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RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
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THE COHOMOLOGY OF $\tilde{G}_{m,2}$ WITH INTEGER COEFFICIENTS

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ABSTRACT. This paper contains the description of the cohomology ring $H^*(\tilde{G}_{n,2};\mathbb{Z})$ of the Grassmannian $\tilde{G}_{n,2}$ of oriented planes with integer coefficients. We describe the ring in terms of generators and relations.

1. INTRODUCTION AND PRELIMINARIES

The description of the cohomology ring $H^*(\tilde{G}_{n,2};\mathbb{Z})$ seems not to be available in the literature. Its knowledge is necessary for example when studying the existence of 2-dimensional subbundles of a vector bundle. Another reason for writing this paper was the necessity to use some results from it in our forthcoming paper [CV] about the cohomology ring of the Grassmannian $G_{n,2}$ of nonoriented planes with integer and twisted integer coefficients.

We shall consider the vector space $\mathbb{R}^n$ with its canonical orientation. The symbol $\tilde{G}_{n,k}$, where $0 < k < n$, will denote the Grassmann manifold of oriented k-dimensional subspaces in $\mathbb{R}^n$, and $\tilde{\gamma}_k$ will be the the canonical oriented k-dimensional vector bundle over $\tilde{G}_{n,k}$. The vector bundle $\tilde{\gamma}_k$ is obviously a subbudle of the trivial n-dimensional vector bundle ε^n with the fiber $\mathbb{R}^n$, and it has a riemannian metric induced from the canonical riemannian metric on ε^n. There is also the orthogonal complement $\tilde{\gamma}_k^\perp$, which is an $(n-k)$-dimensional vector bundle. We orient it in such a way that the orientation of $\tilde{\gamma}_k \oplus \tilde{\gamma}_k^\perp$ coincides with the canonical orientation of ε^n.

From now on we shall assume that $n \geq 4$. We shall consider the unit sphere bundles

$$\pi_1 : S^1\tilde{\gamma}_2 \longrightarrow \tilde{G}_{n,2}$$

and

$$\pi_2 : S^{n-2}\tilde{\gamma}_{n-1} \longrightarrow \tilde{G}_{n,n-1}.$$

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1. Lemma. *The total spaces $S^1\tilde{\gamma}_2$ and $S^{n-2}\tilde{\gamma}_{n-1}$ are homeomorphic.*

Proof. We can easily construct a mapping $\varphi : S^1\tilde{\gamma}_2 \to S^{n-2}\tilde{\gamma}_{n-1}$. An element from $S^1\tilde{\gamma}_2$ is a couple (α, v), where $\alpha \subset \mathbb{R}^n$ is a 2-dimensional oriented subspace and $v \in \alpha$ is a unit vector. In α there is a unique unit vector v' orthogonal to v such that $\{v, v'\}$ is a positive basis in α. Further let $v^\perp$ denote the orthogonal complement to v endowed with such orientation that the natural orientation of $[v] \oplus v^\perp$ coincides with the orientation of $\mathbb{R}^n$. We can now define φ by the formula

$$\varphi(\alpha, v) = (v^\perp, v').$$

It is easy to verify that this mapping is a homeomorphism.

Our next aim is to compute the cohomology ring $H^*(S^{n-2}\tilde{\gamma}_{n-1}; \mathbb{Z})$. We shall use the Gysin sequence for the fibration

$$S^{n-2} \xrightarrow{i_2} S^{n-2}\tilde{\gamma}_{n-1} \xrightarrow{\pi_2} \tilde{G}_{n,n-1} \cong S^{n-1}.$$

We have to distinguish two cases.

