THE LEGENDRE TRANSFORMATION IN DIFFERENTIAL SPACES

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In this paper we present the Legendre transformation on differential spaces in the sense of Sikorski [10], [11], [4]. We investigate some properties of the Legendre transformation for differential spaces with singularities. In Section 2 we give a mechanical interpretation of this transformation.

0. PRELIMINARIES. Let (M,C) be a differential space [10], [11], and let $TM := \bigsqcup_{p\in M} T_pM$ be a disjoint sum of tangent spaces to (M,C). By TC we denote the differential structure on TM [5] generated by the set $\{f\circ\pi: f \in C\} \cup \{df: f \in C\}$, where $\pi: TM \longrightarrow M$ is the natural projection and $df: TM \longrightarrow \mathbb{R}$ is defined by $df(v) = v(f)$, for any $v \in TM$. A smooth tangent vector field to (M,C) is a smooth section of the tangent bundle (TM,π,M). By $\mathfrak{X}(M)$ we denote the C-module of all smooth tangent vector fields to (M,C).

A differential space is said to be of constant differential dimension n if $\dim T_pM = n$ and the C-module $\mathfrak{X}(M)$ is locally free and has the rank n.

A point $p \in M$ is said to be regular if there exists an open neighbourhood $U \in \tau_C$ of p such that the subspace (U,C_U) is of constant differential dimension. A point $p \in M$ is called singular if it is not regular.

If $p \in M$ is a regular point of (M,C) then there exist a set $U \in \tau_C$ containing p, vector fields $X_1,\dots,X_n \in \mathfrak{X}(M)$ for

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some $n \in \mathbb{N}$, and functions $f_1,\dots,f_n \in C$ such that $X_i(q)(f_j) = \delta_{ij}$, for any $i,j = 1,\dots,n$.

One can prove [9]

LEMMA 0.1. Let (M,C) be a differential space generated by a set C_o and $p \in M$ be an arbitrary point. Let $w_o\colon C_o \longrightarrow \mathbb{R}$ be a mapping satisfying the following condition:

(*) for any $\alpha_1,\dots,\alpha_n \in C_o$, $\omega \in \mathcal{E}_n$, $n \in \mathbb{N}$, if

$$\omega\circ(\alpha_1,\dots,\alpha_n) = 0 \text{ then } \sum_{i=1}^{n} \omega'_{|i}(\alpha_1(p),\dots,\alpha_n(p))\cdot w_o(\alpha_i) = 0.$$

Then there is the unique vector $w \in T_pM$ such that $w|C_o = w_o$.

Sketch of the Proof. Let $w\colon C \longrightarrow \mathbb{R}$ be the mapping defined by $w(f) = \sum_{i=1}^{n} \omega'_{|i}(f_1(p),\dots,f_n(p))\cdot w_o(f_i)$, for any $f \in C$, where $f_1,\dots,f_n \in C_o$, $\omega \in \mathcal{E}_n$ are such functions that there is an open neighbourhood U of p and $f|U = \omega\circ(f_1,\dots,f_n)|U$. One can verify that $w \in T_pM$ is the unique vector satisfying the condition $w|C_o = w_o$.

1. MAIN RESULTS. Let (M,C) be a differential space. It is easy to see that, for any $p \in M$, the differential structure TC_{T_pM} induced from TC on T_pM is generated by the set $\{d_p\alpha\colon \alpha \in C\}$, where $d_p\alpha = d\alpha|T_pM$. The tangent space (T_pM, TC_{T_pM}) is a linear differential space i.e. the operation $\mu\colon T_pM \times T_pM \longrightarrow T_pM$ of addition of vectors and $\nu\colon \mathbb{R} \times T_pM \longrightarrow T_pM$ of multiplication of vectors by scalars are smooth.

For an arbitrary $p \in M$, let T'_pM be the set of all smooth linear mappings $\omega\colon T_pM \longrightarrow \mathbb{R}$. Denote by $T^*M = \bigsqcup_{p\in M} T'_pM$ the disjoint union and by $\tilde{\pi}\colon T^*M \longrightarrow \mathbb{R}$ the natural projection.

