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Constructing Subsets of a Given Packing Index in Abelian Groups

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By definition, the sharp packing index $\mathrm{ind}_P^\sharp(A)$ of a subset A of an Abelian group G is the smallest cardinal κ such that for any subset $B \subset G$ of size $|B| \geq \kappa$ the family $\{b + A : b \in B\}$ is not disjoint. We prove that an infinite Abelian group G contains a subset A with given index $\mathrm{ind}_P^\sharp(A) = \kappa$ if and only if one of the following conditions holds: (1) $2 \leq \kappa \leq |G|^+$ and $\kappa \notin \{3,4\}$; (2) $\kappa = 3$ and G is not isomorphic to $\oplus_{i\in I}\mathbb{Z}_3$; (3) $\kappa = 4$ and G is not isomorphic to $\oplus_{i\in I}\mathbb{Z}_2$ or to $\mathbb{Z}_4 \oplus (\oplus_{i\in I}\mathbb{Z}_2)$.

The famous problem of optimal sphere packing traces its history back to B. Pascal and belongs to the most difficult problems of combinatorial geometry [CS]. In this paper we consider an analogous problem in the algebraic setting. Namely, given a non-empty subset A of an Abelian group G we study the cardinal number

$$\mathrm{ind}_P(A) = \sup\{|B| : B \subset G \text{ and } (B - B) \cap (A - A) = \{0\}\}$$

called the *packing index* of A in G. Note that the equality $(B - B) \cap (A - A) = \emptyset$ holds if and only if $(b + A) \cap (b' + A) = \emptyset$ for any distinct points $b, b' \in B$. Therefore, $\mathrm{ind}_P(A)$ can be thought as the maximal number of pairwise disjoint shift copies of A that can be placed in the group G. In this situation it is natural to ask if such a maximal number always exists. In fact, this was a question of D. Dikranjan and I. Protasov who asked in [DP] if for each subset $A \subset \mathbb{Z}$ with $\mathrm{ind}_P(A) \geq \aleph_0$ there exists an infinite family of pairwise disjoint shifts of A. The answer to this problem turned out to be negative, see [BL$_1$], [BL$_2$]. So the supremum in the definition of $\mathrm{ind}_P(A)$ cannot be replaced by the maximum.

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To catch the difference between sup and max, let us adjust the definition of the packing index $\mathrm{ind}_P(X)$ and define the cardinal number

$$\mathrm{ind}_P^\sharp(A) = \min\{\kappa : \forall B \subset G\, |B| \geq \kappa \Rightarrow (B - B \cap A - A \neq \{0\})\}$$

called the *sharp packing index* of A in G. In terms of the sharp packing index the question of D. Dikranjan and I. Protasov can be reformulated as finding a subset $A \subset \mathbb{Z}$ with $\mathrm{ind}_P^\sharp(A) = \aleph_0$. According to [BL2] (and [BL1]) such a set A can be found in each infinite (abelian) group G. Having in mind this result, I. Protasov asked in a private conversation if for any non-zero cardinal $\kappa \leq |G|$ there is a set $A \subset G$ with $\mathrm{ind}_P(A) = \kappa$. In this paper we answer this question affirmatively (with three excepions). Firstly, we treat a similar question for the sharp packing index because its value completely determines the value of $\mathrm{ind}_P(A)$:

$$\mathrm{ind}_P(A) = \sup\{\kappa : \kappa < \mathrm{ind}_P^\sharp(A)\}.$$

Our principal result is

Main Theorem. *An infinite Abelian group G contains a subset $A \subset G$ with sharp packing index $\mathrm{ind}_P^\sharp(A) = \kappa$ if and only if one of the following conditions holds:*
(1) $2 \leq \kappa \leq |G|^+$ *and* $\kappa \notin \{3,4\}$.
(2) $\kappa = 3$ *and G is not isomorphic to* $\bigoplus_{i\in I}\mathbb{Z}_3$.
(3) $\kappa = 4$ *and G is not isomorphic to* $\bigoplus_{i\in I}\mathbb{Z}_2$ *or to* $\mathbb{Z}_4 \oplus (\bigoplus_{i\in I}\mathbb{Z}_2)$.

Using the relation between the packing and sharp packing indices, we can derive from the above theorem an analogous characterization of possible values of the packing index.

Corollary. *An infinite Abelian group G contains a subset $A \subset G$ with packing index $\mathrm{ind}_P(A) = \kappa$ if and only if one of the following conditions holds:*
(1) $1 \leq \kappa \leq |G|$ *and* $\kappa \notin \{2,3\}$.
(2) $\kappa = 2$ *and G is not isomorphic to* $\bigoplus_{i\in I}\mathbb{Z}_3$.
(3) $\kappa = 3$ *and G is not isomorphic to* $\bigoplus_{i\in I}\mathbb{Z}_2$ *or to* $\mathbb{Z}_4 \oplus (\bigoplus_{i\in I}\mathbb{Z}_2)$.

1. Preliminaries

In the proof of Main Theorem we shall exploit a combinatorial lemma proved in this section. For a set A by $[A]^2 = \{B \subset A : |B| = 2\}$ we denote the family of all two element subsets of A.

We shall say that a map $f : [A]^2 \mapsto [B]^2$

- is *separately injective* if for any $a \in A$ the map $f_a : x \mapsto f(\{x,a\})$ is injective;
- *preserves intersection* if for any $a_0, a_1, a_2 \in A$ the intersection $f(\{a_0, a_1\}) \cap \cap f(\{a_0, a_2\})$ is not empty.

