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Theorem on Signatures

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A theorem on signatures presented in this paper generalizes and gives direct proofs for many celebrated theorems due to Brouwer, Kakutani, Nash, Gale, Nikaido and Shapley.

1. Introduction

The main result of this paper is a theorem called here as a theorem on signatures. As app ications we derive some well-known results related to fixed points and equilibria theorems. We extend the classical results onto a class of simplicial spaces being a generalization of convex sets. In 1973 Maynard Smith and Price [13] introduced a concept of evolutionarily stable strategy (ESS) which became a fundamental notion of modern evolutionarily biology and genetic. The area of research of ESS was culminated in John Maynard Smith book [12]. The theory of ESS leads to testable predictions about the evolution of behaviour of sex and genetic systems. According to Maynard Smith ESS is a strategy such that, if all numbers of a population adopt it, then no mutant strategy could invade the population under the influence of natural selection. In this paper we shall investigate ESS from a topological point of view.

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Let $\{p_0, \ldots, p_n\} \subset R^m$ be a collection of linearly independent points of the m-dimensional Euclidean space R^m. The n-dimensional geometric simplex $[p_0, p_1, \ldots, p_n]$ with vertices $p_0, p_1, \ldots, p_n$ is the subspace of R^m given by

$$\left\{x \in R^m : x = \sum_{i=0}^{n} t_i p_i,\ t_i \geq 0 \text{ for each } i = 0, \ldots, n, \quad \text{and} \quad \sum_{i=0}^{n} t_i = 1\right\}.$$

If $\{p_{i_0}, \ldots, p_{i_k}\}$ is a subcollection of $k+1$ points of the collection $\{p_0, \ldots, p_n\}$, then the simplex $[p_{i_0}, \ldots, p_{i_k}]$ is said to be a *k-dimensional face* of the simplex $[p_0, \ldots, p_n]$.

A continuous map $f : [p_0, \ldots, p_n] \to E$ into linear space E is said to be *affine* if $f\left(\sum_{i=0}^{n} t_i p_i\right) = \sum_{i=0}^{n} t_i f(p_i)$, where $t_i \geq 0$ and $\sum_{i=0}^{n} t_i = 1$.

A continuous map $\sigma : [p_0, \ldots, p_n] \to X$ is said to be a *singular* simplex in X. Let us denote vert $\sigma := \{\sigma(p_0), \ldots, \sigma(p_n)\}$ and im $\sigma := \sigma([p_0, \ldots, p_n])$,

A collection S of singular simplices in X is said to be a *simplicial structure* on X if;

(a) For any finite sequence $x_0, \ldots, x_n$ of (not necessarily distinct) points of the space X there exists a $\sigma \in S$, $\sigma : [p_0, \ldots, p_n] \to X$ such that $\sigma(p_0) = x_0, \ldots, \sigma(p_n) = x_n$.

(b) If $\sigma \in S$, then any restriction to any face of the domain of σ belongs to S.

(c) If $l : [q_0, \ldots, q_n] \to [p_0, \ldots, p_n]$ is affine map such that $l(q_i) = p_i$, for each $i \leq n$, then for each $\sigma \in S$, $\sigma : [p_0, \ldots, p_n] \to X$, the composition $\sigma \circ l$ belongs to S.

A topological space X together with a simplicial structure S on X is going be referred to as *simplicial space* (X, S). A subset $A \subset X$ of a simplicial space (X, S) is said to be *simplicially convex* if for each simplex $\sigma \in S$, vert $\sigma \subset A$ implies im $\sigma \subset A$.

It is tacitly assumed (unless otherwise stated) that each convex subset X of linear (= vector) topological space over R is a simplicial space with a simplicial structure consisting of all affine maps.

The following theorem [9] is our main tool in the proof of Theorem on Signatures. We reprove it here for the reader's convenience.