2. n IS EVEN

The Euler class $e = e(\tilde{\gamma}_{n-1}) \in H^{n-1}(S^{n-1}; \mathbb{Z})$ has order two and consequently $e = 0$. Therefore we get the Gysin sequence in the form

$$\dots \xrightarrow{0} H^k(S^{n-1}) \xrightarrow{\pi_2^*} H^k(S^{n-2}\tilde{\gamma}_{n-1}) \to H^{k-n+2}(S^{n-1}) \xrightarrow{0} \dots$$

Thus we can immediately see that

$$H^k(S^{n-2}\tilde{\gamma}_{n-1}; \mathbb{Z}) = 0 \quad \text{for } k \neq 0, n-2, n-1, 2n-3.$$

Because $S^{n-2}\tilde{\gamma}_{n-1}$ is connected, we have

$$H^0(S^{n-2}\tilde{\gamma}_{n-1}; \mathbb{Z}) = \mathbb{Z}.$$

It is also easy to determine the remaining cohomology groups. For the later use we shall write here the relevant parts of the Gysin sequence. For $k = n-2$ we get

$$0 = H^{n-2}(S^{n-1}) \xrightarrow{\pi_2^*} H^{n-2}(S^{n-2}\tilde{\gamma}_{n-1}) \to H^0(S^{n-1}) = \mathbb{Z} \xrightarrow{0} \dots$$

which shows that

$$H^{n-2}(S^{n-2}\tilde{\gamma}_{n-1}; \mathbb{Z}) = \mathbb{Z}.$$

For $k = n-1$ we get

$$\dots \xrightarrow{0} \mathbb{Z} = H^{n-1}(S^{n-1}) \xrightarrow{\pi_2^*} H^{n-1}(S^{n-2}\tilde{\gamma}_{n-1}) \to H^1(S^{n-1}) = 0$$

which gives

$$H^{n-1}(S^{n-2}\tilde{\gamma}_{n-1};\mathbb{Z}) = \mathbb{Z}.$$

Finally for $k = 2n-3$ we have

$$0 = H^{2n-3}(S^{n-1}) \overset{\pi_2^*}{\to} H^{2n-3}(S^{n-2}\tilde{\gamma}_{n-1}) \to H^{n-1}(S^{n-1}) = \mathbb{Z} \overset{0}{\to} \dots$$

which gives

$$H^{2n-3}(S^{n-2}\tilde{\gamma}_{n-1};\mathbb{Z}) = \mathbb{Z}.$$

Because $S^{n-2}\tilde{\gamma}_{n-1}$ is a manifold of dimension $2n-3$ this determines the additive structure of the cohomology in question.

Now we shall describe generators in the above groups. Let us denote $\psi = \varphi^{-1}$: $S^{n-2}\tilde{\gamma}_{n-1} \to S^1\tilde{\gamma}_2$. It is easy to see that for an element $(\beta, w) \in S^{n-2}\tilde{\gamma}_{n-1}$, where $\beta \subset \mathbb{R}^n$ is an oriented $(n-1)$-dimensional subspace and $w \in \beta$ is a unit vector we get

$$\psi(\beta, w) = ([\beta^\perp, w], \beta^\perp),$$

where $\beta^\perp$ denotes the unique unit vector orthogonal to β and such that the natural orientation of $[\beta^\perp] \oplus \beta$ coincides with the orientation of $\mathbb{R}^n$. We orient the 2-dimensional subspace $[\beta^\perp, w]$ by taking $\{\beta^\perp, w\}$ as a positive basis. On $S^{n-2}\tilde{\gamma}_{n-1}$ we have an oriented $(n-2)$-dimensional vector bundle $\psi^*\pi_1^*\tilde{\gamma}_2^\perp$. Its fiber over the point (β, w) has the form