Lemma 1. *If $|A| \geq 5$ and a map $f: [A]^2 \mapsto [B]^2$ is separately injective and preserves intersections, then $|A| \leq |B|$.*

Proof. Fix any point $a_0 \in A$ and consider the family $\{f(\{a,a_0\}): a \in A \setminus \{a_0\}\}$. Since f preserves intersection we have that $f(\{a,a_0\}) \cap f(\{a',a_0\}) \neq \emptyset$ for any distinct $a, a' \in A$. Using the separate injectivity of f and the inequality $|A| \geq 5$ we can prove that the intersection $\bigcap_{a \in A \{a_0\}} f(\{a,a_0\})$ is not empty and hence contains some element b_0. Thus we obtain that $f: \{a,a_0\} \mapsto \{b,b_0\}$. And since f is separately injetive we obtain an injective map from $A \setminus \{a_0\}$ into $B \setminus \{b_0\}$ implying the desired inequality $|A| \leq |B|$. □

We shall also need one structure property of Abelian groups. By $\mathbb{Z}$ we denote te additive group of integer numbers and by

$$\mathbb{Z}(p^\infty) = \{z \in \mathbb{C} : \exists n \in \mathbb{N} \text{ with } z^{p^n} = 1\}$$

the quasicyclic p-group for a prime number p.

Proposition 1. *Each infinite Abelian group G contains an infinite subgroup isomorphic to $\mathbb{Z}$, $\mathbb{Z}(p^\infty)$ or the direct sum of finite cyclic groups.*

Proof If G contains an element g of infinite order, then it generates a cyclic subgroup isomorphic to $\mathbb{Z}$. Otherwise, H is a torsion group and by Theorem 8.4 [Fu] decomposes into the direct sum $G = \bigoplus_p A_p$ of p-groups A_p. If each group A_p is finite, then G contains an infinite direct product of finite cyclic group. If for some prime number p the p-group A_p is infinite, then there are two posibilities. Either A_p contains a copy of the quasicyclic p-group $\mathbb{Z}(p^\infty)$ or else each element of A_p has finite height. In the latter case, take any infinite countable subgroup $H \subset A_p$ and apply Theorem 17.3 of [Fu] to conclude that H is the direct sum of finite cyclic groups. □

2. The proof of the "only if" part of main theorem

The proof of the "only if" part of Main Theorem is divided into two lemmas.

Lemma 2. *If a group G contains a subset $A \subset G$ with $ind_P^\sharp(A) = 3$ (which is equivalent to $\mathrm{ind}_P(A) = 2$), then G is not isomorphic to the direct sum $\bigoplus_{i \in I} \mathbb{Z}_3$.*

Proof. On the contrary suppose that G is isomorphic to the direct sum $\bigoplus_{i \in I} \mathbb{Z}_3$ and take a subset A of G with $\mathrm{ind}_P^\sharp(A) = 3$. The latter is equivalent to $\mathrm{ind}_P(A) = 2$ which means that there is a subset $B_2 \subset G$ of size 2 such that the family $\{b + A \ \ b \in B_2\}$ is disjoint. Note that for every $b' \in G$ the family $\{b + A : b \in \in b' + B_2\}$ is disjoint too. So without loss of generality we can assume that $B_2 = \{0, b_1\}$. The family $\{b + A : b \in B_2\}$ is disjoint and hence

$$A \cap (b_1 + A) = \emptyset.$$

Adding to both sides b_1 and $2b_1$ we get

$$(b_1 + A) \cap (2b_1 + A) = \emptyset;$$
$$(2b_1 + A) \cap (3b_1 + A) = \emptyset.$$

Since G is isomorphic to the direct sum $\bigoplus_{i \in I} \mathbb{Z}_3$ we get $3b_1 = 0$. Thus we conclude that $\{b_1 + A : b \in \{0, b_1, 2b_1\}\}$ is disjoint and so $\mathrm{ind}_P(A) > 2$ and $ind_p^\sharp(A) > 3$, which contradicts our assumptions. □

Lemma 3. *If a group G contains a subset $A \subset G$ with $ind_p^\sharp(A) = 4$ (which is equivalent to $\mathrm{ind}_P(A) = 3$), then G can not be isomorphic neither to the direct sum $\bigoplus_{i \in I} \mathbb{Z}_2$ nor to $\mathbb{Z}_4 \oplus (\bigoplus_{i \in I} \mathbb{Z}_2)$.*

Proof. Conversely suppose that G is isomorphic to $\bigoplus_{i \in I} \mathbb{Z}_2$ or to $\mathbb{Z}_4 \oplus (\bigoplus_{i \in I} \mathbb{Z}_2)$ and there exists a subset A of G with $\mathrm{ind}_P^\sharp(A) = 4$. This is equivalent to $\mathrm{ind}_P(A) = 3$ and from the definition we get that there is a three-element subset $B_3 \subset G$ such that the family $\{b + A : b \in B_3\}$ is disjoint. Note that for any $b' \in G$ the family $\{b + A : b \in b' + B_3\}$ is disjoint too. So, without loss of generality we can assume that $B_3 = \{0, b_1, b_2\}$. Since the family $\{b + A : b \in B_3\}$ is disjoint we conclude that

$$\begin{array}{ll} (1) & A \cap (b_1 + A) = \emptyset; \\ (2) & A \cap (b_2 + A) = \emptyset; \\ (3) & (b_1 + A) \cap (b_2 + A) = \emptyset. \end{array}$$

We consider three cases.

