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CONNECTIONS ON HIGHER ORDER PRINCIPAL PROLONGATIONS

PETR VAŠÍK

ABSTRACT. We introduce geometric constructions of connections on higher order principal prolongations of a principal bundle. We discuss the differences between connections on non-holonomic, semiholonomic and holonomic principal prolongations, respectively.

INTRODUCTION

Let $\mathcal{M}f_m$ be the category of m-dimensional manifolds and local diffeomorphisms, $\mathcal{FM}$ be the category of fibred manifolds and fiber respecting mappings and $\mathcal{FM}_{m,n}$ be the category of fibred manifolds with m-dimensional bases and n-dimensional fibres and locally invertible fiber respecting mappings.

Consider a principal bundle $P \to M$. It is well known that the r-th principal prolongation W^rP of P has many applications in differential geometry. For example, if E is an arbitrary fiber bundle associated to P, then the r-th order jet prolongation J^rE of E is associated to W^rP. The gauge-natural bundle functor W^r plays a fundamental role also in in the theory of gauge-natural bundles. By [7], every gauge-natural bundle is a fibre bundle associated to the bundle W^rP for certain order r.

I. Kolář and G. Virsik have studied connections on the first principal prolongation W^1P, [9]. The aim of this paper is to describe constructions of connections on principal prolongations of higher order. Quite analogously to higher order jet spaces, we will distinguish between non-holonomic, semiholonomic and holonomic principal prolongations $\widetilde{W}^rP$, $\overline{W}^rP$ and W^rP, respectively. Clearly, for $r = 1$ all these prolongations coincide.

It turns out that geometric constructions of connections on higher order principal prolongations are closely related to the prolongation of general connections. That is why we also study constructions of general connections on jet prolongations of a fibred manifold $Y \to M$. In Section 2 we recall some important constructions from this

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area and introduce the iteration method for the construction of general connections on non-holonomic jet prolongations $\widetilde{J}^r Y \to M$ of a fibred manifold $Y \to M$.

Section 3 is devoted to connections on higher order principal prolongations of a principal bundle $P \to M$. We show that an important role will be played by the Ehresmann prolongation of a connection. We introduce some constructions of connections on $\widetilde{W}^r P$, $\overline{W}^r P$ and $W^r P$, respectively. We also show that connections on $\widetilde{W}^r P$ correspond to non-holonomic connections of order $(r+1)$ on P.

All manifolds and mappings are assumed to be infinitely differentiable.

1. Foundations

Let $p : Y \to M$ and $\overline{p} : \overline{Y} \to \overline{M}$ be two fibred manifolds and $s \geq r \leq q$ be three integers. We recall that two morphisms $f, g : Y \to \overline{Y}$ with the base maps $\underline{f}, \underline{g} : M \to \overline{M}$ determine the same (r,s,q)-jet $j_y^{r,s,q} f = j_y^{r,s,q} g$ at $y \in Y$, $p(y) = x$, if

$$j_y^r f = j_y^r g\,, \quad j_y^s(f \mid Y_x) = j_y^s(g \mid Y_x)\,, \quad j_x^q \underline{f} = j_x^q \underline{g}\,.$$

Further, a bundle functor G on $\mathcal{FM}$ is said to be of the order (r,s,q) if $j_y^{r,s,q} f = j_y^{r,s,q} g$ implies $Gf \mid G_y Y = Gg \mid G_y Y$. Then the integer q is called the base order, s is called the fiber order and r is called the total order of G.

It is well known that product preserving functors can be expressed in the terms of Weil algebras. The most important result from this field is that each product preserving functor F on $\mathcal{M}f$ is a Weil functor $F = T^A$ determined by the Weil algebra A. Then the iteration $T^A \circ T^B$ of two Weil functors corresponds to the tensor product $A \otimes B$ of Weil algebras and natural transformations $T^A \to T^B$ are in bijection with algebra homomorphisms $A \to B$.