Theorem 1. *(Theorem on Indexed Families). Let $\sigma : [p_0, \ldots, p_n] \to X$ be a continuous function. For any covering $\{U_0, \ldots, U_n\}$ of the subspace $\sigma([p_0, \ldots, p_n])$ by non-empty open subsets of X there exists a non-empty subset of indices $\{i_0, \ldots, i_k\} \subseteq \{0, \ldots, n\}$ such that $\sigma([p_{i_0}, \ldots, p_{i_k}]) \cap U_{i_0} \cap \ldots \cap U_{i_k} \neq \emptyset$.*

Proof. For $i = 0, 1, \ldots, n$, let d_i be a function on the simplex $[p_0, \ldots, p_n]$ given by

$$d_i(x) = d\left(x, [p_0, \ldots, p_n] \setminus \sigma^{-1}(U_i)\right),$$

where $d(x, Y) = \inf\{\|x - y\| : y \in Y\}$ is the distance between the point x and the subset Y in R^m. Each of the functions d_i is continuous and since the sets $\sigma^{-1}(U_i)$ are open,

$$d_i(x) = 0 \text{ if and only if } x \notin \sigma^{-1}(U_i).$$

The function f given by

$$f(x) = \sum_{i=0}^{n} \left(\frac{d_i(x)}{\sum_{j=0}^{n} d_j(x)} \right) p_i$$

is a continuous function defined on the simplex $[p_0, \ldots, p_n]$ into $[p_0, \ldots, p_n]$. According to the Brouwer Fixed Point Theorem, there exists $a \in [p_0, \ldots, p_n]$ such that $f(a) = a$. Thus

$$a = \sum_{i\ 0}^{n} \left(\frac{d_i(a)}{\sum_{j=0}^{n} d_j(a)} \right) p_i. \tag{1}$$

Let $\{i_0, \ldots, i_k\}$ be the set of all indices i such that

$$\frac{d_i(a)}{\sum_{j=0}^{n} d_j(a)} \neq 0. \tag{2}$$

From (1), $a \in [p_{i_0}, \ldots, p_{i_k}]$. From (2),

$$i \in \{i_0, \ldots, i_k\} \text{ if and only if } a \in \sigma^{-1}(U_i).$$

Subsequently,

$$\sigma(a) \in \sigma([p_{i_0}, \ldots, p_{i_k}]) \cap U_{i_0} \cap \ldots \cap U_{i_k}. \qquad \square$$

2. Theorem on Signatures

A function $\mu : X \times Y \to [0, \infty)$ is said to be *quasi-simplicially convex* with respect to the first variable $x \in X$ if for each $y \in Y$ and $\varepsilon > 0$ the *pseudoball* $A(y, \varepsilon) := := \{x \in X : \mu(x, y) < \varepsilon\}$ is simplicially convex. If μ is a nonnegative function then it is called a *signature.*

The power set of X is denoted by 2^X.

A multivalued map $H : X \to 2^Y$ is said to be a *multivalued limit map* if there exists a sequence $\{h_n : X \to Y | n \in N\}$ of continuous maps such that for each subsequence $\{n_k\} \subset N$;

$$\lim_{k\to\infty} (x_{n_k}, h_{n_k}(x_{n_k})) = (x, y) \text{ implies } y \in H(x).$$

The sequence $\{h_n; n \in N\}$ is said to be *basic* for H.

A set-valued map $H : X \to 2^Y$ is said to be *semicontinuous* if $H^{-1}(V) = \{x \in X : : H(x) \subset V\}$ is an open set in X provided that V is open in Y. In the case when X and Y are compact metric spaces it is equivalent to the fact that the graph $G(H) := \{(x,y) : x \in X,\ y \in H(x)\}$ is closed subset of $X \times Y$.

Theorem 2. *Let (X, d) be a compact metric space and Y convex compact subspace of a normed space. If $H : X \to 2^Y$ is upper semicontinuous and $H(x)$*

is non-empty convex compact set for each $x \in X$, then H is a multivalued limit map.

Proof. Let us denote by $B(x,\varepsilon) := \{y \in X : d(x,y) < \varepsilon\}$ and $B(A,\varepsilon) := \bigcup \{B(x,\varepsilon) : x \in A\}$.