$$((\beta, w), w_\beta^\perp),$$

where $w_\beta^\perp$ denotes the $(n-2)$-dimensional subspace of β orthogonal to w and oriented in such a way that the natural orientation of $[w] \oplus w_\beta^\perp$ coincides with the orientation of β. If we consider again the fibration $S^{n-2} \overset{i_2}{\to} S^{n-2}\tilde{\gamma}_{n-1} \overset{\pi_2}{\to} S^{n-1}$ we can easily see that $i_2^*\psi^*\pi_1^*\tilde{\gamma}_2^\perp \cong TS^{n-2}$, where TS^{n-2} denotes the tangent bundle. Let us orient TS^{n-2} in such a way that this is an orientation preserving isomorphism. We denote $\omega \in H^{n-2}(S^{n-2};\mathbb{Z})$ the generator uniquely determined by the orientation of TS^{n-2}. Obviously we have $e(i_2^*\psi^*\pi_1^*\tilde{\gamma}_2^\perp) = 2\omega$, where e denotes the Euler class. The Serre sequence for this fibration (with $\mathbb{Z}$-coefficients) shows that i_2^* is an isomorphism. We denote by a the unique element $a \in H^{n-2}(S^{n-2}\tilde{\gamma}_{n-1})$ such that $i_2^*a = \omega$. (We shall also write a instead of φ^*a.) Obviously $a \in H^{n-2}(S^{n-2}\tilde{\gamma}_{n-1})$ is a generator. We have then

$$e(\psi^*\pi_1^*\tilde{\gamma}_2^\perp) = 2a.$$

The same Serre sequence shows also that $\pi_2^* : H^{n-1}(S^{n-1}) \to H^{n-1}(S^{n-2}\tilde{\gamma}_{n-1})$ is an isomorphism. Let $\theta \in H^{n-1}(S^{n-1})$ be a generator. We take a generator $b \in H^{n-1}(S^{n-2}\tilde{\gamma}_{n-1})$ such that $b = \pi_2^*\theta$.

We shall now prove that $c = ab$ is a generator of the group $H^{2n-3}(S^{n-2}\tilde{\gamma}_{n-1})$. For this purpose we shall use the oriented vector bundle $\tilde{\gamma}_{n-1}$ over S^{n-1}. Let

$$U \in H^{n-1}(\mathbb{B}^{n-1}\tilde{\gamma}_{n-1}, S^{n-2}\tilde{\gamma}_{n-1})$$

(again with $\mathbb{Z}$-coefficients) denote the Thom class. We then have

$$\delta a = \pm 1 \cup U.$$

Because $b = \pi_2^*\theta$ we have

$$\delta(ab) = \pm\theta \cup U,$$

which shows that ab is a generator. We have thus proved the following proposition.

2. Proposition. *The cohomology ring $H^*(S^{n-2}\tilde{\gamma}_{n-1};\mathbb{Z})$ is isomorphic with the graded ring*

$$\mathbb{Z}[a,b]/(a^2,b^2),$$

where $\deg a = n-2$, $\deg b = n-1$.

We shall now use the above results for the computation of the cohomology ring $H^*(\tilde{G}_{n,2};\mathbb{Z})$. (We recall that $n \geq 4$ is even.) For this purpose we shall use the Gysin sequence for the oriented vector bundle $\tilde{\gamma}_2$ over $\tilde{G}_{n,2}$. We shall denote $e = e(\tilde{\gamma}_2) \in H^2(\tilde{G}_{n,2};\mathbb{Z})$ the Euler class of $\tilde{\gamma}_2$. First we shall prove the following proposition.

3. Proposition.

$$H^{2k}(\tilde{G}_{n,2};\mathbb{Z}) \cong \mathbb{Z} \ni e^k \quad \text{for} \quad 2k < n-2$$

with e^k being a generator,

$$H^{2k+1}(\tilde{G}_{n,2};\mathbb{Z}) = 0 \quad \text{for} \quad 2k+1 < n-2.$$

Proof. Because $\tilde{G}_{n,2}$ is connected we have $H^0(\tilde{G}_{n,2};\mathbb{Z}) \cong \mathbb{Z}$ with the generator $1 = e^0$. Because $\tilde{G}_{n,2}$ is simply connected we have $H^1(\tilde{G}_{n,2};\mathbb{Z}) = 0$. Now it is sufficient to proceed by induction using the Gysin sequence for the vector bundle $\tilde{\gamma}_2$.