Let F be a natural bundle on $\mathcal{M}f_m$. The F-vertical functor is a bundle functor V^F on $\mathcal{FM}_{m,n}$ defined by

$$V^F Y = \bigcup_{x \in M} F(Y_x)\,, \quad V^F f = \bigcup_{x \in M} F(f_x)\,,$$

where f_x is the restriction and corestriction of $f : Y \to \overline{Y}$ over $\underline{f} : M \to \overline{M}$ to the fibers Y_x and $\overline{Y}_{\underline{f}(x)}$. Clearly, if the order of F is s, then the order of V^F is $(0,s,0)$. For the tangent bundle $F = T$ we obtain the classical vertical bundle denoted by V. M. Doupovec and W. M. Mikulski have recently proved

Theorem 1.1. *Let G be a bundle functor on $\mathcal{FM}_{m,n}$. Then the following conditions are equivalent:*

(a) *The order of G is $(0,s,0)$ for some s.*
(b) *The base order of G is zero.*
(c) *G is naturally equivalent to some F-vertical functor V^F.*
(d) *There is an $\mathcal{FM}_{m,n}$-natural operator transforming connections on $Y \to M$ into connections on $GY \to M$.*

It is well known that an arbitrary bundle functor G on $\mathcal{FM}_{m,n}$ admits a natural operator transforming connections on $Y \to M$ into connections on $GY \to M$ by means of an auxiliary higher order linear connection on M, see [1], [7]. By Theorem 1.1, if the base order of G is not zero, then the use of such a higher order linear connection is unavoidable.

Denote by $J^rY \to M$ the r-th jet prolongation of a fibred manifold $p : Y \to M$. Recall that the r-th non-holonomic prolongation $\widetilde{J}^rY$ of Y is defined by iteration

$$\widetilde{J}^1Y = J^1Y\,, \qquad \widetilde{J}^rY = J^1(\widetilde{J}^{r-1}Y \to M)\,.$$

In what follows, J^rY will be called the r-th holonomic prolongation of Y. Clearly, we have an inclusion $J^rY \subset \widetilde{J}^rY$ given by $j^r_x\gamma \mapsto j^1_x(j^{r-1}\gamma)$. Further, the r-th semiholonomic prolongation $\overline{J}^rY \subset \widetilde{J}^rY$ is defined by the following induction. We set $\overline{J}^1Y = J^1Y$ and assume we have $\overline{J}^{r-1}Y \subset \widetilde{J}^{r-1}Y$ such that the restriction of the projection $\beta_{r-1} : \widetilde{J}^{r-1}Y \to \widetilde{J}^{r-2}Y$ maps $\overline{J}^{r-1}Y$ into $\overline{J}^{r-2}Y$. Then we can construct $J^1\beta_{r-1} : J^1\overline{J}^{r-1}Y \to J^1\overline{J}^{r-2}Y$ and define

$$\overline{J}^rY = \{A \in J^1\overline{J}^{r-1}Y;\ \beta_r(A) = J^1\beta_{r-1}(A) \in \overline{J}^{r-1}Y\}\,.$$

Clearly, we have $J^rY \subset \overline{J}^rY \subset \widetilde{J}^rY$. Considering the product fibred manifold $M \times N \to M$, we can define the non-holonomic and semiholonomic jet spaces by

$$\widetilde{J}^r(M,N) := \widetilde{J}^r(M \times N \to M)\,, \quad \overline{J}^r(M,N) := \overline{J}^r(M \times N \to M)\,.$$

In general, an r-th order non-holonomic connection on Y is a section $\Gamma : Y \to \widetilde{J}^rY$. Such a connection is called semiholonomic or holonomic, if it has values in $\overline{J}^rY$ or in J^rY, respectively. Given two higher order connections $\Gamma : Y \to \widetilde{J}^rY$ and $\overline{\Gamma} : Y \to \widetilde{J}^sY$, the product of Γ and $\overline{\Gamma}$ is the $(r+s)$-th order connection $\Gamma * \overline{\Gamma} : Y \to \widetilde{J}^{r+s}Y$ defined by

$$\Gamma * \overline{\Gamma} = \widetilde{J}^s\Gamma \circ \overline{\Gamma}\,.$$

If both Γ and $\overline{\Gamma}$ are of the first order, then $\Gamma * \overline{\Gamma} : Y \to \widetilde{J}^2Y$ is semiholonomic if and only if $\Gamma = \overline{\Gamma}$ and $\Gamma * \overline{\Gamma}$ is holonomic if and only if Γ is curvature-free, [5], [11].