To define a function h_n from a sequence that witnesses limit-valuedness of H, fix $n > 0$. Let $U(x)$ be given by

$$U(x) = B\left(x, \frac{1}{n}\right) \cap \left\{y \in X : H(y) \subset B\left(H(x), \frac{1}{n}\right)\right\}.$$

By compactness of X, the open covering $\{U(x) : x \in X\}$ has a finite star-refinement $\{V_0, \ldots, V_m\}$, i.e., for each $x \in X$ there exists $\bar{x} \in X$ such that $\bigcup \{V_i : x \in V_i\} \subset U(\bar{x})$.

For each $i = 1, \ldots, m$ let p_i be an arbitrary point of the set $H(V_i) := \bigcup \{H(x) : x \in V_i\}$. We set

$$h_n(x) := \sum_{i=0}^{m} \left(\frac{d_i(x)}{\sum_{j=0}^{m} d_j(x)}\right) p_i,$$

where $d_i(x) = d(x, X \setminus V_i)$. The function $h_n : X \to Y$ is continuous.

For a given $x \in X$, choose $\bar{x} \in X$ is such that $\bigcup \{V_i : x \in V_i\} \subset U(\bar{x})$. Then $p_i \in B(H(\bar{x}), \frac{1}{n})$ whenever $x \in V_i$. Since $x \in V_i$ if and only if $d_i(x) \neq 0$, $p_i \in B(H(\bar{x}, \frac{1}{n}))$ whenever $d_i(x) \neq 0$. Since $B(H(\bar{x}, \frac{1}{n}))$ is convex, $h_n(x) = \sum_i^m {}_0 \left(\frac{d_i(x)}{\Sigma_j^m {}_0 d(x)}\right) p_i \in B(H(\bar{x}, \frac{1}{n}))$. Thus we have proved that for each x there is $\bar{x}$ such that

$$\|x - \bar{x}\| < \frac{1}{n} \text{ and } d(h_n(x), H(\bar{x})) < \frac{1}{n}$$

We shall prove that the sequence $\{h_n : n = 1, 2, \ldots\}$ is basic for H. Towards this end, assume that $\lim_{k\to\infty}(x_{n_k}, h_{n_k}(x_{n_k})) = (x, y)$. We have just showed that for each x_{n_k} there is $\bar{x}_{n_k}$ such that $\|x_{n_k} - \bar{x}_{n_k}\| < \frac{1}{nk}$ and $d(h_{n_k}(x_{n_k}), H(\bar{x}_{n_k})) < \frac{1}{nk}$. The latter means that there exists $\bar{y}_{n_k} \in H(\bar{x}_{n_k})$ such that $\|h_{n_k}(x_{n_k}) - \bar{y}_{n_k}\| < \frac{1}{nk}$. Hence $\lim_{k\to\infty}(\bar{x}_{n_k}, \bar{y}_{n_k}) = (x, y)$. Since H is upper semi-continuous, $y \in H(x)$. □

Theorem on Signatures. *Let be given a family of continuous nonnegative functions (signatures) $\mu : X \times Y \to [0, \infty)$, $\mu \in M$, from a product of a compact metric simplicial space (X, S) and a compact metric space Y such that for each finite subset $M_0 \subset M$, $\varepsilon > 0$ and $y \in Y$ the set*

$$\{x \in X : \mu(x,y) < \varepsilon \textit{ for each } \mu \in M_0\}$$

is nonempty and simplicially convex.

Then for each multivaluaed limit map $H : X \to 2^Y$ there exist a point $a \in X$ and $b \in H(a)$ such that

$$\mu(a,b) = 0 \textit{ for each } \mu \in M.$$

Proof. First we shall prove the theorem for a continuous map $h : X \to Y$.

(I). Fix $\varepsilon > 0$ and assume that M is finite. According to assumptions for each $y \in Y$, the set $A(y)$ given by

$$A(y) := \{x \in X : \mu(x, y) < \varepsilon \text{ for each } \mu \in M\}$$

is simplicially convex and nonempty.