4. Proposition.

$$H^{2k}(\tilde{G}_{n,2};\mathbb{Z}) \cong \mathbb{Z} \quad \text{for} \quad 2n-4 \geq 2k > n-2,$$

$$H^{2k+1}(\tilde{G}_{n,2};\mathbb{Z}) = 0 \quad \text{for} \quad 2k+1 > n-2.$$

Proof. $\tilde{G}_{n,2}$ is an orientable compact manifold of dimension $2n-4$. This implies that $H^{2n-4}(\tilde{G}_{n,2};\mathbb{Z}) \cong \mathbb{Z}$. $\tilde{G}_{n,2}$ is simply connected, and consequently $H_1(\tilde{G}_{n,2}) = 0$. The Poincaré duality gives then $H^{2n-5}(\tilde{G}_{n,2};\mathbb{Z}) = 0$. Now, it is again sufficient to proceed by induction (going down) using the same Gysin sequence as above.

It remains to compute the group $H^{n-2}(\tilde{G}_{n,2};\mathbb{Z})$. The relevant part of the Gysin sequence has the form

$$0 = H^{n-3}(S^1\tilde{\gamma}_2) \to H^{n-4}(\tilde{G}_{n,2}) = \mathbb{Z} \xrightarrow{\cup e} H^{n-2}(\tilde{G}_{n,2}) \xrightarrow{\pi_1^*}$$

$$\xrightarrow{\pi_1^*} H^{n-2}(S^1\tilde{\gamma}_2) = \mathbb{Z} \to H^{n-3}(\tilde{G}_{n,2}) = 0,$$

which gives

$$H^{n-2}(\tilde{G}_{n,2};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

It is easy to see that in this group we can choose the generators $e^{(n-2)/2}$ and f', where f' satisfies $\pi_1^* f' = a$. Obviously the Euler class $e(\tilde{\gamma}_2^\perp)$ can be uniquely expressed in the form

$$e(\tilde{\gamma}_2^\perp) = ue^{(n-2)/2} + 2f'.$$

We shall use the standard notation $w_i = w_i(\tilde{\gamma}_2)$ for the Stiefel-Whitney classes. From the relation $\tilde{\gamma}_2 \oplus \tilde{\gamma}_2^{\perp} = \varepsilon^n$, where ε^n denotes the trivial n-dimensional vector bundle, we can easily find that $w_2(\tilde{\gamma}_2^{\perp}) = w_2$. This shows that the integer u is odd, i. e. $u = 2v + 1$. Taking $f = ve^{(n-2)/2} + f'$ we obtain for $H^{n-2}(\tilde{G}_{n,2};\mathbb{Z})$ the generators $e^{(n-2)/2}$ and f, and for the Euler class $e(\tilde{\gamma}_2^{\perp})$ we have the formula

$$e(\tilde{\gamma}_2^{\perp}) = e^{(n-2)/2} + 2f.$$

Moreover the same relation $\tilde{\gamma}_2 \oplus \tilde{\gamma}_2^{\perp} = \varepsilon^n$ shows that

$$e^{n/2} = -2ef.$$

Let us consider now the following part of our Gysin sequence.

$$\mathbb{Z}\oplus\mathbb{Z} = H^{n-2}(\tilde{G}_{n,2}) \overset{\cup e}{\to} H^{n}(\tilde{G}_{n,2}) = \mathbb{Z} \to H^{n}(S^1\tilde{\gamma}_2) = 0$$

We can see that $e^{(n-2)/2} + 2f$ and f is a base of $H^{n-2}(\tilde{G}_{n,2})$ considered as a free module over $\mathbb{Z}$. Because $(e^{(n-2)/2} + 2f) \cup e = 0$ it is obvious that ef is a generator. We have

$$H^{n-2}(\tilde{G}_{n,2};\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z} \text{ with generators } e^{(n-2)/2} \text{ and } f.$$

Using this result and the same Gysin sequence as above we obtain by induction

$$H^{2k}(\tilde{G}_{n,2};\mathbb{Z}) \cong \mathbb{Z} \text{ with generator } e^{(2k-n+2)/2}f \text{ for } 2n-4 \geq 2k > n-2.$$