Considering a connection $\Gamma : Y \to J^1Y$, we can define an r-th order connection $\Gamma^{(r-1)} : Y \to \widetilde{J}^rY$ by

$$\Gamma^{(1)} := \Gamma * \Gamma = J^1\Gamma \circ \Gamma\,, \qquad \Gamma^{(r-1)} := \Gamma^{(r-2)} * \Gamma = J^1\Gamma^{(r-2)} \circ \Gamma\,.$$

The connection $\Gamma^{(r-1)}$ is called the $(r-1)$-st prolongation of Γ in the sense of Ehresmann. By [5], the values of $\Gamma^{(r-1)}$ lie in the semiholonomic prolongation $\overline{J}^rY$ and $\Gamma^{(r-1)}$ is holonomic if and only if Γ is curvature free, [11].

Given a principle bundle $P \to M$ with a structure group G, one can define principal r-th order connections on P as G-invariant sections $P \to \widetilde{J}^rP$, [11]. Let $\dim M = m$. The r-th order principal prolongation W^rP of a principal bundle $P \to M$ is defined as the space of all r-jets at $(0,e) \in \mathbb{R}^m \times G$ of all local principal bundle isomorphisms $\mathbb{R}^m \times G \to P$, where $e \in G$ denotes the unit, [7]. Denoting by P^rM the r-th order frame bundle, we have the natural identification

$$W^rP = P^rM \times_M J^rP\,. \tag{1}$$

Further, $W^rP \to M$ is a principal bundle with the structure group $W^r_mG = J^r_{(0,e)}(\mathbb{R}^m \times G, \mathbb{R}^m \times G)_{(0,-)}$, which coincides with the semidirect product

$$W^r_mG = G^r_m \rtimes T^r_mG\,. \tag{2}$$

Here $G^r_m = \operatorname{inv} J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ and $T^r_mG = J^r_0(\mathbb{R}^m, G)$. If we replace holonomic jets by non-holonomic or semiholonomic ones, we obtain the non-holonomic or semiholonomic principal prolongations $\widetilde{W}^r P$ and $\overline{W}^r P$, respectively. Quite analogously to (1), (2) we have

$$\widetilde{W}^r P = \widetilde{P}^r M \times_M \widetilde{J}^r P, \qquad \overline{W}^r P = \overline{P}^r M \times_M \overline{J}^r P$$

and

$$\widetilde{W}^r_m G = \widetilde{G}^r_m \rtimes \widetilde{T}^r_m G, \qquad \overline{W}^r_m G = \overline{G}^r_m \rtimes \overline{T}^r_m G.$$

Moreover, we have a natural identification

$$\widetilde{W}^r(\widetilde{W}^s P) = \widetilde{W}^{r+s} P$$

of principal bundle structures with corresponding structure groups.

2. Prolongation of general connections

2.1. **Description of operators $P(\Gamma,\Lambda)$ and $\mathcal{J}^1(\Gamma,\Lambda)$.** We recall that a linear r-th order connection on M means a linear base preserving morphism $\Lambda : TM \to J^r TM$ satisfying $\pi^r_0 \circ \Gamma = \mathrm{id}_{TM}$, where π^r_k denotes the canonical projection of r-jets onto k-jets. Clearly, for $r = 1$ this is the classical linear connection on M. By [6], there is a bijection between the linear r-th order connections on M and the principal connections on $P^r M$.

By [7], there are two well known geometric constructions transforming a connection $\Gamma : Y \to J^1 Y$ and a classical linear connection $\Lambda : TM \to J^1 TM$ into a connection on $J^1 Y \to M$. First, denote by $\mathcal{V}\Gamma : VY \to J^1 VY$ the vertical prolongation of Γ and by $\Lambda^* : T^*M \to J^1 T^*M$ the dual connection of Λ. Since $J^1 Y \to Y$ is an affine bundle with the associated vector bundle $VY \otimes T^*M$, the section Γ determines an identification $I_\Gamma : J^1 Y \approx VY \otimes T^*M$. Then the composition

$$J^1 Y \xrightarrow{I_\Gamma} VY \otimes T^*M \xrightarrow{\mathcal{V}\Gamma\otimes\Lambda^*} J^1 VY \otimes J^1 T^*M \xrightarrow{J^1(I_\Gamma)^{-1}} J^1 J^1 Y \tag{3}$$

determines a connection $P(\Gamma,\Lambda)$ on $J^1 Y \to M$. Let $(x^i, y^p, y^p_i = \frac{\partial y^p}{\partial x^i})$ be local coordinates on $J^1 Y$ and denote by $X^i = \mathrm{d}x^i$ the induced coordinates on the tangent bundle TM. If the equations of Γ are $\mathrm{d}y^p = F^p_i(x,y)\,\mathrm{d}x^i$ and the equations of Λ are $\mathrm{d}X^i = \Lambda^i_{jk} X^j \mathrm{d}x^k$, then the connection $P(\Gamma,\Lambda)$ has equations:

