigcap_{\ell \geqslant m}\{x \mid d(f^{\ell}(x),y) \geqslant \frac{1}{m}\}$.

2·1 REMARK The F_σ set $\bigcup_m K_{y,m}$ is the set of points $x \in \mathscr{X}$ that fail to attack y.

Let N_r enumerate a countable basis of the space $\mathscr{X}$, each of finite diameter.

2·2 REMARK A Polish space is separable; therefore we suppose that we have fixed a countable dense subset $\langle p_i \mid i \in \mathbb{N}\rangle$ of $\mathscr{X}$ and whenever we need to choose a member of a basic set N_r, can suppose that we choose the first point p_i to lie in N_r, and therefore avoid appeal to the axiom of choice.

For a closed subset B of $\mathscr{X}$ we define

$$\partial_{f,y}(B) =_{\mathrm{df}} B \setminus \bigcup_r \{N_r \mid \exists m (N_r \cap B \subseteq K_{y,m})\}. \tag{2·3}$$

Whilst y and f are fixed for our discussion, we omit the subscripts from ∂.

$\partial(B)$ is a closed set and a subset of B; the operation is monotone. So iterate, starting from the closed set $\{x \mid c \curvearrowright_f x\}$, often written $\omega_f(c)$, the ω-limit set of c:

$$\begin{aligned} B^0 &= \omega_f(c) \\ B^{\nu+1} &= \partial(B^\nu) \\ B^\lambda &= \bigcap_{\nu<\lambda} B^\nu \qquad \text{for limit } \lambda \end{aligned} \tag{2·4}$$

As we have a shrinking sequence of closed sets in a space with a countable basis, we shall reach in countably many steps a closed set B^∞, with the property that

$$\partial(B^\infty) = B^\infty. \tag{2·5}$$

Note that for any closed set B, if β is in B but not in $\partial(B)$, then β is in some $K_{y,m}$ and therefore cannot attack y; hence any $\beta \in \omega_f(c)$ that attacks y survives each successive application of ∂, and therefore lies in B^∞. Thus if y is attacked by some member of $\omega_f(c)$, B^∞ is not empty, and contains every member of B that attacks y.

By 2·4, B^∞ will have this property, where we write $\mathrm{clos}(A)$ for the closure of a subset A of $\mathscr{X}$:

$$(2{\cdot}6) \qquad \begin{aligned} N_r \cap B^\infty \neq \varnothing \Rightarrow \forall m \exists s\, \mathrm{clos}(N_s) \subseteq N_r \;\&\; \mathrm{diam}(N_s) \leqslant \tfrac{1}{2}\mathrm{diam}(N_r) \;\& \\ \&\; N_s \cap K_{y,m} = \varnothing \;\&\; N_s \cap B^\infty \neq \varnothing \end{aligned}$$

To see that, note that if $N_r \cap B^\infty \neq \varnothing$, then for each m there is a γ in $N_r \cap B^\infty \setminus K_{y,m}$, so for that γ we may choose s with $\gamma \in N_s$, $N_s \cap K_{y,m} = \varnothing$ and $\mathrm{diam}(N_s)$ at most $\frac{1}{2}\mathrm{diam}(N_r)$ and sufficiently small to ensure that $\mathrm{clos}(N_s) \subseteq N_r$.

Given Property 2·6, we may easily find a shrinking sequence N_{s_i} of basic open sets, their diameters tending to 0, the closure of each being a subset of its predecessor, and each containing a point in B^∞, with N_{s_0} any N_r not disjoint from B^∞ and for each $i \geqslant 1$, the set N_{s_i} disjoint from $K_{y,i}$. Let q_i be the canonically chosen (from $\langle p_i \mid i \in \mathbb{N} \rangle$) member of N_{s_i}. The sequence $(q_i)_i$ will be a Cauchy sequence, by the rapid shrinking of the diameters, and its limit point, x^∞ say, will be the sole member of $\bigcap_i \mathrm{clos}(N_{s_i})$; that point x^∞, being a limit of members of B^∞, will itself be in B^∞, since $x^\infty \in \bigcap_i N_{s_i}$ and $N_{s_i} \cap K_{y,m} = \varnothing$, x^∞ will lie outside $K_{y,m}$ for each m and thus will attack y.