For any $X \in \mathfrak{X}(M)$, let $X^*\colon T^*M \longrightarrow \mathbb{R}$ be the function defined by

$$X^*(\omega) = \omega(X(\tilde{\pi}(\omega))), \quad \text{for } \omega \in T^*M. \tag{1.1}$$

Let T^*C be the differential structure on T^*M generated by the set $\{X^*\colon X \in \mathfrak{X}(M)\} \cup \{\alpha\circ\tilde{\pi}\colon \alpha \in C\}$. Now, let $v \in T_pM$ be an arbitrary tangent vector to (M,C) at $p \in M$. It is easy to see that, for any vector $w \in T_v(T_pM)$, the mapping $\ell_v(w)\colon C \longrightarrow \mathbb{R}$ defined by

$$\ell_v(w)(\alpha) = w(d_p\alpha), \quad \text{for } \alpha \in C, \tag{1.2}$$

is a tangent vector to (M,C) at the point p.

Now, in several lemmas we will prove the following proposition.

<u>PROPOSITION 1.1</u>. For an arbitrary $v \in T_pM$, the mapping $\ell_v\colon T_v(T_pM) \longrightarrow T_pM$, $w \longmapsto \ell_v(w)$, is a smooth isomorphism of linear differential spaces.

From Lemma 0.1 it follows

<u>LEMMA 1.2</u>. For any vector $v \in T_pM$, there exists the unique vector $w \in T_0(T_pM)$ such that $w(d_p\alpha) = v(\alpha)$, for any $\alpha \in C$.

<u>Proof</u>. Let $w_0\colon \langle d_p\alpha\colon \alpha \in C\rangle \longrightarrow \mathbb{R}$ be the mapping defined by

$$w_0(d_p\alpha) = v(\alpha), \quad \text{for } \alpha \in C. \tag{1.3}$$

It is enough to verify that w_0 satisfies the condition $(*)$ of Lemma 0.1, for the point $0 \in T_pM$ (the zero vector). Let $\alpha_1,\ldots,\alpha_n \in C$, $\omega \in \mathcal{E}_n$ be such functions that

$$\omega\circ(\alpha_1,\ldots,\alpha_n) = 0. \tag{1.4}$$

Without loosing of generality we can assume that $d_p\alpha_1,\ldots,d_p\alpha_n$ are linearly independent. One can easily see (cf. [12]) that there exist vectors $v_1,\ldots,v_n \in T_pM$ such that $v_i(\alpha_j) = \delta_{ij}$, for $i,j = 1,\ldots,n$. Substituting the vector rv_j to (1.4), where $r \in \mathbb{R}$, one obtains

$$\omega(0,\ldots,0,r,0,\ldots,0) = 0, \tag{1.5}$$

for any $r \in \mathbb{R}$, $j = 1,\ldots,n$. Hence $\omega'_{|j}(0,\ldots,0) = 0$, for $j = 1,\ldots,n$. Thus $\sum_{i=1}^{n}\omega'_{|i}(d_p\alpha(0),\ldots,d_p\alpha(0))\cdot w_0(d_p\alpha_i) = 0$.

So the condition $(*)$ is satisfied. In view of Lemma 0.1, there exists the unique vector $w \in T_0(T_pM)$ such that $w|\langle d_p\alpha\colon \alpha \in C\rangle = w_0$ or, equivalently, $w(d_p\alpha) = v(\alpha)$.

<u>LEMMA 1.3</u>. For an arbitrary vector $v \in T_pM$, the mapping $t_v\colon T_pM \longrightarrow T_pM$ defined by

(1.6) $$t_v(u) = v + u, \quad \text{for } u \in T_pM,$$

is a smooth isomorphism of linear differential spaces.

<u>Proof</u>. It is easy to see that t_{-v} is the inverse map to t_v. It is enough to show that t_v is smooth, for any $v \in T_pM$. In fact, for any $\alpha \in C$,

(1.7) $$d_p\alpha \circ t_v = v(\alpha) + d_p\alpha.$$

Hence $d_p\alpha \circ t_v \in TC_{T_pM}$, for any generator $d_p\alpha$. Thus t_v and t_{-v} are smooth isomorphisms.