Case 1. Suppose one of the element b_1, b_2 is of order 2. Let it be b_1. Then $2b_1 = 0$ and

$$\begin{array}{ll} (2) + b_1: & (b_1 + A) \cap (b_1 + b_2 + A) = \emptyset; \\ (3) + b_1: & A \cap (b_1 + b_2 + A) = \emptyset. \\ (1) + b_2: & (b_2 + A) \cap (b_2 + b_1 + A) = \emptyset. \end{array}$$

Thus we get that the family $\{b + A : b \in \{0, b_1, b_2, b_1 + b_2\}\}$is disjoint and hence $\mathrm{ind}_P(A) > 3$ and $\mathrm{ind}_p^\sharp(A) > 4$, which contradicts our assumption. Thus we complete the proof of the Case 1.

Next we consider two cases both b_1 and b_2 are of order 4. In this case the group G is isomorphic to $\mathbb{Z}_4 \oplus (\bigoplus_{i \in I} \mathbb{Z}_2)$. Therefore there are two possibilities: $b_1 = (g, x)$, $b_2 = (g, y)$ or $b_1 = (g, x)$, $b_2 = (-g, y)$ where $x, y \in \bigoplus_{i \in I} \mathbb{Z}_2$ and $g \in \mathbb{Z}_4$ is order 4.

Case 2. Suppose $b_1 = (g, x)$, $b_2 = (g, y)$ where $x, y \in \bigoplus_{i \in I} \mathbb{Z}_2$ and $g \in \mathbb{Z}_4$ is of order 4.

Recall that $B_3 = \{(0,0), (g, x), (g, y)\}$ and consider the set

$$B_4 = \{(0,0), (g, x), (g, y), (0, x + y)\}.$$

We claim that the family $\{b + A : b \in B_4\}$ is disjoint. Indeed, since $\{b + A : : b \in B_3\}$ is disjoint we have:

$$
\begin{array}{ll}
(1) & A \cap ((g,x) + A) = \emptyset; \\
(2) & A \cap ((g,y) + A) = \emptyset; \\
(3) & ((g,x) + A) \cap ((g,y) + A) = \emptyset.
\end{array}
$$

Then

$$
\begin{array}{lll}
(3) + (3g,y) & : & ((0,x + y) + A) \cap A = \emptyset; \\
(2) + (0,x + y) & : & ((0,x + y) + A) \cap ((g,x) + A) = \emptyset; \\
(1) + (0,x + y) & : & ((0,x + y) + A) \cap ((g,y) + A) = \emptyset;
\end{array}
$$

Hence, the family $\{b + A : b \in B_4\}$ is disjoint which implies $\mathrm{ind}_P(A) > 3$ and $\mathrm{ind}_P^\sharp(A) > 4$, a contradiction with the assumption.

Case 3. Suppose $b_1 = (g,x)$, $b_2 = (-g,y)$ where $x, y \in \oplus_{i \in I}\mathbb{Z}_2$ and $g \in \mathbb{Z}_2$ and $g \in \mathbb{Z}_4$ is of order 4.

In this case $B_3 = \{(0,0),(g,x),(-g,y)\}$. Put $B_4 = \{(0,0),(g,x),(-g,y),(2g,x + y)\}$. We claim that the family $\{b + A : b \in B_4\}$ is disjoint. Indeed, since $\{b + A : : b \in B_3\}$ is disjoint we have:

$$
\begin{array}{ll}
(1) & A \cap ((g,x) + A) = \emptyset; \\
(2) & A \cap ((-g,y) + A) = \emptyset; \\
(3) & ((g,x) + A) \cap ((-g,y) + A) = \emptyset.
\end{array}
$$

Then

$$
\begin{array}{lll}
(3) + (g,y) & : & ((2g,x + y) + A) \cap A = \emptyset; \\
(2) + (2g,x + y) & : & ((2g,x + y) + A) \cap ((g,x) + A) = \emptyset; \\
(1) + (2g,x + y) & : & ((2g,x + y) + A) \cap ((-g,y) + A) = \emptyset.
\end{array}
$$

Hence the family $\{b + A : b \in B_4\}$ is disjoint and thus $\mathrm{ind}_P(A) > 3$ and $\mathrm{ind}_P^\sharp(A) > 4$, which contradicts our assumption. □

Thus if G contains a subset $A \subset G$ with $\mathrm{ind}_P^\sharp(A) = \kappa$ then one of the condition 1)$-$3) holds.

3. The proof of the if part "of" main theorem

To prove the "if" part of the Main Theorem, we should construct a subset $A \subset G$ with $\mathrm{ind}_P^\sharp(A) = \kappa$ for any cardinal κ satisfying one of the conditions 1)$-$3) First we shall construct a subset A_κ assuming that we have in disposal an auxiliary subset $\mathbb{B}_\kappa$ with some properties. Next, a subset $\mathbb{B}_\kappa$ with the desired properties will be constructed in each group G.