$$\mathrm{d}y^p = F^p_i(x,y)\,\mathrm{d}x^i \tag{4}$$

$$\mathrm{d}y^p_i = \Big(\frac{\partial F^p_i}{\partial x^j} + \frac{\partial F^p_i}{\partial y^q} F^q_j - \frac{\partial F^p_j}{\partial y^q}(F^q_i - y^q_i) + (F^p_k - y^p_k)\Lambda^k_{ij}\Big)\,\mathrm{d}x^j . \tag{5}$$

On the other hand, consider the r-th order linear connection $\Sigma : TM \to J^r TM$, the lifting map $\gamma : Y \times_M TM \to TY$ of Γ and its r-th jet extension $J^r\gamma : J^r Y \times_M J^r TM \to J^r TY$. Denoting by $\mu_r : J^r TY \to TJ^r Y$ the flow natural transformation from [10], [7], the composition

$$J^r Y \times_M TM \xrightarrow{\mathrm{id}\times\Sigma} J^r Y \times_M J^r TM \xrightarrow{J^r\gamma} J^r TY \xrightarrow{\mu_r} TJ^r Y \tag{6}$$

is the lifting map of a connection on $J^rY \to M$, which will be denoted by $\mathcal{J}^r(\Gamma,\Sigma)$. For $r=1$ we obtain a connection $\mathcal{J}^1(\Gamma,\Lambda)$ on $J^1Y \to M$ with the coordinate expression

$$\text{(7)} \qquad dy^p = F_i^p(x,y)\,dx^i$$

$$\text{(8)} \qquad dy_i^p = \Big(\frac{\partial F_j^p}{\partial x^i} + \frac{\partial F_j^p}{\partial y^q}y_i^q + (F_k^p - y_k^p)\Lambda_{ji}^k\Big)\,dx^j\,.$$

If $\widetilde{\Lambda} : TM \to J^1TM$ is the conjugate connection of Λ, then we have the following result, [7]:

Proposition 2.1. $P(\Gamma,\Lambda) = \mathcal{J}^1(\Gamma,\widetilde{\Lambda})$ *if and only if* Γ *is curvature free.*

2.2. **The flow prolongation $\mathcal{G}(\Gamma,\Sigma)$.** The connection $\mathcal{J}^r(\Gamma,\Sigma)$ is a particular case of the following general construction. Let $G : \mathcal{FM}_{m,n} \to \mathcal{FM}$ be an arbitrary bundle functor of the base order q. Then the couple of a connection $\Gamma : Y \to J^1Y$ and a q-th order linear connection $\Sigma : TM \to J^qTM$ induces a connection $\mathcal{G}(\Gamma,\Sigma)$ on $GY \to M$ in the following way. Take a vector field X on M and denote by $\Gamma X : Y \to TY$ its Γ-lift. The flow prolongation $\mathcal{G}(\Gamma X)$ of such a vector field gives rise to a map

$$\mathcal{G}(\Gamma X) : GY \times_M J^qTM \to TGY\,,$$

which is linear in the second factor, see [8]. Then the composition $\mathcal{G}(\Gamma X)\circ(\mathrm{id}\times_M\Sigma) : GY\times_M TM \to TGY$ is the lifting map of a connection $\mathcal{G}(\Gamma,\Sigma)$ on $GY \to M$. In what follows the connection $\mathcal{G}(\Gamma,\Sigma)$ will be called the flow prolongation of Γ by means of Σ. Clearly, for $G = J^r$ we obtain the connection $\mathcal{J}^r(\Gamma,\Sigma)$ on $J^rY \to M$, which was constructed above and for $G = \widetilde{J}^r$ we get a connection $\widetilde{\mathcal{J}}^r(\Gamma,\Sigma)$ on $\widetilde{J}^rY \to M$.