<u>Proof of Proposition 1.1</u>. By simple calculation one can check that

(1.8) $$\ell_v = \ell_o \circ (t_v)^{-1}_{*o} = \ell_o \circ (t_{-v})_{*v},$$

(1.9) $$d_p\alpha \circ \ell_v = d_v(d_p\alpha),$$

for any $\alpha \in C$. From (1.9) it follows that, for any $v \in T_pM$, the mapping ℓ_v is smooth. It remains to show that ℓ_o and $(t_{-v})_{*v}$ are smooth linear isomorphisms. It is easy to see that ℓ_o: $T_o(T_oM) \longrightarrow T_pM$ is a monomorphism. From Lemma 1.2 it follows that ℓ_o is an epimorphism. Thus ℓ_o is an isomorphism of linear spaces. From (1.9) it follows the following equality

(1.10) $$d_o(d_p\alpha) \circ \ell_o^{-1} = d_p\alpha, \quad \text{for any } \alpha \in C.$$

Thus ℓ_o^{-1} is smooth and ℓ_o is a smooth linear isomorphism. The smoothness of $(t_{-v})_{*v}$ follows from

(1.11) $$d_o(d_p\alpha) \circ (t_{-v})_{*v} = d_v(d_p\alpha), \quad \text{for any } \alpha \in C.$$

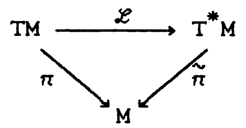
From Lemma 1.3 it follows that $(t_{-v})_{*p}$ is an isomorphism. Thus ℓ_v is a smooth isomorphism as a composition of smooth isomorphisms. Of course $\ell_v^{-1} = (t_v)_{*o} \circ \ell_o^{-1}$. Hence ℓ_v^{-1} is an isomorphism in the category of linear differential spaces.

Now, let L: $TM \longrightarrow \mathbb{R}$ be a smooth mapping. The mapping $\mathcal{L}$: $TM \longrightarrow T^*M$ defined by

(1.12) $$\mathcal{L}(v) = d_v(L|T_{\pi(v)}M) \circ \ell_v^{-1}, \quad \text{for } v \in TM,$$

is called the <u>Legendre transformation</u> on (M,C) corresponding to L.

<u>PROPOSITION 1.4</u>. The mapping $\mathcal{L}$ defined by (1.12) is smooth and the following diagram



commutes.

Proof. From (1.12) it follows

$$\tilde{\pi} \circ \mathcal{L} = \pi. \tag{1.13}$$

It is enough to show the smoothness of $\mathcal{L}$. Since $L \in TC$, for an arbitrary $v \in TM$ there exist functions $\alpha_1, \dots, \alpha_k \in C$, $\omega \in \mathcal{E}_{2k}$ and an open set $\pi^{-1}(U) \ni v$, where $U \in \tau_C$ is a neighbourhood of the point $\Pi(v)$ such that

$$L|\pi^{-1}(U) = \theta \circ (d\alpha_1, \dots, d\alpha_k, \ \alpha_1 \circ \pi, \dots, \alpha_k \circ \pi)|\pi^{-1}(U). \tag{1.14}$$

From (1.12), for any $X \in \mathfrak{X}(M)$ and $v \in TM$, it follows

$$(X^* \circ \mathcal{L})(v) = \ell_v^{-1}(X(\pi(v)))(L|T_{\pi(v)}M). \tag{1.15}$$

From (1.14) and (1.15) it follows that

$$X^* \circ \mathcal{L}|\pi^{-1}(v) =$$
$$= \sum_{i=1} \omega'|_i (d\alpha_1, \dots, d\alpha_k, \ \alpha_1 \circ \pi, \dots, \alpha_k \circ \pi)|\pi^{-1}(U) \cdot (X\alpha_i \circ \pi)|\pi^{-1}(U).$$

Thus $X \circ \mathcal{L}$ is smooth for any $X \in \mathfrak{X}(M)$. By (1.13) $\alpha \circ \tilde{\pi} \circ \mathcal{L} = \alpha \circ \pi$, for any $\alpha \in C$. Therefore $\mathcal{L}$ is smooth. This ends the proof.