Proposition 2. *An infinite Abelian group G contains a subset A_κ with $\mathrm{ind}_P^\sharp(A_\kappa) = \kappa$ if there exists a subset $\mathbb{B}_\kappa = -\mathbb{B}_\kappa$ of G with the following properties:*

(1_κ) *for every cardinal* $\alpha < \kappa$ *there is a subset* B_α *of size* $|B_\alpha| = \alpha$ *such that* $B_\alpha - B_\alpha \subset \mathbb{B}_\kappa$;

(2_κ) $B_\kappa - B_\kappa \not\subset \mathbb{B}_\kappa$ *for any subset* $B_\kappa \subset G$ *of size* κ;

(3_κ) $F + \mathbb{B}_\kappa \neq G$ *for any subset* $F \subset G$ *of size* $|F| < |G|$.

By $|A|$ we denote the cardinality of a set A.

Proof. Let $\mathbb{B}_\kappa = \mathbb{B}_\kappa \backslash \{0\}$. We shall construct a subset $A_\kappa \subset G$ such that $(\mathbb{B}_\kappa + A_\kappa) \cap A_\kappa = \emptyset$. Moreover, the subset A_κ will be constructed so that $G \backslash \mathbb{B} \subset \subset A_\kappa - A_\kappa$.

Let $\lambda = |G \backslash \mathbb{B}_\kappa|$ and $G \backslash \mathbb{B}_\kappa = \{g_\alpha : \alpha < \lambda\}$ be an enumeration of $G \backslash \mathbb{B}_\kappa$ by ordinals $\alpha < \lambda$.

We put $A_\kappa = \bigcup_{\alpha<\lambda} \{a_\alpha, g_\alpha + a_\alpha\}$, where a sequence $(a_\alpha)_{\alpha<\lambda}$ is to be defined later. This clearly forces that $G \backslash \mathbb{B}_\kappa \subset A_\kappa - A_\kappa$.

The task is now to find a sequence $(a_\alpha)_{\alpha<\lambda}$ such that $(\mathbb{B}_\kappa + A_\kappa) \cap A_\kappa = \emptyset$. We define this sequence by induction.

We start with $a_0 = 0$. Assuming that for some α the points $a_\beta, \beta < \alpha$, have been constructed, put $F_\alpha = \{a_\beta, g_\beta + a_\beta : \beta < \alpha\}$.

According to the property (3_κ) of the set $\mathbb{B}_\kappa$ we can pick a point $a_\alpha \in G$ so that

$$a_\alpha \notin (F_\alpha + \mathbb{B}_\kappa) \cup (F_\alpha - g_\alpha + \mathbb{B}_\kappa).$$

This gives $(\mathbb{B}_\kappa^\circ + A_\kappa) \cap A_\kappa = \emptyset$.

It remains to show that A_κ satisfies the conclusion of the theorem.

According to the property (1_κ) of the set $\mathbb{B}_\kappa$, for any cardinal $\alpha < \kappa$ there is B_α such that $B_\alpha - B_\alpha \subset \mathbb{B}_\kappa$. From the fact that $\mathbb{B}_\kappa + A_\kappa \cap A_\kappa = \emptyset$ we conclude that $(b - b' + A_\kappa) \cap A_\kappa = \emptyset$ for all distinct $b, b' \in B_\alpha$. Thus for any cardinal $\alpha < \kappa$ there is B_α such that the family $\{b + A_\kappa : b \in B_\alpha\}$ is disjoint and so $\mathrm{ind}_P^\sharp(A_\kappa) \geq \kappa$.

Let us show that $\mathrm{ind}_P^\sharp(A_\kappa) = \kappa$. According to the property (2_κ), for any subset $B_\kappa \subset G$ of size κ there are $b, b' \in B_\kappa$ such that $b, b' \notin \mathbb{B}_\kappa$. Therefore $b - b' \in G \backslash \mathbb{B}_\kappa \subset A_\kappa - A_\kappa$. Hence $b + A_\kappa \bigcap b' + A_\kappa \neq \emptyset$, which yields $\mathrm{ind}_P^\sharp(A_\kappa) \leq \kappa$. Combining the two inequalities, we get $\mathrm{ind}_P^\sharp(A_\kappa) = \kappa$. $\square$

The proof of the Main Theorem will be completed as soon as we construct a subset $\mathbb{B}_\kappa$ with properties $(1_\kappa) - (3_\kappa)$. This will be done in the following five lemmas.

Lemma 4. *Let* $\kappa = 3$ *and* G *be an infinite Abelian group which is not isomorphic to the direct sum* $\oplus_{i\in I}\mathbb{Z}_3$. *Then* G *contains a subset* $\mathbb{B}_3$ *with the properties* $(1_3) - (3_3)$.

Proof. Pick any nonzero point $g \in G$ whose order is not equal to 3 and consider the set $\mathbb{B}_3 = B_2 - B_2 = \{0, \pm g\}$ where $B_2 = \{0, g\}$. It is clear that $\mathbb{B}_3$ has the properties (1_3), (3_3). So it is enough to show that $\mathbb{B}_3$ satisfies the property (2_3). Note that if $2g = 0$ then $\mathbb{B}_3 = \{0, g\}$ is a subgroup of G and hence has the property (2_3).

