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ON THE CONFORMAL THEORY OF ICHIJYŌ MANIFOLDS

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ABSTRACT. First, we introduce the concept of *Ichijyō manifolds*: these are Finsler manifolds endowed with an Ichijyō connection. (An Ichijyō connection is a special type of "horizontally basic" or "linear" Finsler connections, first described axiomatically in [8].) Second, we show that in the context of Ichijyō manifolds a quite natural and efficient concept of conformal equivalence is available, under which the curvature becomes invariant. Generalized Berwald manifolds and, in particular, Wagner manifolds can be interpreted as special Ichijyō manifolds. Finally, to demonstrate the power of the theory developed in [8], [9] and this paper, we present new intrinsic proofs for *Hashiguchi-Ichijyō theorems* concerning the conformal change of Finsler (and special Finsler) metrics.

1. Introduction. In our earlier works [8], [9] we found that the so-called Ichijyō connections play a distinguished role in the calculus of a large class of special Finsler manifolds. In this note we present a "reasonable" concept of the conformal equivalence of Finsler manifolds endowed with an Ichijyō connection. These Finsler manifolds will be mentioned as *Ichijyō manifolds* in the sequel. We show that the curvature tensor of an Ichijyō manifold remains invariant under conformal changes. This nice behaviour seems to be by itself a convincing argument for the new concept. However, the most telling argument is the surprisingly simple and transparent description of the change of a generalized Berwald structure presented in the last section. Here we obtain as an easy corollary that the class of generalized Berwald manifolds (with common carrier manifold) is closed under the conformal changes of the structure. This implies immediately a celebrated result of M. Hashiguchi and Y. Ichijyō (Corollary 3). (For another coordinate-free deduction of this nice theorem we refer to [12].)

In Sections 2-4 the most necessary preparations are summarized. (A detailed background and supplementary material can be found in our works [8],[9]; see also

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[10] and [11].) Section 5 is devoted to a concise description of the conformal equivalence of Ichijyō manifolds. Finally, in Section 6 some characteristic applications to special Finsler manifolds are presented.

2. Generalities.

(a) In what follows M will denote a 2nd countable, connected, smooth manifold of dimension $n \geqq 2$. $C^\infty(M)$ is the ring of smooth functions on M, $\mathfrak{X}(M)$ is the $C^\infty(M)$-module of the (smooth) vector fields on M, $\mathfrak{X}^*(M)$ is the dual of $\mathfrak{X}(M)$. $\mathcal{T}_1^1(M)$ denotes the module of $(1,1)$ tensor fields (or vector 1-forms) on M.

(b) (TM, π, M) is the tangent bundle of M, $\overset{\circ}{T}M := TM \setminus \{0\}$. The *vertical lift* of a function $f \in C^\infty(M)$ into TM is $f^{\mathrm{v}} := f \circ \pi$, the *complete lift* of f is the function f^c defined by $v \in TM \mapsto f^c(v) := (df)_{\pi(v)}(v)$. The vertical lift X^{v} and the complete lift X^c of a vector field $X \in \mathfrak{X}(M)$ are determined by $X^{\mathrm{v}} f^c := (Xf)^{\mathrm{v}}$ and $X^c f^c = (Xf)^c$ ($f \in C^\infty(M)$) respectively.

The *vertical endomorphism* $J \in \mathcal{T}_1^1(TM)$ can be given by the rules $J(X^{\mathrm{v}}) := 0$, $J(X^c) := X^{\mathrm{v}}$ ($X \in \mathfrak{X}(M)$). A vector field $Z \in \mathfrak{X}(TM)$ is *vertical* if $JZ = 0$; the submodule of vertical vector fields is denoted by $\mathfrak{X}^{\mathrm{v}}(TM)$. There exists a distinguished vertical vector field on TM, the *Liouville vector field* C, determined uniquely by $Cf^c = f^c$ ($f \in C^\infty(M)$).