2.3. **Iteration method for higher order non-holonomic prolongation.** Obviously, the r-th non-holonomic prolongation $\widetilde{J}^rY$ is defined by iteration. The same method can be used to construct connections on $\widetilde{J}^rY \to M$. Consider a natural operator A transforming a connection Γ on $Y \to M$ and a linear connection Λ on M into a connection $A(\Gamma,\Lambda)$ on $J^1Y \to M$. Write

$$\begin{aligned} A_1(\Gamma,\Lambda) &= A(\Gamma,\Lambda)\\ A_2(\Gamma,\Lambda) &= A(A_1(\Gamma,\Lambda),\Lambda)\\ &\;\;\vdots\\ A_r(\Gamma,\Lambda) &= A(A_{r-1}(\Gamma,\Lambda),\Lambda). \end{aligned}$$

Then $A_r(\Gamma,\Lambda)$ is a connection on $\widetilde{J}^rY \to M$.

Let us now consider the case $r=2$. Applying the above iteration process to the connections $P(\Gamma,\Lambda)$ and $\mathcal{J}^1(\Gamma,\Lambda)$ on $J^1Y \to M$, we obtain the following connections on $\widetilde{J}^2Y \to M$:

$$P(P(\Gamma,\Lambda),\Lambda)\,,\quad \mathcal{J}^1(\mathcal{J}^1(\Gamma,\Lambda),\Lambda)\,,\quad P(\mathcal{J}^1(\Gamma,\Lambda),\Lambda)\quad\text{and}\quad \mathcal{J}^1(P(\Gamma,\Lambda),\Lambda)\,.$$

For example, to obtain the connection $P(P(\Gamma,\Lambda),\Lambda)$, the composition (3) should be replaced with

$$(9)\quad \widetilde{J}^2Y \xrightarrow{I_{P(\Gamma,\Lambda)}} VJ^1Y\otimes T^*M \xrightarrow{\mathcal{V}P(\Gamma,\Lambda)\otimes\Lambda^*} J^1VJ^1Y\otimes J^1T^*M$$
$$\approx J^1(VJ^1Y\otimes T^*M) \xrightarrow{J^1(I_{P(\Gamma,\Lambda)})^{-1}} J^1\widetilde{J}^2Y\,,$$

where $I_{P(\Gamma,\Lambda)}$ is the identification of the affine bundle $\widetilde{J}^2Y\to J^1Y$ with the associated vector bundle $VJ^1Y\otimes T^*M$. Quite analogously to (6), the lifting map of $\mathcal{J}^1(\mathcal{J}^1(\Gamma,\Lambda),\Lambda)$ is of the form

$$(10)\qquad \widetilde{J}^2Y\times_M TM \xrightarrow{\mathrm{id}\times\Lambda} \widetilde{J}^2Y\times_M J^1TM \xrightarrow{J^1\gamma_{\mathcal{J}^1(\Gamma,\Lambda)}} J^1TJ^1Y \xrightarrow{\mu} T\widetilde{J}^2Y\,,$$

where $\gamma_{\mathcal{J}^1(\Gamma,\Lambda)}$ is the lifting map of the connection $\mathcal{J}^1(\Gamma,\Lambda)$. Quite analogously we obtain the remaining mixed operators.

The above iteration process is not the only way to construct a connection on $\widetilde{J}^2Y\to M$. Indeed, taking a second order linear connection $\overline{\Lambda}: TM\to J^2TM$, we have a connection $\widetilde{\mathcal{J}}^2(\Gamma,\overline{\Lambda})$ on $\widetilde{J}^2Y\to M$. In the rest of this section we find the coordinate expression of the connection $\mathcal{J}^2(\Gamma,\overline{\Lambda})$ on the second holonomic prolongation $J^2Y\to M$. Denoting by $X^i=\mathrm{d}x^i$, $X^i_j=\frac{\partial X^i}{\partial x^j}$, $X^i_{jk}=\frac{\partial^2X^i}{\partial x^j\partial x^k}$ the induced coordinates on J^2TM, the equations of $\overline{\Lambda}$ are

$$\mathrm{d}X^i=\Lambda^i_{jk}X^j\mathrm{d}x^k\,,\quad \mathrm{d}X^i_j=\Lambda^i_{jkl}X^k\mathrm{d}x^l\,.$$