So we assume that $2g \neq 0$ which yields that $\mathbb{B}_3 = \{0, g, -g\}$ contains three elements. To prove that $\mathbb{B}_3$ has property (2_3) fix some subset $B_3 \subset G$ of size 3 and pick any point $b_0 \in B_3$. If there is $b \in B_3$ with

$$b = b_0 \notin \mathbb{B}_3 = \{0, g, -g\}$$

then there is nothing to prove. Otherwise we have that

$$B_3 - b_0 \subset \mathbb{B}_3.$$

Since $|B_3| = 3$ there are $b, b' \in B_3$ such that $b - b_0 = g$; $b' - b_0 = -g$. Hence we get $b - b' = 2g$. From the choice of elements g we get that $2g \notin \mathbb{B}_3$. Hence $b_2 - b \notin \mathbb{B}_3$ and $\mathbb{B}$ has the property (2_3) which completes the proof of the lemma. □

Lemma 5. *Let* $\kappa = 4$ *and* G *be an infinite Abelian group which is not isomorphic to* $\oplus_{\iota \in I} \mathbb{Z}_2$ *or to* $\mathbb{Z}_4 \oplus (\oplus_{\iota \in I} \mathbb{Z}_2)$. *Then* G *contains a subset* $\mathbb{B}_4$ *with properties* $(1_4) - (3_4)$.

Proof. We consider three cases.

Case 1. Suppose a group G contains an element g with order >5.

Put $\mathbb{B}_4 = B_3 - B_3 = \{0, \pm g, \pm 2g\}$ where $B_3 = \{0, g, -g\}$. It is easily to check that $\mathbb{B}_4$ has the properties $(1_4), (3_4)$. We claim that $\mathbb{B}_4$ satisfies the property (2_4).

To derive a contradiction, suppose that there is a subset $B_4 \subset G$ of size $|B_4| = 4$ such that $B_4 - B_4 \subset \mathbb{B}_4 = \{0, g, -g, 2g, -2g\}$.

Fix some element $b_0 \in B_4$. Since $B_4 - b_0 \subset \mathbb{B}_4$ there are $b, b' \in B_4$ such that $b - b_0 = -g$; $b' - b_0 = 2g$ or $b - b_0 = g$; $b' - b_0 = -2g$.

Then $b' - b = 3g$ or $b' - b = -3g$

Note that since the order of g is greater than 5, neither $3g \in \mathbb{B}_4$ nor $-3g \in \mathbb{B}_4$. Thus we get $b' - b \notin \mathbb{B}_4$, a contradiction with the assumption. Hence $\mathbb{B}_4$ satisfies the property (2_4) and we complete the proof of Case 1.

Case 2. Assume that G contains no element of order greater that 5. Then G is the direct sum of cyclic groups according to Theorem 17.2 of [Fu]. More precisely, G is isomorphic either to $(\oplus_{i \in I} \mathbb{Z}_2) \oplus (\oplus_{j \in J} \mathbb{Z}_4)$ or to $\oplus_{i \in I} \mathbb{Z}_3$ or to $\oplus_{i \in I} \mathbb{Z}_5$. Since G is not isomorphic to $\oplus_{i \in I} \mathbb{Z}_2$ or $\mathbb{Z}_4 \oplus (\oplus_{i \in I} \mathbb{Z}_2)$, we have to consider the following two cases: G contains a subgroup isomorphic to $\mathbb{Z}_3$ and G contains a subgroup isomorphic to $\mathbb{Z}_i \oplus \mathbb{Z}_j$ for some $4 \leq i, j \leq 5$.

Case 2a. Suppose that G contains a subgroup H isomorphic to $\mathbb{Z}_3$.

In this case we put $\mathbb{B}_4 = H$ and see that $\mathbb{B}_4$ has the properties $(1_4) - (3_4)$.

Case 2b. Suppose G contains a subgroup isomorphic to the direct sum of $\mathbb{Z}_i \oplus \mathbb{Z}$ for some $4 \leq i, j \leq 5$.

We shall identify $\mathbb{Z}_i \oplus \mathbb{Z}_j$ with a subgroup of G and shall find a subset $\mathbb{B}_4 \subset \mathbb{Z}_i \oplus \mathbb{Z}_j$ with the properties $(1_4) - (3_4)$. Obviously $\mathbb{B}_4$ has the same properties in the whole group G.

Put $\mathbb{B}_4 = B_3 - B_3$ where $B_3 = \{(0,0),(g_1,0),(0,g_2)\}$. It is clear that $\mathbb{B}_4$ has the properties $(1_4),(3_4)$. We claim that $\mathbb{B}_4$ has property (2_4). Indeed, assuming the converse, we would find a subset $B_4 \subset G$ of size $|B_4| = 4$ with $B_4 - B_4 \subset \subset B_3 - B_3$.

Fix any point $b_0 \in B_4$. Then

$$B_4 - b_0 \subset \mathbb{B}_4 = \{(0,0),(g_1,0),(0,g_2),(-g_1,0),(0,-g_2),(g_1,-g_2),(-g_1,g_2)\}.$$

Let us show that $(g_1,0) \notin B_4 - b_0$. Since the elements g_1 and g_2 have order ≥ 4,

$$\begin{aligned}(g_1,0) - (-g,0) &\notin \mathbb{B}_4;\\ (g_1,0) - (-g_1,g_2) &\notin \mathbb{B}_4;\\ (g_1,0) - (0,-g_2) &\notin \mathbb{B}_4.\end{aligned}$$

Thus if there is $b \in B_4$ with $b - b_0 = (g_1,0)$ then

$$B_4 - b_0 \subset \mathbb{B}_4 = \{(0,0),(g_1,0),(0,g_2),(g_1,-g_2)\}.$$

From the above and the fact that $|B_4| = 4$ we get that there are $b_1, b_2 \in B_4$ such that $b_1 - b_0 = (0,g_2)$ and $b_2 - b_0 = (g_1,-g_2)$. Hence $b_2 - b_1 = (g_1,-2g_2) \notin \mathbb{B}_4$, a contradiction with the assumption that $B_4 - B_4 \subset \mathbb{B}_4$. So, we conclude that $(g_1,0) \notin B_4 - b_0$.