(c) A vector 1-form $h \in \mathcal{T}_1^1(TM)$, continuous on TM, smooth on $\overset{\circ}{T}M$, is said to be a *horizontal endomorphism* on M if $\operatorname{Ker} h = \mathfrak{X}^{\mathrm{v}}(TM)$ and $h^2 = h$. $\mathfrak{X}^h(TM) := \operatorname{Im} h$ is the module of *horizontal vector fields*; then $\mathfrak{X}(TM) = \mathfrak{X}^{\mathrm{v}}(TM) \oplus \mathfrak{X}^h(TM)$. $X^h := hX^c$ is the *horizontal lift* of $X \in \mathfrak{X}(M)$; the $(1,1)$ tensor $F \in \mathcal{T}_1^1(TM)$ determined by $FX^{\mathrm{v}} = X^h$, $FX^h = -X^{\mathrm{v}}$ is the *almost complex structure* induced by h. The vector forms

$$t := [J,h], \quad H := [h,C], \quad \Omega := -\frac{1}{2}[h,h] \tag{1a–c}$$

($[\,,\,]$ is the Frölicher–Nijenhuis bracket) are the *torsion vector 2-form*, the *tension 1-form* and the *curvature vector 2-form* of h, respectively. A horizontal endomorphism is said to be *homogeneous* if its tension vanishes.

(d) Suppose that $S : TM \to TTM$ is a C^1 vector field which is smooth on $\overset{\circ}{T}M$. S is called a *semispray* on M if it satisfies the condition $JS = C$. A semispray is said to be a *spray* if it is homogeneous of degree 2 in the sense that $[C,S] = S$. If S is semispray on the manifold M, then for any vector field X on M we have

$$J[X^{\mathrm{v}}, S] = X^{\mathrm{v}}\,; \tag{2}$$

this useful observation is an immediate consequence of Prop. I.7. of [1].

Lemma 1. *Let h and $\tilde{h}$ be homogeneous horizontal endomorphisms on the manifold M. Assume that*

$$[X^h, Y^{\mathrm{v}}] = [X^{\tilde{h}}, Y^{\mathrm{v}}] \tag{3}$$

for any vector fields X, Y on M. Then $h = \tilde{h}$.

Proof. The difference $X^h - X^{\tilde{h}}$ is clearly vertical for all vector fields $X \in \mathfrak{X}(M)$. By condition (3) this vertical vector field commutes with any vertically lifted vector field. Hence, in view of Lemma 1.16 of [11], $X^h - X^{\tilde{h}}$ is also a vertical lift. This implies immediately that $[J, X^h - X^{\tilde{h}}] = 0$, so for any semispray S we have

$$\begin{aligned} 0 &= [J, X^h - X^{\tilde{h}}]S = [JS, X^h - X^{\tilde{h}}] - J[S, X^h - X^{\tilde{h}}] \\ &\overset{(2)}{=} X^h - X^{\tilde{h}} + [C, X^h] - [C, X^{\tilde{h}}] = X^h - X^{\tilde{h}}, \end{aligned}$$

since h and $\tilde{h}$ are homogeneous. □

(e) Recall that any linear connection ∇ on M gives rise naturally to a horizontal endomorphism h_∇; the data corresponding to (1a–c) will be denoted by t_∇, H_∇ and Ω_∇, respectively. From ∇ a spray S_∇ can also be canonically derived.

3. Finsler manifolds. (a) A function $E : TM \to \mathbb{R}$ is said to be an *energy function* on TM, if it is smooth over $\overset{\circ}{T}M$; for any vector $v \in TM$, $E(v) \geqq 0$; $E(v) = 0 \iff v = 0$; $CE = 2E$ (i.e., E is homogeneous of degree 2), and the 2-form $\omega := d(dE \circ J)$ is symplectic. A pair (M, E) is a *Finsler manifold*, if E is an energy function on TM. The mapping $\overline{g}$ given by $\overline{g}(JX, JY) := \omega(JX, Y)$ $(X, Y \in \mathfrak{X}(TM)$ is a well-defined, nondegenerate $C^\infty(TM)$-bilinear form, called the *vertical metric* of the Finsler manifold. The covariant *first Cartan tensor* $\mathcal{C}_\flat$ of (M, E) can be given by the requirements

$$\begin{cases} 2\mathcal{C}_\flat(X^c, Y^c, Z^c) := X^v[Y^v(Z^v E)]; & X, Y, Z \in \mathfrak{X}(M); \\ \mathcal{C}_\flat(\cdot, \cdot, \cdot) := 0, & \text{if one of the arguments is vertical.} \end{cases}$$

Then the vector-valued first Cartan tensor $\mathcal{C}$ is determined by the conditions $J \circ \mathcal{C} = 0$ and $\overline{g}(\mathcal{C}(X^c, Y^c), Z^v) = \mathcal{C}_\flat(X^c, Y^c, Z^c)$.

(b) Two Finsler manifolds (M, E) and $(M, \overline{E})$ are said to be *conformally equivalent*, if their energy functions are related by $\overline{E} = \varphi E$, where $\varphi \in C^\infty(TM)$ is a positive function. It is well-known that in this case *φ is a vertical lift* (Knebelman's observation), so it can be written in the form $\exp \circ \sigma^v$, $\sigma \in C^\infty(M)$.