Putting $r=2$ and $\Sigma=\overline{\Lambda}$ into (6), we obtain the lifting map of $\mathcal{J}^2(\Gamma,\overline{\Lambda})$. To find equations of the flow natural transformation $\mu_2: J^2TY\to TJ^2Y$ from [7],[10], it is useful to use another geometric construction of this map, which was recently introduced by I. Kolář, [3]. Let $(x^i,y^p,X^i=\mathrm{d}x^i,Y^p=\mathrm{d}y^p,y^p_i,y^p_{ij},X^i_j,X^i_{jk},Y^p_i,Y^p_{ij})$ be the induced coordinates on J^2TY and denote by

$$\left(x^i,y^p,y^p_i=\frac{\partial y^p}{\partial x^i},y^p_{ij}=\frac{\partial y^p}{\partial x^i\partial x^j},X^i=\mathrm{d}x^i,Y^p=\mathrm{d}y^p,Y^p_i=\mathrm{d}y^p_i,Y^p_{ij}=\mathrm{d}y^p_{ij}\right)$$

the coordinates on TJ^2Y. Using [3], we find directly

$$(11)\quad \mu_2\left(x^i,y^p,X^i,Y^p,y^p_i,y^p_{ij},X^i_j,X^i_{jk},Y^p_i,Y^p_{ij}\right)$$
$$=\left(x^i,y^p,y^p_i,y^p_{ij},X^i,Y^p,Y^p_i-X^j_iy^p_j,Y^p_{ij}-X^k_{ij}y^p_k-y^p_{jk}X^k_i-y^p_{ik}X^k_j\right)\,.$$

Then the equations of $\mathcal{J}^2(\Gamma,\overline{\Lambda})$ are (7), (8) and

$$\begin{aligned}\mathrm{d}y^p_{ij}=\Big[&\frac{\partial^2F^p_k}{\partial x^i\partial x^j}+\frac{\partial^2F^p_k}{\partial x^i\partial y^q}y^q_j+\frac{\partial^2F^p_k}{\partial y^q\partial x^j}y^q_i+\frac{\partial^2F^p_k}{\partial y^q\partial y^r}y^q_iy^r_j\\&+\frac{\partial F^p_k}{\partial y^q}y^q_{ij}+\left(\frac{\partial F^p_l}{\partial x^i}+\frac{\partial F^p_l}{\partial y^q}y^q_i-y^p_{il}\right)\Lambda^l_{jk}\\&+\left(\frac{\partial F^p_l}{\partial x^j}+\frac{\partial F^p_l}{\partial y^q}y^q_j-y^p_{jl}\right)\Lambda^l_{ik}+\left(F^p_l-y^p_l\right)\Lambda^l_{ijk}\Big]\,\mathrm{d}x^k\end{aligned}$$

3. Prolongation of principal connections

In what follows all connections are supposed to be principal.

3.1. **Connections on the first principal prolongation W^1P.** Let $\Gamma : P \to J^1P$ be a connection on $P \to M$ and $\Lambda : P^1M \to J^1P^1M$ be a linear connection. By [9], Γ and Λ induce the connection $p(\Gamma, \Lambda)$ on $W^1P \to M$, which is defined in the following way. First, we define a subspace

$$R(\Gamma) := P^1M \times_M \Gamma(P) \subset P^1M \times_M J^1P = W^1P\,.$$

This is a reduction of the principal bundle $W^1P \to M$ to the subgroup $G^1_m \rtimes i(G) \subset W^1_mG$, where i is an injection $G \hookrightarrow T^1_mG$. Therefore $R(\Gamma)$ can be identified with $P^1M \times_M P$ and the product connection $\Lambda \times \Gamma$ on $P^1M \times_M P$ can be identified with a connection in $R(\Gamma)$. Finally, this connection can be uniquely extended into a connection $p(\Gamma, \Lambda)$ in W^1P.

On the other hand, the couple (Γ, Λ) induces the connection $\mathcal{W}^1(\Gamma, \Lambda)$ on $W^1P \to M$ by means of the flow prolongation.

3.2. **Construction of connections on non-holonomic principal prolongations $\widetilde{W}^rP$.**

I. First, given two connections $\Gamma : P \to \widetilde{J}^{r+1}P$ and $\Lambda : \widetilde{P}^rM \to J^1\widetilde{P}^rM$, we can construct a connection

$$\varkappa_{r+1}(\Gamma, \Lambda) : \widetilde{W}^rP \to J^1\widetilde{W}^rP\,, \tag{12}$$

for the groupoid form see also [4]. Let us denote by $\Gamma_1 := \pi^{r+1}_r\Gamma : P \to \widetilde{J}^rP$ the underlying connection of order r, where $\pi^{r+1}_r : \widetilde{J}^{r+1}P \to \widetilde{J}^rP$ is the projection. Write

$$R(\Gamma_1) := \widetilde{P}^rM \times_M \Gamma_1(P) \subset \widetilde{P}^rM \times_M \widetilde{J}^rP = \widetilde{W}^rP\,.$$