Now, for a smooth function $L\colon TM \longrightarrow \mathbb{R}$ and for vectors $u, v \in T_pM$, $p \in M$, let $\Phi_{u,v}\colon \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$\Phi_{u,v}(t) = L(u+tv), \qquad \text{for } t \in \mathbb{R}. \tag{1.16}$$

One can check that

$$\mathcal{L}(u)(v) = \frac{d\Phi_{u,v}}{dt}\bigg|_{t=0}, \qquad \text{for any } v, u \in T_pM,\ p \in M. \tag{1.17}$$

Now we prove

PROPOSITION 1.5. Let g be a smooth Riemannian metric on (M,C) and $L\colon TM \longrightarrow \mathbb{R}$ be the smooth function defined by

$$L(u) = \frac{1}{2}\, g(u,u), \qquad \text{for } u \in TM. \tag{1.18}$$

Then the Legendre transformation $\mathcal{L}$ corresponding to L is a bijection and satisfies

$$\mathcal{L}(u)(v) = g(u,v), \qquad \text{for any } (u,v) \in (TM)^2. \tag{1.19}$$

If (M,C) is of constant differential dimension then $\mathscr{L}$ is a diffeomorphism.

Proof. To prove (1.19) let us notice that, for any $u,v \in T_pM$, $p \in M$, the function $\Phi_{u,v}$ defined by (1.16) satisfies

(1.20) $$\Phi_{u,v}(t) = \frac{1}{2}\, g(v,v)t^2 + g(u,v)t + \frac{1}{2}\, g(u,u).$$

Hence

$$\mathscr{L}(u,v) = \left.\frac{d\Phi_{u,v}}{dt}\right|_{t=0} = g(u,v).$$

It is evident that $\mathscr{L}$ is a bijection.

Now, let (M,C) be of constant differential dimension n. We shall show that $\mathscr{L}$ is a diffeomorphism. It is enough to show that $\mathscr{L}^{-1}$ is smooth. We should prove that, for an arbitrary $\alpha \in C$, the compositions $d\alpha\circ\mathscr{L}^{-1}$ and $\alpha\circ\pi\circ\mathscr{L}^{-1}$ are smooth. It is evident that $\alpha\circ\pi\circ\mathscr{L}^{-1} = \alpha\circ\tilde{\pi} \in T^*C$. It remains to show the smoothness of $d\alpha\circ\mathscr{L}^{-1}$. Let $W_1,\dots,W_n \in \mathfrak{X}(U)$ be a local vector basis given on an open set $U \in \tau_C$, and let $W_1^*,\dots,W_n^*$ be smooth 1-forms such that $W_i^*(p)(W_j(p)) = \delta_{ij}$, $i,j = 1,\dots,n$. We will verify the smoothness of the composition $d\alpha\circ\mathscr{L}^{-1}\circ\psi$, where $\psi\colon U\times\mathbb{R}^n \longrightarrow T^*U$ is a diffeomorphism defined by

(1.21) $$\psi(p,r_1,\dots,r_n) = \sum_{i=1}^{n} r_i W_i^*(p).$$

One can see that there exist the unique vector fields $A_1,\dots,A_n \in \mathfrak{X}(U)$ such that $W_i^*(X) = g(A_i,X)$, for any $X \in \mathfrak{X}(U)$, $i = 1,\dots,n$. From (1.19) it follows that $\mathscr{L}^{-1}(W_i^*(p)) = A_i(p)$, for any $p \in U$. Now, by simple calculation we obtain

$$(d\alpha\circ\mathscr{L}^{-1}\circ\psi)(p,r) = \Big(\sum_{i=1}^{n} r_i\mathscr{L}^{-1}(W_i^*(p))\Big)(\alpha) = \sum_{i=1}^{n} r_i A_i(p)(\alpha),$$

for any $(p,r) \in U\times\mathbb{R}$. Therefore $d\alpha\circ\mathscr{L}^{-1}\circ\psi$ is smooth.

Now we prove

PROPOSITION 1.6. Let (M,C) be a differential space of class D_0 and $L\colon TM \longrightarrow \mathbb{R}$ be a smooth function. If the Legendre transformation corresponding to L is a local diffeomorphism then (M,C) is regular (has locally constant differential dimension).