Lemma 2. *The vector-valued first Cartan tensor is conformally invariant.*

A standard *proof* of this fact can be found in [2]; for a coordinate-free reasoning we refer to [10]. □

4. Ichijyō connections ([8]). Let (M, E) be a Finsler manifold. Suppose that ∇ is a linear connection on M and consider the horizontal endomorphism h_∇. Let g be the prolongation of the vertical metric to $\mathfrak{X}(TM)$ along h_∇.

(a) There exists a unique linear connection D on TM satisfying the following conditions:

(I) $Dh_\nabla = 0$, $DF_\nabla = 0$ (i.e., the pair (D, h_∇) is a Finsler connection).

(II) D is *v-metrical*, i.e., $D_{v_\nabla} g = 0$ $\quad (v_\nabla := 1 - h_\nabla)$.

(III) If $\mathbb{T}$ is the torsion of D, then the so-called *$(v)v$-torsion* $\mathcal{S}^1$ given by $\mathcal{S}^1(X,Y) := v_\nabla \mathbb{T}(JX, JY)$ $(X, Y \in \mathfrak{X}(TM))$ *vanishes.*

(IV) If $\widetilde{D}_X Y := D_{h_\nabla X} Y + J[v_\nabla X, F_\nabla Y] + h_\nabla[v_\nabla X, Y]$ $(X, Y \in \mathfrak{X}(TM))$, then the so-called *mixed curvature* (see below) *of $\widetilde{D}$ vanishes.*

(V) For any vector field $X \in \mathfrak{X}(TM)$ we have $D_{h_\nabla X} \mathcal{C} = 0$.

Then the pair (D, h_∇) is called the *Ichijyō connection* induced by ∇. The triplet (M, E, ∇) is said to be an *Ichijyō manifold* if (M, E) is a Finsler manifold endowed with the Ichijyō connection (D, h_∇).

(b) From axioms **(I)**–**(V)** the following rules of calculation for the covariant derivatives with respect to D can be deduced: for any vector fields $X, Y \in \mathfrak{X}(TM)$,

$$\boxed{\begin{array}{ll} D_{JX} JY = J[JX, Y] + \mathcal{C}(X,Y), & D_{h_\nabla X} JY = v_\nabla [h_\nabla X, JY], \\ D_{JX} h_\nabla Y = h_\nabla [JX, Y] + F_\nabla \mathcal{C}(X,Y), & D_{h_\nabla X} h_\nabla Y = h_\nabla F_\nabla [h_\nabla X, JY] \end{array}}$$

(c) Suppose that (M, E, ∇) is an Ichijyō manifold. Let us denote by $\mathbb{K}$ and $\mathbb{T}$ the curvature and the torsion of D, respectively. Then $\mathbb{K}$ can be decomposed into three *partial curvatures* and $\mathbb{T}$ can be decomposed into five *partial torsions* as follows:

Partial curvature	$(X, Y, Z \in \mathfrak{X}(TM))$
horizontal	$\mathcal{R}(X,Y)Z := \mathbb{K}(h_\nabla X, h_\nabla Y)JZ$ $= [J, \Omega_\nabla(X,Y)]h_\nabla Z + \mathcal{C}(F\Omega_\nabla(X,Y), Z)$
mixed	$\mathcal{P}(X,Y)Z := \mathbb{K}(h_\nabla X, JY)JZ = (D_{h_\nabla X}\mathcal{C})(h_\nabla Y, h_\nabla Z)$
vertical	$\mathcal{Q}(X,Y)Z := \mathbb{K}(JX, JY)JZ$ $= \mathcal{C}(F\mathcal{C}(X,Z), Y) - \mathcal{C}(X, F\mathcal{C}(Y,Z))$

Partial torsion	$(X, Y \in \mathfrak{X}(M))$
h-horizontal	$\mathcal{A}(X^{h_\nabla}, Y^{h_\nabla}) := h_\nabla \mathbb{T}(X^{h_\nabla}, Y^{h_\nabla}) = (\mathbb{T}_\nabla(X,Y))^{h_\nabla}$
h-mixed	$\mathcal{B}(X^{h_\nabla}, Y^{h_\nabla}) := h_\nabla \mathbb{T}(X^{h_\nabla}, Y^{\mathrm{v}}) = -F_\nabla \mathcal{C}(X^{h_\nabla}, Y^{h_\nabla})$
v-horizontal	$\mathcal{R}^1(X^{h_\nabla}, Y^{h_\nabla}) := v_\nabla \mathbb{T}(X^{h_\nabla}, Y^{h_\nabla}) = \Omega_\nabla(X^{h_\nabla}, Y^{h_\nabla})$
v-mixed	$\mathcal{P}^1(X^{h_\nabla}, Y^{\mathrm{v}}) := v_\nabla \mathbb{T}(X^{h_\nabla}, Y^{\mathrm{v}}) = 0$
v-vertical	$\mathcal{S}^1(X^{\mathrm{v}}, Y^{\mathrm{v}}) := v_\nabla \mathbb{T}(X^{\mathrm{v}}, Y^{\mathrm{v}}) = 0$