It remains to determine f^2. Obviously $f^2 = te^{(n-2)/2}f$, where t is an integer. Our next aim is to determine this integer. We shall apply the Gysin sequence of the vector bundle

$$\pi^{\perp} : \tilde{\gamma}_2^{\perp} \to \tilde{G}_{n,2}$$

with $\mathbb{Z}$-coefficients. More precisely, we shall need only the following piece of the Gysin sequence.

$$H^{n-2}(\tilde{G}_{n,2}) \overset{\cup e^{\perp}}{\to} H^{2n-4}(\tilde{G}_{n,2}) \to H^{2n-4}(S^{n-3}\tilde{\gamma}_2^{\perp}),$$

where $S^{n-3}\tilde{\gamma}_2^{\perp}$ is the sphere bundle of $\tilde{\gamma}_2^{\perp}$ and $e^{\perp} = e(\tilde{\gamma}_2^{\perp})$. We know already that $e^{\perp} = e^{(n-2)/2} + 2f$.

Obviously, we must first calculate $H^{2n-4}(S^{n-3}\tilde{\gamma}_2^{\perp})$. Let us notice first that $S^{n-3}\tilde{\gamma}_2^{\perp}$ is a compact connected orientable manifold and $\dim S^{n-3}\tilde{\gamma}_2^{\perp} = 3n-7$. For this purpose we shall consider the following part of the Gysin sequence for the vector bundle $\tilde{\gamma}_2^{\perp}$.

$$0 = H^{n-3}(\tilde{G}_{n,2}) \to H^{n-3}(S^{n-3}\tilde{\gamma}_2^{\perp}) \to H^{0}(\tilde{G}_{n,2}) = \mathbb{Z} \overset{\cup e^{\perp}}{\to}$$
$$\overset{\cup e^{\perp}}{\to} H^{n-2}(\tilde{G}_{n,2}) = \mathbb{Z}\oplus\mathbb{Z} \to H^{n-2}(S^{n-3}\tilde{\gamma}_2^{\perp}) \to H^{1}(\tilde{G}_{n,2}) = 0$$

From this sequence we get immediately

$$H^{n-3}(S^{n-3}\tilde{\gamma}_2^{\perp}) = 0$$

$$H^{n-2}(S^{n-3}\tilde{\gamma}_2^{\perp}) = \mathbf{Z} \quad \text{with the generator } (\pi^{\perp})^* f.$$

Using the Poincare duality on $S^{n-3}\tilde{\gamma}_2^{\perp}$ we get

$$H_{2n-5}(S^{n-3}\tilde{\gamma}_2^{\perp}) = \mathbf{Z}, \quad H_{2n-4}(S^{n-3}\tilde{\gamma}_2^{\perp}) = 0.$$

Now it is sufficient to apply the universal coefficients theorem. We can write the exact sequence

$$0 \to Ext(H_{2n-5}(S^{n-3}\tilde{\gamma}_2^{\perp}), \mathbf{Z}) \to H^{2n-4}(S^{n-3}\tilde{\gamma}_2^{\perp}; \mathbf{Z}) \to$$
$$\to Hom(H_{2n-4}(S^{n-3}\tilde{\gamma}_2^{\perp}), \mathbf{Z}) \to 0,$$

which shows that

$$H^{2n-4}(S^{n-3}\tilde{\gamma}_2^{\perp}; \mathbf{Z}) = 0.$$

Now we can see that the mapping

$$\cup e^{\perp} : H^{n-2}(\tilde{G}_{n,2}) \to H^{2n-4}(\tilde{G}_{n,2})$$

is surjective. We have

$$(e^{(n-2)/2} + 2f)(e^{(n-2)/2} + 2f) = 2e^{(n-2)/2}f + 4f^2 = (2 + 4t)e^{(n-2)/2}f$$
$$f(e^{(n-2)/2} + 2f) = e^{(n-2)/2}f + 2f^2 = (1 + 2t)e^{(n-2)/2}f$$

Consequently, there exists integers r and s such that

$$r(2 + 4t) + s(1 + 2t) = 1.$$

This equation can be written in the form

$$(1 + 2t)(2r + s) = 1,$$

which shows that $t = 0$. We have thus proved that

$$f^2 = 0.$$

We have obtained in this way a description of the integral cohomology ring of $\tilde{G}_{n,2}$.