By continuity of μ's, the dual sets

$$B(x) := \{y \in Y : \mu(x, y) < \varepsilon \text{ for each } \mu \in M\}$$

are open for each $x \in X$. Observe that

$$x \in A(y) \text{ if and only if } y \in B(x).$$

Since each set $A(y)$ is nonempty we infer that

$$Y = \bigcup \{B(x) : x \in X\}.$$

Compactness of $h(X)$ implies that there is a finite set of points $x_0, \ldots, x_n \in X$ such that

$$h(X) \subset B(x_0) \cup \ldots \cup B(x_n).$$

Now, choose a singular simplex $\sigma \in S$, $\sigma : [p_0, \ldots, p_n] \to X$, such that $\sigma(p_0) = x_0, \ldots, \sigma(p_n) = x_n$.

According to Theorem on Indexed Families there is a point $a \in X$ and a set of indices $0 \leq i_0 < \ldots < i_k \leq n$ such that

$$a \in \sigma([p_{i_0}, \ldots, p_{i_k}]) \cap h^{-1}(B(x_{i_0})) \cap \ldots \cap h^{-1}(B(x_{i_k})).$$

Let $\eta := \sigma \mid [p_{i_0}, \ldots, p_{i_k}]$. Then $\eta \in S$, $a \in \operatorname{im} \eta$, $h(a) \in B(x_{i_0}) \cap \ldots \cap B(x_{i_k})$ and hence $\operatorname{vert} \eta = \{x_{i_0}, \ldots, x_{i_k}\} \subset A(h(a))$. Since $A(h(a))$ is simplicially convex we infer that

$$a \in A(h(a))$$

and this means that

$$\mu(a, h(a)) < \varepsilon \text{ for each } \mu \in M.$$

(II). From the above it follows that for each $\varepsilon > 0$ the set

$$K(\varepsilon) := \{x \in X : \mu(x, h(x)) \leq \varepsilon \text{ for each } \mu \in M\}$$

is nonempty and compact. Therefore there is a point

$$a \in \bigcap \{K(\varepsilon) : \varepsilon > 0\}$$

and this means that

$$\mu(a, h(a)) = 0 \text{ for each } \mu \in M.$$

(III). Now assume that M is infinite. For each finite set $M_0 \subset M$ let

$$L(M_0) := \{x \in X : \mu(x, h(x)) = 0 \text{ for each } \mu \in M_0.$$

The family

$$\{L(M_0) : M_0 \text{ is finite subset of } M\}$$

is a centered family of nonempty compact sets. It is clear that each point $a \in X$ belonging to the intersection of this family satisfies for each $\mu \in M$ the equation $\mu(a, h(a)) = 0$.

(IV). Finally, assume a sequence $\{h_n : n \in N\}$ is basic for a multivalued limit map $H : X \to 2^Y$. For each $n \in N$ choose a point $a_n \in X$ satisfying

$$\mu(a_n, h_n(a_n)) = 0 \text{ for each } \mu \in M.$$

From compactness of X and Y there are two points $a \in X$ and $b \in Y$ and subsequence $\{a_{n_k}\}$ such that

$$\lim_{k\to\infty} a_{n_k} = a \text{ and } \lim_{k\to\infty} h_{n_k}(a_{n_k}) = b.$$

By definition of multivalued limit map, $b \in H(a)$.

Continuity of functions $\mu \in M$ implies that

$$m(a, b) = 0 \text{ for each } \mu \in M.$$

Remark. In the case when $H : X \to Y$ is a map the assumption that the spaces X and Y are metric is superfluous.

3. Applications

In this part we are going to present some consequences of the theorem on signatures. Another applications the reader will find in [10]. To be near classical results we mostly state the theorems in terms of convex sets and continuous maps.

If $M = \{\mu\}$,where $\mu(x, y) := \|x - y\|$ is the metric induced by a norm then we obtain Kakutani's theorem [4].

Kakutani Theorem. *Let $H : X \to 2^X$ be an upper semicontinous multivalued map from a convex compact subset X of a normed space.*

Then H has a fixed point, i.e., there is an $a \in X$ such that $a \in H(a)$.