($F \in \mathcal{T}_1^1(TM)$ is an arbitrary almost complex structure).

Remark. The Finsler connection described by axioms **(I)**–**(V)** in [8], first appeared in Y. Ichijyō's excellent papers [5], [6], given explicitly by means of classical tensor calculus.

5. Conformal equivalence of Ichijyō manifolds. Two Ichijyō manifolds (M, E, ∇) and $(M, \overline{E}, \overline{\nabla})$ are said to be *conformally equivalent* if

(i) (M, E) and $(M, \overline{E})$ are conformally equivalent Finsler manifolds, i. e. $\overline{E} = (\exp \circ \sigma^{\mathrm{v}})E$, $\sigma \in C^\infty(M)$;

(ii) $\overline{\nabla} = \nabla + \frac{1}{2} d\sigma \otimes \mathrm{id}$.

Proposition 1. *We have the following relations:*

$$h_{\overline{\nabla}} = h_\nabla - \frac{1}{2} d\sigma^{\mathrm{v}} \otimes C\,; \tag{4}$$

$$\overline{D}_{JX} JY = D_{JX} JY\,, \tag{5}$$

$$\overline{D}_{h_{\overline{\nabla}} X} JY = D_{h_\nabla X} JY - \frac{1}{2} d\sigma^{\mathrm{v}}(X)[C, JY] \;\; (X, Y \in \mathfrak{X}(TM))\,; \tag{6}$$

$$t_{\overline{\nabla}} = t_\nabla + \frac{1}{2}(d\sigma^c \circ J) \wedge J\,; \tag{7}$$

$$\mathring{t}_{\overline{\nabla}} = \mathring{t}_\nabla + \frac{1}{2}\left(\sigma^c J - (d\sigma^c \circ J) \otimes C\right) \tag{8}$$

($\mathring{t} = i_S t$, $S \in \mathfrak{X}(TM)$ *is an arbitrary semispray*);

$$H_{\overline{\nabla}} = H_\nabla = 0, \quad \Omega_{\overline{\nabla}} = \Omega_\nabla\,; \tag{9a-b}$$

$$S_{\overline{\nabla}} = S_\nabla - \frac{1}{2}\sigma^c C, \text{ and so } S_{\overline{\nabla}} \text{ and } S_\nabla \text{ are projectively equivalent.} \tag{10}$$

Proof. (a) It can be seen immediately that $h_\nabla - \frac{1}{2} d\sigma^{\mathrm{v}} \otimes C$ is indeed a horizontal endomorphism, smooth on the whole tangent manifold TM. Since

$$\begin{aligned}[h_\nabla - \frac{1}{2} d\sigma^{\mathrm{v}} \otimes C, C] &= [h_\nabla, C] - \frac{1}{2}[d\sigma^{\mathrm{v}} \otimes C, C] = \frac{1}{2}[C, d\sigma^{\mathrm{v}} \otimes C] \\ &\overset{(21)\text{ of }[9]}{=} \frac{1}{2}(\mathcal{L}_C d\sigma^{\mathrm{v}} \otimes C + [C, C] \otimes d\sigma^{\mathrm{v}}) = 0\,,\end{aligned}$$

$h_\nabla - \frac{1}{2} d\sigma^{\mathrm{v}} \otimes C$ is homogeneous. Now we check that it coincides with the horizontal endomorphism $h_{\overline{\nabla}}$. For any vector fields X, Y on M we have on the one hand

$$[X^{h_{\overline{\nabla}}}, Y^{\mathrm{v}}] = (\overline{\nabla}_X Y)^{\mathrm{v}} = (\nabla_X Y)^{\mathrm{v}} + \frac{1}{2}(d\sigma(X)Y)^{\mathrm{v}} = [X^{h_\nabla}, Y^{\mathrm{v}}] + \frac{1}{2}(X\sigma)^{\mathrm{v}} Y^{\mathrm{v}}\,;$$