5. Theorem. *The cohomology ring $H^*(\tilde{G}_{n,2}; \mathbf{Z})$ with n even, $n \geq 4$ is isomorphic with the graded ring*

$$\mathbf{Z}[e, f]/(e^{n/2} + 2ef, f^2),$$

where $\deg e = 2$, $\deg f = n - 2$. *Moreover, under this isomorphism, $e(\tilde{\gamma}_2^{\perp})$ correspond to the class determined by the element $e^{(n-2)/2} + 2f$.*

3. n IS ODD

Let us denote by $\omega \in H^{n-1}(S^{n-1};\mathbf{Z})$ a generator of S^{n-1}. Because $\tilde{\gamma}_{n-1}$ is isomorphic with TS^{n-1} we have $e = e(\tilde{\gamma}_{n-1}) = \pm 2\omega$. Obviously, we can choose ω in such a way that $e = e(\tilde{\gamma}_{n-1}) = 2\omega$. The same Gysin sequence as in the even case gives us first

$$H^k(S^{n-2}\tilde{\gamma}_{n-1};\mathbf{Z}) = 0 \quad \text{for } k \neq 0, n-2, n-1, 2n-3.$$

We have obviously

$$H^0(S^{n-2}\tilde{\gamma}_{n-1};\mathbf{Z}) \cong \mathbf{Z}.$$

For $k = n-2$ we get

$$0 = H^{n-2}(S^{n-1}) \to H^{n-2}(S^{n-2}\tilde{\gamma}_{n-1}) \to H^0(S^{n-1}) = \mathbf{Z} \overset{\cup 2\omega}{\to} H^{n-1}(S^{n-1}) = \mathbf{Z},$$

which shows that

$$H^{n-2}(S^{n-2}\tilde{\gamma}_{n-1};\mathbf{Z}) = 0.$$

Next for $k = n-1$ we have

$$\mathbf{Z} = H^0(S^{n-1}) \overset{\cup 2\omega}{\to} H^{n-1}(S^{n-1}) = \mathbf{Z} \to H^{n-1}(S^{n-2}\tilde{\gamma}_{n-1}) \to H^1(S^{n-1}) = 0,$$

which gives

$$H^{n-1}(S^{n-2}\tilde{\gamma}_{n-1};\mathbf{Z}) \cong \mathbf{Z}_2.$$

Finally for $k = 2n-3$ we get

$$0 = H^{2n-3}(S^{n-1}) \to H^{2n-3}(S^{n-2}\tilde{\gamma}_{n-1}) \to H^{n-1}(S^{n-1}) = \mathbf{Z} \to H^{2n-2}(S^{n-1}) = 0,$$

which gives

$$H^{2n-3}(S^{n-2}\tilde{\gamma}_{n-1};\mathbf{Z}) \cong \mathbf{Z}.$$

The following proposition is obvious.

6. Proposition. *The cohomology ring $H^*(S^{n-2}\tilde{\gamma}_{n-1};\mathbf{Z})$ is isomorphic with the graded ring*

$$\mathbf{Z}[x,y]/(2x, x^2, xy, y^2),$$

where deg $x = n-1$ and $\deg y = 2n-3$.

We shall now again compute the integral cohomology ring of $\tilde{G}_{n,2}$. We start with the following proposition.

7. Proposition.

$$H^i(\tilde{G}_{n,2};\mathbf{Z}) = 0 \quad \textit{for } i \textit{ odd.}$$

Proof. $\tilde{G}_{n,2}$ is simply connected. Therefore $H_1(\tilde{G}_{n,2};\mathbf{Z}) = 0$ and $H^1(\tilde{G}_{n,2};\mathbf{Z}) = 0$. The Poincaré duality gives $H^{2n-5}(\tilde{G}_{n,2};\mathbf{Z}) = 0$. Proceeding now by induction (first going up, then going down), and using the Gysin sequence for the vector bundle $\tilde{\gamma}_2$, we get easily the assertion.