Proof. Let $p \in M$ be an arbitrary point. Assume that $\dim T_pM = k$. Let $v_1,\dots,v_k$ be a vector basis of T_pM. There exist functions $\alpha_1,\dots,\alpha_n \in C$ such that $v_i(\alpha_j) = \delta_{ij}$, for $i,j = 1,\dots,k$. Of course, $d_p\alpha_1,\dots,d_p\alpha_k$ is a basis of $T_p'M$. Any element $\theta \in T_p'M$ has a form $\theta = \sum_{i=1}^{n} \lambda^i d_p\alpha_i$, where $\lambda^1,\dots,\lambda^n \in \mathbb{R}$ and θ can be prolonged to the covector field $\sum_{i=1}^{n} \lambda^i d\alpha_i$. Since $\mathscr{L}$ is a local diffeomorphism, every vector $v \in T_pM$ can be prolonged to a smooth vector field $X \in \mathfrak{X}(M)$, i.e. $X(p) = v$. In fact, if ω is a smooth covector field such that $\mathscr{L}^{-1}(v) = \omega(p)$, then $X = \mathscr{L}^{-1}\circ\omega$ is a smooth vector field such that $X(p) = v$.

Now, let $X_1,\dots,X_k$ be such smooth vector fields that $X_i(p) = v_i$, for $i = 1,\dots,k$. Clearly, $X_i(p)(\alpha_i) = \delta_{ij}$, for $i,j = 1,\dots,k$. There exists an open neighbourhood V of p such that $\det(X_i(q)(\alpha_j)) \neq 0$, for $q \in V$. Therefore the vectors $X_1(q),\dots,X_k(q)$ are linearly independent and $\dim T_qM \geq k$, for any $q \in V$. Since (M,C) is of class D_0, there is an open neighbourhood U of p such that $\dim T_qM \leq k$, for $q \in U$ [13]. Thus $\dim T_qM = k$, for $q \in U\cap V$. Now, it is clear that $X_1|U\cap V,\dots,X_k|U\cap V$ is a local vector basis of $\mathfrak{X}(M)$ in a neighbourhood of p. Thus p is regular. This ends the proof.

Now, let $L\colon TM\times\mathbb{R} \longrightarrow$ be a smooth function. The mapping $\mathscr{L}\colon TM\times\mathbb{R} \longrightarrow T^*M\times\mathbb{R}$, defined by

$$\mathscr{L}(v,t) = (d_v(L_t|T_{\pi(v)}M)\circ\ell_v^{-1},\ t), \tag{1.22}$$

where $L_t(v) = L(v,t)$, for $v \in TM$, $t \in \mathbb{R}$, is called the <u>time-dependent Legendre transformation</u> corresponding to L.

Analogously as in Proposition 5.1, one can verify the smoothness of $\mathscr{L}$. Let $p \in M$ be a regular point of (M,C). Let $W_1,\dots,W_n$ be a local vector basis of $\mathfrak{X}(M)$ defined on an open neighbourhood U of p. Let $\dot{x}_i\colon \pi^{-1}(U) \longrightarrow \mathbb{R}$, $i = 1,\dots,n$, be the functions defined by

$$v = \sum_{i=1}^{n} \dot{x}_i(v)W_i(\pi(v)), \quad \text{for } v \in \pi^{-1}(U), \tag{1.23}$$

and let $y_i\colon \tilde{\pi}^{-1}(U) \longrightarrow \mathbb{R}$, $i = 1,\dots,n$, be the functions given by

$$\omega = \sum_{i=1} y_i(\omega)W_i^*(\pi(v)), \tag{1.24}$$

where $W_1^*,\dots,W_n^*$ is the dual basis to $W_1,\dots,W_r$. One can verify

[12] that the mapping $(\pi|U,\dot{x}_1,\dots,\dot{x}_n): \pi^{-1}(U) \longrightarrow U\times\mathbb{R}^n$ and $(\tilde{\pi}|U,y_1,\dots,y_n): \pi^{-1}(U) \longrightarrow U\times\mathbb{R}^n$ are diffeomorphisms. Denote by $\bar{W}_1,\dots,\bar{W}_n$ the lifts of $W_1,\dots,W_n$ onto $\pi^{-1}(U)\times\mathbb{R}$. From definition (1.22) it follows that

$$\mathscr{L}(v,t) = (\sum_{i=1}^{n}(\partial_{\bar{W}_i}L)(v,t)\cdot W_i^*(\pi(v)),t),$$