Indeed, our discussion has yielded this result:

2·7 Proposition *B^∞ is the closure of the set of those points in $\omega_f(c)$ that attack y.*

Proof: the sequence of N_{s_i} can start inside an arbitrarily small neighbourhood of any given member of B^∞. ⊣(2·7)

2·8 Thus we have this entirely explicit procedure for finding a particular x in $\omega_f(c)$ attacking a given point y: first form the set $B^\infty_{f,y}$: if that is empty, no such x exists. Otherwise, form the set $\{r \in \mathbb{N} \mid N_r \cap B^\infty_{f,y} \neq \varnothing\}$; then using that set as an oracle, and repeatedly invoking Property 2·6, find a sequence (s_i) as above, choosing the smallest appropriate s at each step; then the desired point x will be the limit of the Cauchy sequence $(q_i)_i$ of canonical members of N_{s_i}.

If there are many choices at each stage, one might build a tree of possibilities, arranging that each infinite branch through the tree will lead to a possible x; in such a case the particular point x described in the previously paragraph will correspond to the left-most branch of that tree.

Thus we have a method for determining for each y an appropriate x without appeal to the axiom of choice.

We note some further properties of our definition.

2·9 Proposition *If B is f-closed so is $\partial_f(B)$.*

Proof: Let B be f-closed, and suppose that $x \in \partial(B)$ but $f(x) \notin \partial(B)$. Since $\partial(B) \subseteq B$, $x \in B$ and therefore $f(x) \in B$ and B is f-closed. Thus there exist r, m such that $f(x) \in N_r$ and $N_r \cap B \subseteq K_{y,m}$. So $x \in f^{-1}[N_r]$ which is an open set by the continuity of f. So pick s with $x \in N_s \subseteq f^{-1}[N_r]$.

Now for any $z \in N_s \cap B$, $f(z) \in N_r \cap B \subseteq K_{y,m}$. So $\forall \ell \geqslant m(d(f^{\ell+1}(z),y) \geqslant \frac{1}{m} > > \frac{1}{m+1}$. So for all $\ell \geqslant m+1$, $d(f^{\ell},y) > \frac{1}{m+1}$, and thus $z \in K_{y,m+1}$. Hence $N_s \cap B \subseteq K_{y,m+1}$ and so $x \notin \partial(B)$, a contradiction. $\dashv$(2·9)

2·10 Corollary *Starting from $B^0 = \omega_f(c)$, all B^ν and B^∞ are f-closed.*

2·11 Remark If a set B is both closed and f-closed it is $\curvearrowright_f$-closed, in the sense that $x \in B$ and $x \curvearrowright_f z$ implies $z \in B$. Hence all the B^ν's and B^∞ are $\curvearrowright_f$-closed.

3. Choosing a recurrent attacker

This time we wish to start from $\{\varrho \mid c \curvearrowright_f \varrho \curvearrowright_f \varrho\}$; but that, very probably, is not a closed set. It is however a G_δ. We appeal to a general result from descriptive set theory, originating in Lusin [4] and due perhaps jointly to Lusin and Souslin, which we take in the form stated as Exercise 13·10 of Kechris' treatise [3]:

3·0 Proposition *If P is a Borel subset of a Polish space $\mathscr{X}$, then there is a closed subset C of $\mathscr{X} \times \mathscr{N}$ such that the projection $\pi : (x,\alpha) \mapsto x$ from $\mathscr{X} \times \mathscr{N}$ to $\mathscr{X}$ is one-to-one on C and $\pi[C] = P$.*

For the proof of the general case, we refer the reader to [4] or [8]. However we note that when the Borel set is a G_δ set in a space, such as $\mathscr{N}$, with a countable basis of clopen sets, there is a very easy proof: represent the G_δ set P as $\bigcap_m \mathscr{O}_m$ where each $\mathscr{O}_m$ is open; write $\mathscr{O}_m = \bigcup_{k \in I_m} Q_k$, where Q_k is a basic clopen set, and arrange matters, by having repetitions if necessary, so that each I_m is an infinite set of positive integers. Let ψ be $\prod_m I_m$; ψ is evidently homeomorphic to $\mathscr{N}$. Then

$$\begin{aligned} x \in P &\Leftrightarrow \forall m \exists n :\in I_m \quad x \in Q_n \\ &\Leftrightarrow \exists \psi :\in \Psi\ \forall m \ \ x \in Q_{\psi(m)} \\ &\Leftrightarrow \exists \psi :\in \Psi\ \forall m \ \ \big(x \in Q_{\psi(m)}\ \&\ \forall n :\in I_m(n < \psi(m) \Rightarrow x \notin Q_n)\big), \end{aligned}$$

the third line representing a minimization of the values of ψ. So if we take C to be the following closed subset of $\mathscr{X} \times \Psi$:

$$\{(x,\psi) \mid \forall m(x \in Q_{\psi(m)}\ \&\ \forall n :\in I_m(n < \psi(m) \Rightarrow x \notin Q_n)\},$$

the projection $\pi : (x, \psi) \mapsto \varrho$ will be injective on C and have image P.