8. Proposition.

$$H^{2k}(\tilde{G}_{n,2};\mathbb{Z}) \cong \mathbb{Z} \quad \textit{for } 2k < n-1.$$

with the generator being e^k.

Proof. Obviously $H^0(\tilde{G}_{n,2};\mathbb{Z}) \cong \mathbb{Z}$ with the generator $1 = e^0$. Now, it suffices to proceed by induction (going up) and use the same Gysin sequence as above.

9. Proposition.

$$H^{2k}(\tilde{G}_{n,2};\mathbb{Z}) \cong \mathbb{Z} \quad \textit{for } n-1 < 2k \leq 2n-4.$$

Proof. We proceed in the same way as above with the induction going down.

It remains to determine the group $H^{n-1}(\tilde{G}_{n,2};\mathbb{Z})$. The Gysin sequence gives here

$$0 = H^{n-2}(S^1\tilde{\gamma}_2) \to H^{n-3}(\tilde{G}_{n,2}) = \mathbb{Z} \overset{\cup e}{\to} H^{n-1}(\tilde{G}_{n,2}) \to$$
$$\to H^{n-1}(S^1\tilde{\gamma}_2) = \mathbb{Z}_2 \to H^{n-2}(\tilde{G}_{n,2}) = 0.$$

The last group vanishes because $n-2$ is odd. From this exact sequence we can see that $H^{n-1}(\tilde{G}_{n,2}) \cong \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}_2$. Associated with the exact coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$ we have the exact sequence

$$0 = H^{n-2}(\tilde{G}_{n,2};\mathbb{Z}_2) \overset{\beta}{\to} H^{n-1}(\tilde{G}_{n,2};\mathbb{Z}) \overset{2\times}{\to} H^{n-1}(\tilde{G}_{n,2};\mathbb{Z}),$$

where β is the Bockstein homomorphism. We can see that the homomorphism $2\times$ is injective and consequently $H^{n-1}(\tilde{G}_{n,2};\mathbb{Z}) \cong \mathbb{Z}$. We have the following exact sequence.

$$0 = H^{n-2}(S^1\tilde{\gamma}_2) \to H^{n-3}(\tilde{G}_{n,2}) = \mathbb{Z} \overset{\cup e}{\to} H^{n-1}(\tilde{G}_{n,2}) = \mathbb{Z} \to$$
$$\to H^{n-1}(S^1\tilde{\gamma}_2) = \mathbb{Z}_2 \to H^{n-2}(\tilde{G}_{n,2}) = 0$$

Now it is obvious that we can choose a generator $f \in H^{n-1}(\tilde{G}_{n,2})$ in such a way that there is

$$e^{(n-1)/2} = 2f.$$

10. Proposition.

$$H^{2k}(\tilde{G}_{n,2};\mathbb{Z}) \cong \mathbb{Z} \ni e^{(2k-n+1)/2}f \quad \textit{for } n-1 \leq 2k \leq 2n-4.$$

with $e^{(2k-n+1)/2}f$ *being a generator.*

Proof. It is of the same type as before.

Now we get easily

11. Theorem. *The cohomology ring* $H^*(\tilde{G}_{n,2};\mathbb{Z})$ *with* n *odd,* $n \geq 5$ *is isomorphic with the graded ring*

$$\mathbb{Z}[e,f]/(e^{(n-1)/2} - 2f, f^2),$$

where $\deg e = 2$, $\deg f = n-1$. *Moreover, there is* $e(\tilde{\gamma}_2^{\perp}) = 0$.

References

[CV] M. Čadek, J. Vanžura, *The cohomology of $G_{m,2}$ with twisted integer coefficients*, to appear.

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