on the other hand

$$\begin{aligned}[X^{h_\nabla} - \frac{1}{2} d\sigma^{\mathrm{v}}(X^c)C, Y^{\mathrm{v}}] &= [X^{h_\nabla}, Y^{\mathrm{v}}] - \frac{1}{2}[(X\sigma)^{\mathrm{v}} C, Y^{\mathrm{v}}] \\ &= [X^{h_\nabla}, Y^{\mathrm{v}}] - \frac{1}{2}(X\sigma)^{\mathrm{v}}[C, Y^{\mathrm{v}}] + \frac{1}{2} Y^{\mathrm{v}}(X\sigma)^{\mathrm{v}} C = [X^{h_\nabla}, Y^{\mathrm{v}}] + \frac{1}{2}(X\sigma)^{\mathrm{v}} Y^{\mathrm{v}}\,,\end{aligned}$$

therefore $[X^{h_{\overline{\nabla}}}, Y^{\mathrm{v}}] = [X^{h_\nabla} - \frac{1}{2} d\sigma^{\mathrm{v}}(X^c)C, Y^{\mathrm{v}}]$. This implies by Lemma 1 the desired relation (4).

(b) Let X and Y be arbitrary vector fields on M. Taking into account the rules for calculation described in 4/(b), we obtain:

$$\overline{D}_{X^{\mathrm{v}}} Y^{\mathrm{v}} = \overline{\mathcal{C}}(X^c, Y^c) \overset{\text{Lemma 2}}{=} \mathcal{C}(X^c, Y^c) = D_{X^{\mathrm{v}}} Y^{\mathrm{v}}$$

whence (5) ;

$$\overline{D}_{X^{h_{\overline{\nabla}}}} Y^{\mathrm{v}} = (\overline{\nabla}_X Y)^{\mathrm{v}} = (\nabla_X Y)^{\mathrm{v}} + \frac{1}{2}(d\sigma(X)Y)^{\mathrm{v}} = D_{X^{h_\nabla}} Y^{\mathrm{v}} + \frac{1}{2} d\sigma^{\mathrm{v}}(X^c) Y^{\mathrm{v}} \ ,$$

and so for any function $f \in C^\infty(TM)$,

$$\begin{aligned} \overline{D}_{X^{h_{\overline{\nabla}}}} fY^{\mathrm{v}} - D_{X^{h_\nabla}} fY^{\mathrm{v}} &= ((X^{h_{\overline{\nabla}}} - X^{h_\nabla})f)Y^{\mathrm{v}} + f(\overline{D}_{X^{h_{\overline{\nabla}}}} Y^{\mathrm{v}} - D_{X^{h_\nabla}} Y^{\mathrm{v}}) \\ &= ((X^{h_{\overline{\nabla}}} - X^{h_\nabla})f)Y^{\mathrm{v}} + \frac{1}{2} f d\sigma^{\mathrm{v}}(X^c) Y^{\mathrm{v}} \overset{(4)}{=} -\frac{1}{2}(d\sigma^{\mathrm{v}}(X^c)C)fY^{\mathrm{v}} + \frac{1}{2} d\sigma^{\mathrm{v}}(X^c) fY^{\mathrm{v}} \\ &= -\frac{1}{2} d\sigma^{\mathrm{v}}(X^c)((Cf)Y^{\mathrm{v}} - fY^{\mathrm{v}}) = -\frac{1}{2} d\sigma^{\mathrm{v}}(X^c)[C, fY^{\mathrm{v}}] \ , \end{aligned}$$

whence (6).

(c)

$$\begin{aligned} t_{\overline{\nabla}} &\overset{(1a)}{:=} [J, h_{\overline{\nabla}}] \overset{(4)}{=} [J, h_\nabla - \frac{1}{2} d\sigma^{\mathrm{v}} \otimes C] = t_\nabla - \frac{1}{2}[J, d_J\sigma^c \otimes C] \\ &\overset{(22) \text{ of } [9]}{=} t_\nabla - \frac{1}{2}(d_J d_J \sigma^c \otimes C - dd_J\sigma^c \otimes JC - d_J\sigma^c \wedge [J, C]) \\ &= t_\nabla + \frac{1}{2}(d_J\sigma^c \wedge J) \ , \end{aligned}$$

whence (7).