In the same manner we can show that none of the elements $(0,g_2)$, $(-g_1,0)$, $(0,-g_2)$ belong to $B_4 - b_0$, which contradicts the fact that $B_4 - B_4 \subset \mathbb{B}_4$. This completes the proof of Lemma. □

Lemma 6. *If $\kappa > 4$ is a finite cardinal, then each infinite Abelian group G contains a subset $\mathbb{B}_\kappa$ with the properties* $(1_\kappa) - (3_\kappa)$.

Proof. It is easy to check that each subset $\mathbb{B}_\kappa$ with the properties $(1_\kappa) - (3_\kappa)$ in a subgroup $H \subset G$ has these properties in the whole group G. This observation combined with Proposition 1 reduces the problem to constructing a set $\mathbb{B}_\kappa$ in the groups $\mathbb{Z}, \mathbb{Z}(p^\infty)$ or the direct sum of finite cyclic groups. This will be done separately in the following three cases.

Case 1. We construct a subset $\mathbb{B}_k$ in the group $\mathbb{Z}$.

In this case put $\mathbb{B}_\kappa = B_{\kappa\ 1} - B_{\kappa-1}$ where $B_{\kappa\ 1} = \{i \colon 1 \leq i \leq \kappa - 1\}$. It is easy to check that $\mathbb{B}_\kappa$ has properties $(1_\kappa) - (3_\kappa)$ in $\mathbb{Z}$.

Case 2. We construct a sbset $\mathbb{B}_\kappa$ in the quasicyclic p-group $\mathbb{Z}(p^\infty)$.

Choose n such that $z^{p^n} \in \{e^{i\phi} : \frac{2\pi}{\kappa} < \phi < \frac{2\pi}{\kappa-1}\}$. Then put $\mathbb{B}_\kappa = B_{\kappa-1} - B_{\kappa\ 1}$ where

$$B_{\kappa\ 1} = \left\{e^{i\phi} : \phi = \frac{2\pi l}{p^n}, 1 \leq l \leq \kappa - 1\right\}.$$

It is easy to check that $\mathbb{B}_\kappa$ has the properties $(1_\kappa) - (3_\kappa)$ in $\mathbb{Z}(p^\infty)$.

Case 3. We construct a subset $\mathbb{B}_\kappa$ in the direct sum of cyclic groups $\bigoplus_{i\in\omega}\langle g_i\rangle$.

Put $\mathbb{B}_\kappa = B_{\kappa-1} - B_{\kappa-1} = \{g_i : 1 \leq i \leq \kappa - 1\}$. Obviously $\mathbb{B}_\kappa$ has properties $(1_\kappa), (3_\kappa)$. We claim that $\mathbb{B}_\kappa$ has property (2_κ). To obtain a contradiction assume that there exists a subset $B_\kappa \subset G$ with size $|B_\kappa| = \kappa$ such that

$$B_\kappa - B_\kappa \subset B_{\kappa-1} - B_{\kappa-1}.$$

Consider the sets $S = \{i : 1 \leq i \leq \kappa - 1\}$ and $F = \{i : 1 \leq i \leq \kappa\}$. We can enumerate the sets $B_{\kappa-1}$ and B_κ as $B_{\kappa-1} = \{g_i : i \in S\}$ and $B_\kappa = \{b_i : i \in F\}$.

Since $B_\kappa - B_\kappa \subset B_{\kappa-1} - B_{\kappa-1}$ we can define a map $f : [F]^2 \mapsto [S]^2$ assigning to each pair $\{i,j\} \in [F]^2$ a unique pair $\{k,l\} \in [S]^2$ such that $b_i - b_j = \pm(g_k - g_l)$. A desired contradiction will follow from Lemma 1 as soon as we check that f is separately injective and preserves intersections.

Claim 1. The map f preserves intersection.

To derive a contradiction, suppose that there are distinct $i, i' \in F$ and $j \in F$ such that

$$f(\{i,j\}) \cap f(\{i',j\}) = \emptyset.$$

Then

$$b_i - b_j = g_k - g_l \text{ and } b_{i'} - b_j = g_n - g_m$$

where k, l, n, m are pairwise distinct.

Hence $b_{i'} - b_i = g_n - g_m - g_k + g_l \notin B_{\kappa-1} - B_{\kappa-1}$, which contradicts the assumption that $B_\kappa - B_\kappa \subset B_{\kappa-1} - B_{\kappa-1}$.

Claim 2. The map f is separately injective.

To derive a contradiction, suppose that there are distinct $i, i' \in F$ and $j \in F$ such that

$$f(\{i,j\}) = f(\{i',j\}) = \{k,l\}.$$

Since $b_i, b_{i'}$ are distinct we get

$$b_i - b_j = g_k - g_l \text{ and } b_{i'} - b_j = g_l - g_k$$

and thus $b_{j'} - b_\iota = 2(g_l - g_k) \neq 0$. Note that $2(g_l - g_k) \in B_{\kappa-1} - B_{\kappa-1}$ iff $2(g_l - g_\kappa) = g_k - g_l$. Thus we get that $3g_k = 0$ and $3g_l = 0$.