One finds easily, that $R(\Gamma_1)$ is a reduction of the principal bundle $\widetilde{W}^rP$ to the subgroup $\widetilde{G}^r_m \times i(G) \subset \widetilde{W}^r_mG$, where $i : G \to \widetilde{T}^r_mG$ is an injection. As every $\Gamma(v) \in \widetilde{J}^{r+1}P$ can be considered as an element of $J^1\widetilde{J}^rP$ over $\Gamma_1(v)$, we obtain in such a way a map

$$\varphi : R(\Gamma_1) \to J^1\widetilde{P}^rM \times_M J^1\widetilde{J}^rP = J^1(\widetilde{P}^rM \times_M \widetilde{J}^rP) = J^1\widetilde{W}^rP$$

defined by

$$\varphi(u, \Gamma_1(v)) = (\Lambda(u), \Gamma(v)) \quad \text{for} \quad (u, v) \in \widetilde{P}^rM \times_M P\,.$$

Then φ is right invariant and thus it can be extended into the connection on $\widetilde{W}^rP$, which will be denoted by $\varkappa_{r+1}(\Gamma, \Lambda)$.

II. Now let $\Gamma : P \to J^1P$ be a connection on $P \to M$ and $\Lambda : \widetilde{P}^rM \to J^1\widetilde{P}^rM$ be a connection on $\widetilde{P}^rM$. Using (12) and the Ehresmann prolongation of Γ, we have the connection

$$\widetilde{p}_r(\Gamma, \Lambda) := \varkappa_{r+1}(\Gamma^{(r)}, \Lambda) \tag{13}$$

on $\widetilde{W}^rP \to M$.

III. Suppose we have a connection $\Gamma : P \to J^1P$ and an r-th order connection $\Lambda : TM \to J^rTM$. Then the flow prolongation of Γ with respect to Λ induces the connection $\widetilde{\mathcal{W}}(\Gamma, \Lambda)$ on $\widetilde{W}^rP \to M$.

IV. Further, quite analogously to the non-holonomic jet prolongation $\widetilde{J}^rY \to M$, we can construct connections on $\widetilde{W}^rP \to M$ by means of iteration. Indeed, we have $\widetilde{W}^1P = W^1P$ and $\widetilde{W}^rP = W^1(\widetilde{W}^{r-1}P)$. Starting from connections $\Gamma : P \to J^1P$,

$\Lambda : P^1M \to J^1P^1M$ and using the basic operators $p(\Gamma,\Lambda)$ and $\mathcal{W}^1(\Gamma,\Lambda)$ on $\widetilde{W}^1P$, we have the following connections on $\widetilde{W}^2P$:

$$p(p(\Gamma,\Lambda),\Lambda)\,,\quad p(\mathcal{W}^1(\Gamma,\Lambda),\Lambda)\,,\quad \mathcal{W}^1(p(\Gamma,\Lambda),\Lambda)\quad\text{and}\quad \mathcal{W}^1(\mathcal{W}^1(\Gamma,\Lambda),\Lambda)\,.$$

Obviously, such an iteration process can be applied for an arbitrary order r.

3.3. Construction of connections on semiholonomic principal prolongations $\overline{W}^r P$.

I. Given two principal connections $\Gamma : P \to J^1P$ and $\Lambda : \overline{P}^r M \to J^1\overline{P}^r M$ we construct a connection

$$\overline{p}_r(\Gamma,\Lambda) : \overline{W}^r P \to J^1\overline{W}^r P\,. \tag{14}$$

Denote by $\Gamma^{(r-1)} : P \to \widetilde{J}^r P$ the Ehresmann prolongation of Γ. By Section 1, this connection has values in $\overline{J}^r P$. Further, $\Gamma^{(r)} : P \to \widetilde{J}^{r+1}P$ is of the form

$$\Gamma^{(r)} = \Gamma^{(r-1)} * \Gamma = J^1\Gamma^{(r-1)} \circ \Gamma : P \to \overline{J}^{r+1}P\,.$$

This yields that for $v \in P$, $\Gamma^{(r)}(v) \in \overline{J}^{r+1}P$ is the element of $J^1\overline{J}^r P$ over $\Gamma^{(r-1)}(v) \in \overline{J}^r P$. Write

$$R(\Gamma^{(r-1)}) := \overline{P}^r M \times_M \Gamma^{(r-1)}(P) \subset \overline{P}^r M \times_M \overline{J}^r P = \overline{W}^r P\,.$$