(d) The verification of (8) is also an easy calculation. If $S : TM \to TTM$ is a semispray and X is a vector field on M, then

$$\begin{aligned} \overset{\circ}{t}_{\overline{\nabla}}(X^c) &:= t_{\overline{\nabla}}(S, X^c) \overset{(7)}{=} \overset{\circ}{t}_\nabla(X^c) + \frac{1}{2} d_J\sigma^c(S)JX^c - \frac{1}{2} d_J\sigma^c(X^c)JS \\ &= (\overset{\circ}{t}_\nabla + \frac{1}{2}(\sigma^c J - d_J\sigma^c \otimes C))(X^c) \ . \end{aligned}$$

(e) Above we have already seen that $h_{\overline{\nabla}}$ is also homogeneous, and so $H_{\overline{\nabla}} = H_\nabla = 0$. Using (4) and formulas (22), (23) of [14], (9b) can also easily be deduced.

(f) Let $\tilde{S}$ be a semispray on M. Then, in view of Prop. I.38. of [1],

$$S_{\overline{\nabla}} = h_{\overline{\nabla}}\tilde{S} \overset{(4)}{=} h_\nabla \tilde{S} - \frac{1}{2} d\sigma^{\mathrm{v}}(\tilde{S})C = S_\nabla - \frac{1}{2}(\tilde{S}\sigma^{\mathrm{v}})C = S_\nabla - \frac{1}{2}\sigma^c C \ ,$$

which completes the proof. □

Corollary 1. *Suppose that (M, E, ∇) and $(M, \overline{E}, \overline{\nabla})$ are conformally equivalent Ichijyō manifolds; let $\overline{E} = (\exp \circ \sigma^{\mathsf{v}})E$, $\sigma \in C^{\infty}(M)$. Then the following assertions are equivalent:*

(i) *The function σ is constant, i.e., the conformal change is homothetic.*

(ii) $h_{\nabla} = h_{\overline{\nabla}}$. **(iii)** $S_{\nabla} = S_{\overline{\nabla}}$. **(iv)** $t_{\nabla} = t_{\overline{\nabla}}$.

Proof. The implication **(i)**$\Rightarrow$ **(ii)** is an immediate consequence of (4), while the implication **(ii)**$\Rightarrow$ **(iii)** is evident. If $S_{\nabla} = S_{\overline{\nabla}}$, then we infer from (10) that $\sigma^c C = 0$. Thus, in particular, $0 = (\sigma^c C)E = \sigma^c(CE) = 2\sigma^c E$, therefore $\sigma^c = 0$. This means that σ is constant; so the implication **(iii)**$\Rightarrow$ **(i)** is also true. It remains to check that the assertions **(i)** and **(iv)** are equivalent. - If $\sigma \in C^{\infty}(M)$ is constant then $\sigma^c = 0$, and we obtain from (7) that $t_{\overline{\nabla}} = t_{\nabla}$. Conversely, if $t_{\overline{\nabla}} = t_{\nabla}$ then $\overset{\circ}{t}_{\overline{\nabla}} = \overset{\circ}{t}_{\nabla}$, and (8) yields the relation $\sigma^c J = d_J \sigma^c \otimes C$. Forming the semibasic trace ([14], p.134) of both sides it follows that

$$n\sigma^c = i_{\tilde{S}} d_J \sigma^c \text{ , i.e. , } n\sigma^c = \sigma^c$$

($\tilde{S}$ is an arbitrary semispray). Since $n \geq 2$ (**2**(b)), we conclude that $\sigma^c = 0$, and consequently σ is constant.

Thus the proof is complete. □

Theorem. *Conformally equivalent Ichijyō manifolds have the same partial curvatures.*

Proof. Keeping the previous notations, consider the Ichijyō manifolds (M, E, ∇) and $(M, \overline{E}, \overline{\nabla})$.

(a) The coincidence of the horizontal curvatures is an immediate consequence of (4) and (9b), while it is clear from Lemma 2 that the v-curvatures are also the same.