Since $|F| = \kappa > 4$ we can chose $r \in F \setminus \{i, i', j\}$. The map f preserves intersection so $f(\{r,j\}) \cap \{k,l\} \neq \emptyset$. Also note that $f(\{r,j\}) \cap \{k,l\} \neq \{k,l\}$ otherwise $b_r = b_i$ or $b_r = b_{i'}$. So without loss of generality we can assume that $f(\{r,j\}) \cap \cap \{k,l\} = \{k\}$.

Hence $b_r - b_j = g_s - b_k$ or $b_r - b_j = g_k - g_s$ for some s.

Consequently, $b_r - b_i = g_s - 2g_k + g_l$ or $b_r - b_{i'} = 2g_k - g_s - g_l$.

Note that $2g_k \neq 0$ since $3g_k = 0$.

So we get $b_r - b_i = g_s - 2g_k + g_l \notin B_{\kappa-1} - B_{\kappa-1}$ or $b_r - b_{i'} = 2g_k - g_s - - g_l \notin B_{\kappa-1} - B_{\kappa-1}$.

This contradicts the assumption that $B_\kappa - B_\kappa \subset B_{\kappa-1} - B_{\kappa-1}$. □

Lemma 7. *Let κ be an infinite non-limit cardinal with $\kappa \leq |G|$ where G is an infinite Abelian group. Then there exists a subset $\mathbb{B}_\kappa$ with the properties $(1_\kappa) - (3_\kappa)$.*

Proof. Since κ is infinite non-limit cardinal there exists cardinal α such that $\kappa = \alpha^+$. Put $\mathbb{B}_\kappa = B_\alpha - B_\alpha$ where B_α is any subset of G with size $|B_\alpha| = \alpha$. Obviously $\mathbb{B}_\kappa$ satisfies property (1_κ).

Since $|\mathbb{B}_\kappa| = \alpha$ and $|B_\kappa - B_\kappa| = \kappa = \alpha^+$ for any subset $B_\kappa \subset G$ of size κ we get $B_\kappa - B_\kappa \not\subset \mathbb{B}_\kappa$. Therefore $\mathbb{B}_\kappa$ has property (2_κ).

The last property (3_κ) follows from the fact $|F| + |\mathbb{B}_\kappa| \leq |F| \cdot |\mathbb{B}_\kappa| < |G|$. $\square$

Lemma 8. *Let κ be a limit cardinal and G be an infinite Abelian group with $\kappa \leq |G|$. Then there exists a subset $\mathbb{B}_\kappa \subset G$ with the properties $(1_\kappa) - (3_\kappa)$.*

Proof. Note that it is enough to show that each group G of size κ contains a subset $\mathbb{B}_\kappa$ with properties $(1_\kappa) - (3_\kappa)$. When $|G| > \kappa$ then we can take any subgroup $H \subset G$ of size $|H| = \kappa$ and find a subset $\mathbb{B}_\kappa$ of with properties $(1_\kappa) - (3_\kappa)$ in H. Then the subset $\mathbb{B}_\kappa$ will have the properties $(1_\kappa) - (3_\kappa)$ in the whole group.

So it remains to prove that such a set $\mathbb{B}_\kappa$ exists in each group G of size κ.

First we describe a sequence of symmetric subsets $F_\alpha \subset G$ of size α such that $G = \bigcup_{\alpha<\kappa} F_\alpha$ and $F_\alpha \supset \bigcup_{\beta<\alpha} F_\beta$. Enumerate the group G so that $G = \{g_\alpha : \alpha < \kappa\}$ and $g_0 = e$. Then put $F_\alpha = \{g_\beta, -g_\beta : \beta < \alpha\}$ for all $\alpha < \kappa$.

We put

$$\mathbb{B}_\kappa = \bigcup_{\alpha<\kappa} B_\alpha - B_\alpha$$

where a set $B_\alpha = \{b_\alpha^\beta : \beta < \alpha\} \subset G$ of size α will be chosen later.

To simplify notation we write $\mathbb{B}_{<\alpha}$ instead of $\bigcup_{\beta<\alpha}(B_\beta - B_\beta)$ and $\mathbb{B}_{>a}$ instead of $\bigcup_{\alpha<\beta<\kappa}(B_\beta - B_\beta)$. By $B_\alpha^{<\beta}$ we shall denote the initial interval $\{b_\alpha^\gamma : \gamma < \beta\}$ of B_α.

Now we are in a position to define a sequence of sets B_α forcing the set $\mathbb{B}_\kappa$ to satisfy the properties (2_κ) and (3_κ). To ensure property (3_κ) we will also construct a transfinite sequence of points $(h_\alpha)_{\alpha<\kappa}$ of G such that $h_\alpha \notin F_\alpha + \mathbb{B}_\kappa$.

We start putting $B_0 = \{e\}$and taking any non-zero point $h_0 \in G$. Assume that for some ordinal $\alpha < \kappa$ the sets B_β and the points h_β, $\beta < \alpha$, have been constructed. Then pick any point $h_\alpha \in G$ with

$$h_\alpha \notin F_\alpha + \mathbb{B}_{<\alpha}.$$

Such a point exists because the size of the set $F_\alpha + \mathbb{B}_{<\alpha}$ is equals $\alpha < \kappa = |G|$. Let

$$H_\alpha = \{h_\beta, -h_\beta : \beta \leq \alpha\}.$$

Next we define inductively elements of $B_\alpha = \{b_\alpha^\beta : \beta < \alpha\}$.