So we start from a closed set C_0 of $\mathscr{X} \times \mathscr{N}$ with $\pi[C_0] = \{\varrho \mid c \curvearrowright_f \varrho \curvearrowright_f \varrho\}$. We define $K_{y,m}$ as before but this time we enumerate a countable basis of $\mathscr{X} \times \mathscr{N}$ as O_r, (again regarding each as furnished with a canonical member of a fixed countable dense subset of $\mathscr{X} \times \mathscr{N}$) and define the *π-derivative at y* of a closed subset C of $\mathscr{X} \times \mathscr{N}$ thus:

3·1 Definition $\partial^\pi_{f,y}(C) =_{\mathrm{df}} C \setminus \bigcup\{O_r \mid \exists m\, \pi[O_r \cap C] \subseteq K_{y,m}\}$

3·2 Remark $\partial^\pi(C)$ is closed.

3·3 Remark If $\alpha \in C \setminus \partial^\pi(C)$ then for some r and m, $\alpha \in O_r \cap C$ and $\pi[O_r \cap C] \subseteq K_{y,m}$, so $\pi(\alpha)$ cannot attack y. Hence if $\alpha \in C$ and $\pi(\alpha) \rightharpoondown_f y$ then $\alpha \in \partial^\pi(C)$.

Thus if we iterate starting from C_0,

$$
\begin{aligned}
C^0 &= C_0 \\
C^{\nu+1} &= \partial^\pi(C^\nu) \\
C^\lambda &= \bigcap_{\nu<\lambda} C^\nu \qquad \text{for limit } \lambda
\end{aligned}
\tag{3·4}
$$

reaching a final closed set C^∞, each $\alpha \in C_0$ with $\pi(\alpha)$ attacking y will be in C^∞, and therefore its projection $\pi(\alpha)$ will be in $\pi[C^\infty]$, which thus contains all the recurrent points, attacked by c, that attack y.

In particular if y is in the abode, centre, or kernel of c, $C^\infty_{f,y}$ will be non-empty.

$C^\infty = \partial^\pi(C^\infty)$: so if $O_r \cap C^\infty$ is not empty, then for all m,$\pi[O_r \cap C^\infty]$ is not included in $K_{y,m}$, so for some $\gamma \in O_r \cap C^\infty$, $\pi(\gamma) \notin K_{y,m}$. For such m and γ, choose $\mathcal{O}$ open in $\mathscr{X}$ with $\pi(\gamma) \in \mathcal{O}$ and $\mathcal{O} \cap K_{y,m}$ empty: possible as $K_{y,m}$ is closed. Then pick O_s with $\gamma \in O_s$, (so that $O_s \cap C^\infty$ is not empty), $\mathrm{clos}(O_s) \subseteq O_r \cap \pi^{-1}[\mathcal{O}]$, and such that $\mathrm{diam}(O_s) \leqslant \frac{1}{2}\mathrm{diam}(O_r)$. Thus $\pi[O_s \cap C^\infty] \cap K_{y,m} = \varnothing$, and we have established the following property:

$$
\begin{aligned}
O_r \cap C^\infty \neq \varnothing \Rightarrow \forall m \exists s\, \mathrm{clos}(O_s) \subseteq O_r \,\&\, \mathrm{diam}(O_s) \leqslant \tfrac{1}{2}\mathrm{diam}(O_r) \,\& \\
\&\, \pi[O_s \cap C^\infty] \cap K_{y,m} = \varnothing \,\&\, O_s \cap C^\infty \neq \varnothing
\end{aligned}
\tag{3·5}
$$

Now given α_0 *in* C^∞, find a sequence O_{s_i} each containing the closure of the next, with diameters finite and shrinking to 0, with $\pi[O_{s_i}] \cap K_{y,i} = \varnothing$ for $i \geqslant 1$, and with α_0 *in* O_{s_0}, the latter to have arbitrarily small diameter. The intersection will be non-empty, since it will contain the limit of the Cauchy sequence of canonical members of the sets O_{s_i}, and the intersection will be a single point, α^∞, which will be in C^∞. Further, for each m, $\pi(\alpha^\infty)$ will not be in $K_{y,m}$ and therefore $\pi(\alpha) \rightharpoondown_f y$.