(b) For any vector fields $X, Y, Z \in \mathfrak{X}(M)$, we have

$$\begin{aligned}
\overline{\mathcal{P}}(X^c, Y^c)Z^c &= \overline{D}_{X^{h_{\overline{\nabla}}}} \overline{D}_{Y^{\mathsf{v}}} Z^{\mathsf{v}} - \overline{D}_{Y^{\mathsf{v}}} \overline{D}_{X^{h_{\overline{\nabla}}}} Z^{\mathsf{v}} - \overline{D}_{[X^{h_{\overline{\nabla}}}, Y^{\mathsf{v}}]} Z^{\mathsf{v}} \overset{(4),\, (5),\, (6)}{=} \\
&= \overline{D}_{X^{h_{\overline{\nabla}}}} D_{Y^{\mathsf{v}}} Z^{\mathsf{v}} - \overline{D}_{Y^{\mathsf{v}}} \left(D_{X^{h_{\nabla}}} Z^{\mathsf{v}} + \frac{1}{2} d\sigma^{\mathsf{v}}(X^c) Z^{\mathsf{v}} \right) - \overline{D}_{[X^{h_{\nabla}}, Y^{\mathsf{v}}]} Z^{\mathsf{v}} \\
&\quad - \frac{1}{2} d\sigma^{\mathsf{v}}(X^c) \overline{D}_{Y^{\mathsf{v}}} Z^{\mathsf{v}} \overset{(5)}{=} \overline{D}_{X^{h_{\overline{\nabla}}}} D_{Y^{\mathsf{v}}} Z^{\mathsf{v}} - \overline{D}_{Y^{\mathsf{v}}} D_{X^{h_{\nabla}}} Z^{\mathsf{v}} - \frac{1}{2} d\sigma^{\mathsf{v}}(X^c) \overline{D}_{Y^{\mathsf{v}}} Z^{\mathsf{v}} \\
&\quad - D_{[X^{h_{\nabla}}, Y^{\mathsf{v}}]} Z^{\mathsf{v}} - \frac{1}{2} d\sigma^{\mathsf{v}}(X^c) D_{Y^{\mathsf{v}}} Z^{\mathsf{v}} \overset{(5),\, (6)\ 4(b)}{=} D_{X^{h_{\nabla}}} D_{Y^{\mathsf{v}}} Z^{\mathsf{v}} - D_{Y^{\mathsf{v}}} D_{X^{h_{\nabla}}} Z^{\mathsf{v}} \\
&\quad - D_{[X^{h_{\nabla}}, Y^{\mathsf{v}}]} Z^{\mathsf{v}} - \frac{1}{2} d\sigma^{\mathsf{v}}(X^c) [C, D_{Y^{\mathsf{v}}} Z^{\mathsf{v}}] - d\sigma^{\mathsf{v}}(X^c) \mathcal{C}(Y^c, Z^c) \\
&= \mathcal{P}(X^c, Y^c)Z^c - d\sigma^{\mathsf{v}}(X^c) \left(\frac{1}{2} [C, \mathcal{C}(Y^c, Z^c)] + \mathcal{C}(Y^c, Z^c) \right) = \mathcal{P}(X^c, Y^c)Z^c,
\end{aligned}$$

taking into account that the vector field $\mathcal{C}(X^c, Y^c)$ is homogeneous of degree -1. □

6. Applications to generalized Berwald manifolds.

Definitions. (1) An Ichijyō manifold (M, E, ∇) is said to be a *generalized Berwald manifold* if the horizontal endomorphism h_∇ is conservative in the sense that $d_{h_\nabla} E = 0$, i.e. , for any vector field X on TM, $(d_{h_\nabla} E)(X) = (h_\nabla X)E = 0$.

(2) A quadruple (M, E, ∇, α) is called a *Wagner manifold* if (M, E, ∇) is a generalized Berwald manifold, and α is a smooth function on M satisfying the condition

$$\mathbb{T}_\nabla(X, Y) = d\alpha(X)Y - d\alpha(Y)X \quad (X, Y \in \mathfrak{X}(M) ,$$

where $\mathbb{T}_\nabla$ is the classical torsion of the linear connection ∇.

Remarks. (1) It can easily be seen that a generalized Berwald manifold reduces to a Berwald manifold if ∇ is torsion-free, and the converse is also true. In this case the linear connection ∇ is unique.

(2) The concept of generalized Berwald manifolds is due to V.V. Wagner [13]. Their importance for the conformal theory of Finsler manifolds was discovered by M. Hashiguchi and Y. Ichijyō ([3], [4]). The definition presented here was motivated by Z. I. Szabó's paper [7].

Lemma 3. *If (M, E, ∇, α) is a Wagner manifold then*

$$t_\nabla = d\alpha^{\mathrm{v}} \wedge J := d\alpha^{\mathrm{v}} \otimes J - J \otimes d\alpha^{\mathrm{v}} . \tag{11}$$

Proof. See [9], 4.2.