We pick any b_α^0 with $b_\alpha^0 \in G \backslash \mathbb{B}_{<\alpha}$. Next, we choose b_α^β with

(a) $b_\alpha^\beta \notin B_\alpha^{<\beta} + F_\alpha + \mathbb{B}_{<\alpha}$;
(b) $b_\alpha^\beta \notin B_\alpha^{<\beta} - B_\alpha^{<\beta} + B_\alpha^{<\beta} + F_\alpha$;
(c) $b_\alpha^\beta \notin B_\alpha{}^\beta + F_\alpha + H_\alpha$.

To ensure properties (a), (b), (c) we have to avoid the sets of size α, which is possible because $|G| = \kappa$.

Now let us prove that the constructed set $\mathbb{B}_\kappa$ satisfies the properties $(1_\kappa) - (3_\kappa)$. In fact the property (1_κ) is evident while (3_κ) follows immediately from (c). It remains to prove

Claim. *The set $\mathbb{B}_\kappa$ has property* (2_κ).

Let B_κ be a subset of G of size $|B_\kappa| = \kappa$. Fix any pairwise distinct points $c_1, c_2, c \in B_\kappa$.

If $B - B_\kappa \subset \mathbb{B}_\kappa$ then $B_\kappa \subset \bigcap_{i=1}^3 c_i + \mathbb{B}_\kappa$ and $\kappa = |B_\kappa| \leq |\bigcap_{i=1}^3 \; c_i + \mathbb{B}_\kappa|$.

So to prove our claim it is enough to show that $|\bigcap_{i=1}^3 c_i + \mathbb{B}_\kappa| < \kappa$. Find an ordinal $\alpha < \kappa$ such that $c_p - c_q \in F_\alpha$ for any $1 \leq p, q \leq 3$. Assuming that $|\bigcap_i^3 {}_1 c_i + \mathbb{B}_\kappa| = \kappa$ we may find a point $b \in \bigcap_{i=1}^3 (c_i + \mathbb{B}_{>\alpha}) \setminus \{c_i\}$. A contradiction will be reached in three steps.

Steps 1. *First show that there is $\beta > \alpha$ with $b \in \bigcap_{i=1}^3 (c_i + B_\beta)$.*

Otherwise, $b - c_p \in B_\gamma - B_\gamma$ and $b - c_q \in B_\beta$ for some $\gamma > \beta > \alpha$ and some $p \neq q$. Find $i, j < \gamma$ with $b - c_p = b_\gamma^i - b_\gamma^j$. The inequality $b \neq c_p$implies $i \neq j$.

If $i < j$ then $b_\gamma^j = b_\gamma^i - b + c_p = b_\gamma^i - b + c_q - c_q + c_p \subset b_\gamma^i - B_\beta + B_\beta + {}$ ${} + F_\gamma \subset B_\gamma^{<j} + \mathbb{B}_{<\gamma} + F_\gamma$, which contradicts (a).

If $i < j$ then $b_\gamma^i = b_\gamma^j + b - c_p = b_\gamma^i + B_\beta - B_\beta + c_q - c_p \subset B_\gamma^{<j} + \mathbb{B}_{<\gamma} + {}$ ${} + F_\gamma$, which again contradicts (a).

Step 2. *We claim that if $b - c_p = b_\beta^i - b_\beta^j$ and $b - c_q = b_\beta^s - b_\beta^t$ then* $\max\{i,j\} = \max\{s,t\}$.

It follows from the hypothesis that $c_q - c_p = b_\beta^i - b_\beta^j + b_\beta^t - b_\beta^s$. To obtain a contradiction assume that $\max\{i,j\} > \max\{s,t\}$.

If $j < i$ then $b_\beta^i = c_q - c_p + b_\beta^j - b_\beta^t + b_\beta^s \in F_\beta + B_\beta^{<i} - B_\beta^{<i} + B_\beta^{<i}$, which contradicts (b).

If $i < j$ then $b_\beta^j = c_p - c_q + b_\beta^i + b_\beta^t - b_\beta^s \in F_\beta + B_\beta^{<i} - B_\beta^{<i} - B_\beta^{<i}$, again a contradiction with (b).

Step 3. According to the previous step there exist $\beta > \alpha$ and l such that

$b - c_1 = b_\beta^i - b_\beta^j$ where $\max\{i,j\}$ is equal to l;
$b - c_2 = b_\beta^s - b_\beta^t$ where $\max\{s,t\}$ is equal to l;
$b - c_3 = b_\beta^q - b_\beta^r$ where $\max\{q,r\}$ is equal to l.

In this case we obtain a dichotomy: either among three numbers i, s, q two are equal to l or among j, t, r two are equal to l.

In the first case we lose no generality assuming that $i = s = l$; in the second, that $j = t = l$.

In the first case we get $F_\alpha \ni c_2 - c_1 = b^i_\beta - b^s_\beta$, which contradicts (a).

In the second case we get $F_\alpha \ni c_2 - c_1 = b^t_\beta - t^j_\beta$, which contradicts (a) again.

Therefore, there is no $b \in \bigcap_{i=1}^3 (c_i + \mathbb{B}_{>\alpha}) \setminus \{c_i\}$ and hence $|\bigcap_i^3 {}_1 c_i + \mathbb{B}_{>\alpha}| < \kappa$. $\square$

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