Quite analogously to the non-holonomic principal prolongation we prove that $R(\Gamma^{(r-1)})$ is a reduction of $\overline{W}^r P$ to the subgroup $\overline{G}^r_m \times i(G) \subset \overline{W}^r_m G$, where i is the injection of G into $\overline{T}^r_m G$. Then we can define a map

$$\varphi : R(\Gamma^{(r-1)}) \to J^1\overline{P}^r M \times_M J^1\overline{J}^r P = J^1\overline{W}^r P$$

by

$$\varphi(u, \Gamma^{(r-1)}(v)) = (\Lambda(u), \Gamma^{(r)}(v))\,.$$

This defines a connection $\overline{p}_r(\Gamma,\Lambda)$ on $\overline{W}^r P \to M$.
II. Let $\Gamma : P \to J^1P$ be a connection and $\Lambda : TM \to J^rTM$ be an r-th order linear connection on M. Using the flow prolongation of Γ with respect to Λ, we have the connection $\overline{\mathcal{W}}^r(\Gamma,\Lambda)$ on $\overline{W}^r P \to M$.

3.4. Construction of connections on holonomic principal prolongations W^rP.

I. Let $\Gamma : P \to J^1P$ and $\Lambda : P^rM \to J^1P^rM$ be principal connections and suppose that Γ is curvature-free. By [11], the Ehresmann prolongation $\Gamma^{(r-1)} : P \to \widetilde{J}^r P$ is holonomic. Quite analogously to the connection (14) from 3.3 we can construct the connection $p_r(\Gamma,\Lambda) : W^rP \to J^1W^rP$.
II. The flow prolongation of $\Gamma : P \to J^1P$ with respect to an r-th order linear connection $\Lambda : TM \to J^rTM$ defines the connection $\mathcal{W}^r(\Gamma,\Lambda)$ on $W^rP \to M$.

3.5. **Properties and identifications.** First, given a couple of connections (Λ,Γ), where $\Lambda : \widetilde{P}^rM \to J^1\widetilde{P}^rM$ and $\Gamma : P \to \widetilde{J}^{r+1}P$, we have shown the construction of a connection $\varkappa_{r+1}(\Gamma,\Lambda)$ on $\widetilde{W}^rP$. On the other hand, let $\Omega : \widetilde{W}^rP \to J^1\widetilde{W}^rP$ be a connection. We are going to find a couple (Λ,Γ), which corresponds to Ω.

Write $p_1 : \widetilde{W}^rP \to \widetilde{P}^rM$ and $p_2 : \widetilde{W}^rP \to P$ for the projections and set

$$\Lambda := p_1\Omega,\ \Gamma := p_2\Omega\,.$$

Then Λ is the connection on $\widetilde{P}^rM$ and Γ is the connection on P. Furthermore, consider the Ehresmann prolongation $\Gamma^{(r-1)} : P \to \widetilde{J}^rP$. For $(u,v) \in \widetilde{P}^rM \times_M P$ we have

$$\Omega(u,\Gamma^{(r-1)}(v)) \in J^1\widetilde{W}^rP = J^1\widetilde{P}^rM \times_M J^1\widetilde{J}^rP\,.$$

The second projection pr_2 yields

$$pr_2\Omega(u,\Gamma^{(r-1)}(v)) \in J^1\widetilde{J}^rP = \widetilde{J}^{r+1}P\,. \tag{15}$$

One verifies directly that this is independent of u. Hence (15) determines a map

$$\Omega^* : P \to \widetilde{J}^{r+1}P\,.$$

Obviously, this map is G-invariant, so it is an $(r+1)$-st order connection on P. We have proved

Proposition 3.1. *The mapping $\Omega \mapsto (p_1\Omega,\Omega^*)$ determines a bijection between the connections on $\widetilde{W}^rP$ and the couples consisting of the connection on $\widetilde{P}^rM$ and the non-holonomic connection of order $(r+1)$ on P.*

We have

Corollary 3.1. *Let $\widetilde{p}_r(\Gamma,\Lambda)$ be the connection (13) on $\widetilde{W}^rP$. Then we have $\big(\widetilde{p}_r(\Gamma,\Lambda)\big)^* = \Gamma^{(r)}$. In particular, for $r=1$ we obtain $\big(p(\Gamma,\Lambda)\big)^* = \Gamma * \Gamma$.*

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