for $v \in \pi^{-1}(U)$, $t \in \mathbb{R}$, or equivalently

$$\mathscr{L}|\pi^{-1}(U)\times\mathbb{R} = (\sum_{i=1}^{n}\partial_{\bar{W}_i}L\cdot W_i^*\circ\pi, \tau_{\pi^*_M}|\pi^{-1}(U)\times\mathbb{R}), \tag{1.25}$$

where $\tau_{\pi^*_M}: T^*M\times\mathbb{R} \longrightarrow \mathbb{R}$ is the projection onto $\mathbb{R}$. Now, it is obvious that

$$y^i\circ pr_1\circ\mathscr{L} = \partial_{W_i}L, \quad \text{for } i = 1,\dots,n, \tag{1.26}$$

$$\tau_{\pi^*_M}\circ\mathscr{L} = \tau_{TM}, \quad \tilde{\pi}\circ pr_1\circ\mathscr{L} = \pi. \tag{1.27}$$

2. PHYSICAL INTERPRETATIONS. The fundamental equations of dynamics may essentially be expressed in the two forms, closely related with each other. One can follow either Lagrange and formulate the equations in terms of a set of generalized coordinates and velocities, or Hamilton who gave them an alternative form in terms of generalized coordinates and momenta. One needs both forms to work with. The postulate of the relativistic invariance can easily be discussed within the framework of the Lagrange formulation, and for the purpose of quantum theory the Hamiltonian form is required. Geometrically, the two formulations are organized on the tangent and cotangent bundle, correspondingly, and are connected with each other by the so-called Legendre transformation.

The macroscopic spacetime of contemporary relativistic physics should be doubtlessly modeled by a four-dimensional differentiable manifold, but sufficiently near to cosmological singularities or at extremely small scales the differentiable manifold model is commonly believed to break down. Also the space of states of many physical systems does not allow for the differentiable manifold structure. It turns out that the differential space approach is a very efficient tool in dealing with the above mentioned problems.

Analogously as it was for the differentiable manifold case, one can consider dynamical systems on differential spaces. In particular they may by Hamiltonian or Lagrangian dynamical systems. Generally, a Hamiltonian dynamical system on a differential space (M,C) may be defined in the following way. Let $\langle\cdot,\cdot\rangle\colon C\times C \longrightarrow C$ be a mapping such that $(C,\langle\cdot,\cdot\rangle)$ is a real Lie algebra and $\langle fg,h\rangle = f\langle g,h\rangle + g\langle f,h\rangle$, for any $f,g,h \in C$. Then, of course, $X_h := \langle\cdot,h\rangle$ is a smooth vector field on (M,C), for $h \in C$. The pair $((M,C),\langle\cdot,\cdot\rangle)$ is called the Poisson differential space.

EXAMPLE. Let X_i, Y_i, $i = 1,\dots,n$, be smooth and commutative vector fields on (M,C), i.e. $[X_i,X_k] = [X_i,Y_k] = [Y_i,Y_k] = 0$, $i,k = 1,\dots,n$. Then the mapping $\langle\cdot,\cdot\rangle\colon C\times C \longrightarrow C$, defined as $\langle f,g\rangle := \sum_i X_i f\cdot Y_i - Y_i f\cdot X_i g$, is a Poisson structure on (M,C).

From the physical point of view, a Poisson structure defined on a differential space determines certain kinematic conditions on the space, i.e. it determines the general form of the equations responsible for the evolution of a process whose states are considered to be points of the differential space in question. A chosen function $h \in C$, which is assumed to be fixed for further considerations, determines the so-called Hamiltonian dynamical system $X_h := \langle\cdot,h\rangle$ on (M,C). The differential space (M,C) is called then the space of states or the phase space of the Hamiltonian system X_h. The chosen function h, the so-called hamiltonian, is responsible for the dynamics of our system. A function $f \in C$ is called the Casimir function of the given Poisson structure $\langle\cdot,\cdot\rangle$ if $\langle g,f\rangle = 0$, for any $g \in C$. Two distinct functions h_1, $h_2 \in C$ determine the same dynamics if the difference $h_1 - h_2$ is a Casimir function, i.e. $\langle f,h_1 - h_2\rangle = 0$, for any $f \in C$.