Proposition 2. *Suppose that (M, E, ∇) and $(M, \overline{E}, \overline{\nabla})$ are conformally equivalent Ichijyō manifolds. If (M, E, ∇) is a generalized Berwald manifold, then $(M, \overline{E}, \overline{\nabla})$ is also a generalized Berwald manifold.*

Proof. Let $\overline{E} = \varphi E$, $\varphi = \exp \circ \sigma^{\mathrm{v}}$, $\sigma \in C^\infty(M)$. Then

$$d_{h_{\overline{\nabla}}} \overline{E} = d_{h_{\overline{\nabla}}}(\varphi E) = \varphi d_{h_{\overline{\nabla}}} E + E d_{h_{\overline{\nabla}}} \varphi ,$$

so we get for any vector field X on M

$$\begin{aligned}
(d_{h_{\overline{\nabla}}} \overline{E})(X^c) &= \varphi(d_{h_{\overline{\nabla}}} E)(X^c) + E(d_{h_{\overline{\nabla}}} \varphi)(X^c) = \varphi(h_{\overline{\nabla}} X^c)E + E(h_{\overline{\nabla}} X^c)\varphi \\
&\overset{(4)}{=} \varphi(X^{h_\nabla} E - \frac{1}{2}(X\sigma)^{\mathrm{v}} CE) + E(X^{h_\nabla} \varphi - \frac{1}{2}(X\sigma)^{\mathrm{v}} C\varphi) \\
&= -\varphi(X\sigma)^{\mathrm{v}} E + E X^{h_\nabla}(\exp \circ \sigma^{\mathrm{v}}) = -\varphi(X\sigma)^{\mathrm{v}} E + E\varphi(X^{h_\nabla} \sigma^{\mathrm{v}}) \\
&= -\varphi(X\sigma)^{\mathrm{v}} E + \varphi(X\sigma)^{\mathrm{v}} E = 0 .
\end{aligned}$$

This means that $h_{\overline{\nabla}}$ is also conservative as we claimed. $\square$

Corollary 2. *Let (M, E, ∇) and $(M, \overline{E}, \overline{\nabla})$, $\overline{E} = (\exp \circ \sigma^{\mathsf{v}})E$ $(\sigma \in C^\infty(M))$ be conformally equivalent Ichijyō manifolds. If (M, E, ∇, α) is a Wagner manifold, then $(M, \overline{E}, \overline{\nabla}, \overline{\alpha})$ is also a Wagner manifold with $\overline{\alpha} = \alpha + \frac{1}{2}\sigma$.*

Proof. $(M, \overline{E}, \overline{\nabla})$ is a generalized Berwald manifold by Proposition 2. Since

$$t_{\overline{\nabla}} \overset{(7)}{=} t_\nabla + \frac{1}{2}(d\sigma^{\mathsf{v}} \wedge J) \overset{(11)}{=} d\alpha^{\mathsf{v}} \wedge J + \frac{1}{2}(d\sigma^{\mathsf{v}} \wedge J) = d(\alpha + \frac{1}{2}\sigma)^{\mathsf{v}} \wedge J\,,$$

it follows that $(M, \overline{E}, \overline{\nabla}, \overline{\alpha})$, $\overline{\alpha} := \alpha + \frac{1}{2}\sigma$ is indeed a Wagner manifold. □

Corollary 3. (Theorem of M. Hashiguchi and Y. Ichijyō.) *The necessary and sufficient condition for a Finsler manifold to be conformally equivalent to a Berwald manifold is that the Finsler manifold is a Wagner manifold.*

Proof. Let a Finsler manifold (M, E) be given.

(1) *Necessity.* Suppose that $(M, \overline{E}, \overline{\nabla})$ is a Berwald manifold conformally equivalent to (M, E), namely $\overline{E} = (\exp \circ \sigma^{\mathsf{v}})E$, $\sigma \in C^\infty(M)$. If $\nabla := \overline{\nabla} - \frac{1}{2}d\sigma \otimes \mathrm{id}$, then by corollary 2 $(M, E, \nabla, -\frac{1}{2}\sigma)$ is a Wagner manifold.

(2) *Sufficiency.* Now we assume that (M, E, ∇, α) is a Wagner manifold. Let $\overline{E} := (\exp \circ \sigma^{\mathsf{v}})E$, $\sigma := -2\alpha$, $\overline{\nabla} = \nabla + \frac{1}{2}d\sigma \otimes \mathrm{id}$. Then $(M, \overline{E}, \overline{\nabla})$ is an Ichijyō manifold conformally equivalent to (M, E, ∇), moreover $(M, \overline{E}, \overline{\nabla}, \alpha + \frac{1}{2}\sigma)$ is a Wagner manifold. But $\alpha + \frac{1}{2}\sigma = \alpha - \alpha = 0$ by the construction, therefore $\overline{\nabla}$ is torsion-free, and so $(M, \overline{E}, \overline{\nabla})$ is a Berwald manifold. □

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