Thus we have proved the following

3·6 Proposition $\pi[C^\infty_{f,y}]$ *will be the closure of the set of recurrent points attacked by c and attacking y.*

3·7 Remark As before, we have a procedure for choosing a recurrent point in $\omega_f(c)$ attacking a given point $y \in \omega_f(c)$, when such exist: first form $C^\infty_{f,y}$. If it is empty, no such ϱ exists. Otherwise form the set $\{r \mid O_r \cap C^\infty_{f,y} \neq \varnothing\}$ and use it with the Property 3·5 to build, entirely constructively, a tree all of whose infinite branches lead to recurrent points x attacking the given y; and we may follow the left-most branch of that tree to obtain a definable such x.

4. Further remarks and problems

Local derivations

Our functions ∂ and ∂^π are *derivations* in the sense of Dellacherie's 1977 paper [1] and of Kechris' treatise [3], pp 270 et seq, where they are called *derivatives* and where numerous examples are discussed. For our purposes we may take a derivation in a space $\mathscr{X}$ to be a map δ defined on the closed subsets of $\mathscr{X}$ with the properties that for closed sets D and E, $\delta(D)$ is a closed subset of D, and $D \subseteq E \Rightarrow \delta(D) \subseteq \delta(E)$.

Dellacherie in his course [1] page 287 discusses the concept of a *local* derivation, which is one satisfying

$$\partial(X) = \{x \in X \mid \forall U :\in \mathscr{B} \,(x \in U \Rightarrow \partial(\mathrm{clos}(U) \cap X) \neq \varnothing)\}$$

where $\mathscr{B}$ is some countable basis of $\mathscr{X}$.

4·0 Remark Our derivations ∂_y and ∂_y^π are both local; hence if $B_{f,y}^\infty$ is empty, $B_{f,y}^0$ is the union of countably many closed sets B with $\partial(B) = \varnothing$, namely those sets sets $B^0 \cap N_r$ which are subsets of some $K_{y,m}$. In the C case, if $C_{f,y}^\infty$ is empty, then C_0 will also be covered by countably many sets with ∂^π-derivate empty, namely the sets $C_0 \cap O_r$ with $\pi[C_0 \cap O_r]$ a subset of some $K_{y,m}$.

Some characterizations

In [6] we defined this sequence of sets:

$$\begin{aligned} A^0(c,f) &= \omega_f(c) \\ A^{\beta+1}(c,f) &= \{y \mid \exists x :\in A^\beta(c,f)\, x \rightharpoonup_f y\} \\ A^\lambda(c,f) &= \bigcap_{\nu<\lambda} A^\nu(c,f) \qquad \text{for } \lambda \text{ a limit ordinal} \end{aligned}$$

A^0 is always a closed set; the others are all analytic, and the first construction in [7] gave a recursive $a \in \mathscr{N}$ with $A^1(a,\mathfrak{s})$ not Borel.

We also defined $\theta(a,f)$ to be the least ordinal θ with $A^\theta(c,f) = A^{\theta+1}(c,f)$ and showed in [6] that θ never exceeds ω_1, but, again by [7], for some recursive $b \in \mathscr{N}$, $\theta(b,\mathfrak{s}) = \omega_1$.

Hence although the sequence A^ν is shrinking, it is not composed of closed sets; nor in general need its terms be Borel sets, nor need it stop at a countable ordinal; so that attempts to represent its stages as projections of a shrinking sequence of closed sets are in general hopeless.

The arguments of §§2, 3 show the following:

4·1 PROPOSITION *$B^{\infty}_{f,y}$ is non-empty if and only if $y \in A^1(c, f)$.*

4·2 PROPOSITION *$C^{\infty}_{f,y}$ is non-empty if and only if $y \in A^{\theta(c,f)}(c, f)$.*

Two open problems

4·3 PROBLEM Is there a counterpart of 2·9 for ∂^π?

4·4 PROBLEM To what more general problem of picking members of sets does this method apply?

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