Let $X_h = \langle\cdot,h\rangle$ be a Hamiltonian system with a hamiltonian $h \in C$. Then the time dependence of states of the system (evolution of the process) is described by the integral curve $\gamma\colon I \longrightarrow M$ of the field X_h, i.e. $\dot\gamma(t) \equiv d\gamma(e_t) = X_h(\gamma(t))$, where $e_t := \frac{d}{ds}\Big|_{s=t} \in T_tI$, $I \subset \mathbb{R}$ is an interval in $\mathbb{R}$ considered here to be a differential subspace of the differential space of real numbers $(\mathbb{R}, C^\infty(\mathbb{R}))$. The functions of the family C we call the dynamical quantities or observables. The time dependence of a dynamical quantity $f \in C$ along a trajectory γ of the system X_h is determined by the function $\dot{f}: M \longrightarrow \mathbb{R}$, defined as $\dot{f}(\gamma(t)) := e_t(f \circ \gamma)$. One can easily check that the function $\dot{f}$ fulfills the equation $\dot{f} = \{f, h\}$, which is known as the evolution equation for the dynamical quantity f.

The above presented approach to Hamiltonian dynamical systems is far beyond the domain of Hamiltonian systems formulated with the help of a chosen symplectic form on the space of states. In particular, the above formulation allows us to consider Hamiltonian systems on a phase space of odd dimension [3], infinite dimension, or even variable (depending on a point) dimension.

Let (M, C) be a differential space of constant differential dimension n. If a symplectic form ω is given on (M, C), then it defines a relation $\alpha = i_X\omega$ $(X \equiv X_\alpha)$ of smooth vector fields and 1-forms. A form ω equips (M, C) with the Poisson structure $\{f, g\} := -\omega(X_{df}, X_{dg})$, $f, g \in C$. There is the particular situation when the space of states (M, C) is a cotangent space of some (configurational) differential space (Q, F), i.e. $M = T^*Q$. The Legendre transformation, defined above, allows us to pass from the Hamiltonian formulation of the problem on T^*M to the corresponding Lagrangian formulation on the tangent space TQ. Let $L: TQ \longrightarrow \mathbb{R}$ be a smooth function, the so called lagrangian. The Legendre transformation $\mathcal{L}: TQ \longrightarrow T^*Q$ allows for the definition $\omega_L := \mathcal{L}^*\omega$. If L is a regular lagrangian, i.e. if $\mathcal{L}$ is a diffeomorphism, ω_L is a symplectic form on TQ. For the lagrangian L we define its action $A: TQ \longrightarrow \mathbb{R}$, $A(v) := (\mathcal{L}(v))(v)$, and the energy $E := A - L$. The vector field $X_L: TQ \longrightarrow T(TQ)$ is said to be the lagrangian vector field if

$$i_{X_L}\omega_L = -dE.$$

If L is a regular lagrangian, the hamiltonian corresponding to L is given by $H = E \circ \mathcal{L}^{-1}$, and the Hamiltonian dynamical system

$X_H = \langle \cdot, H\rangle$ is related with the Lagrangian system X_L by

$$\mathcal{L}_* X_L = X_H.$$

In principle, we can consider an abstract Hamiltonian system on a tangent space TQ to a differential space (Q,F) which is assumed to be specified with the help of a chosen Poisson structure $\langle \cdot, \cdot \rangle_{TQ}$ on TQ. Namely, for a lagrangian L we define the energy E = A − L, and the dynamical system $X_L := \langle \cdot, E\rangle_{TQ}$. If the lagrangian L is regular, the Legendre transformation $\mathcal{L}$ allows us for the equivalent formulation on the cotangent space T^*Q:

$$H := \mathcal{L}^* E,$$

$$\langle \mathcal{L}^* f, \mathcal{L}^* g\rangle_{T^*Q} := \mathcal{L}^* \langle f, g\rangle_{TQ}, \text{ for } f, g \in F,$$

$$X_H := \langle \cdot, H\rangle_{T^*Q},$$

$$\mathcal{L}_* X_L = X_H.$$

Thus we see that, for a regular lagrangian case, the Legendre transformation turns out to be the isomorphism of Poisson spaces, also called the canonical transformation.

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