The above theorem can be extended to

Brouwer-Schauder-Tichonov-Kakutani Theorem. *Let X be a compact metric simplicial space and M be a set of continuous functions (signatures) $\mu : X \times X \to$ $\to [0, \infty)$ quasi-simplicially convex with respect to the first variable such that;*

1. for each $\mu \in M$, $\mu(x, x) = 0$,

2. for each two distinct points $x, y \in X$ there is $\mu \in M$ with $\mu(x, y) > 0$.

Then any multivalued limit map $H : X \to 2^X$ has a fixed point.

A function $f : X \to R$ from a simplicial space is said to be *quasi-concave* [resp. *quasi-convex*] if the set $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is simplicially convex for each $r \in R$.

A function $f : X \to R$ from a convex subset of linear space is said to be *concave* [resp. *convex*] if for each points $x, y \in X$ and $t \in [0,1]$ the following inequality holds $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$ [resp. $f(tx + (1-t)y) \leq$ $\leq tf(x) + (1-t)f(y)$.

Since many results are stated in terms of concave or affine maps for the reader's convenience we shall prove the following.

Lemma. *If a function $f : X \to R$ from a convex subset of linear space is concave [convex] then f is quasi-concave [quasi-convex].*

Proof. Assume that f is concave. Fix $r \in R$ and let $C(r) := \{x \in X : f(x) > r\}$. Then for each $x, y \in C(r)$, $z := tx + ((1-t)y$, where $t \in [0,1]$ we have; $f(z) =$ $= f(tx + (1-t)y) \geq tf(x) + (1-t)f(y) > tr + (1-t)r = r$. And this means that $z \in C(r)$. If f is convex the proof is similar. □

Let us notice that any monotonic function $f : R \to R$ is quasi-concave and quasi-convex. This leads to an observation that for example the class of quasi-concave functions is larger than the class of concave functions.

A point $a \in X$ is said to be an ESS point (or *evolutionarily stable strategy*) for a function $f : X \times X \to R$ if $f(x,a) \leq f(a,a)$ for each $x \in X$.

Maynard Smith Theorem. *Let X be a compact simplicial space and $f : X \times X \to R$ a continuous map which is quasi-concave with respect to the first variable.*

Then f has an ESS point.

Proof. Let us define $\mu(x,y) := -f(x,y) + \sup_{z \in X} f(z,y)$,

Fix $y \in X$ and $r > 0$. Let $s := \sup_{z \in X} f(z,y)$. Observe that the pseudoball $B(y,r) := \{x \in X : \mu(x,y) < r\}$ is simplicially convex because $B(y,r) = \{x \in X :$ $: s - r < f(x,y)\}$ and according to the assumption this set is simplicially convex. The pseudoball $B(y,r)$ is nonempty because by continuity and compactness for each point y there is a point x such that $\mu(x,y) = 0$.

Applying Theorem on Signatures to the identity map $h : X \to X$, $h(x) = x$, we infer that there is a point $a \in X$ such that $\mu(a,a) = 0$ and therefore we get $f(a,a) = \sup_{x \in X} f((x,a))$. □

Remark. The Maynard Smith Theorem implies the Brouwer Fixed Point Theorem. To see this consider a continuous map $g : X \to X$, where X is a nonempty compact convex subset of a normed space. According to the Maynard Smith Theorem for the map

$$f(x,y) := -\,|\,x - y\|$$

there exists a point $a \in X$ such that for each $x \in X$;

$$f(x,a) \leq f(a,a).$$

Setting $x = g(a)$ we obtain $\|a - g(a)\| = 0$, and in consequence, $a = g(a)$. $\square$

Nash Theorem. *Let $f_i : X_1 \times \ldots \times X_n \to R$, $i = 1, \ldots n$, be a family of continuous functions from a Cartesian product of compact simplicial spaces and let us assume that each function f_i is quasi-concave with respect to the variable $x_i \in X_i$.*

Then there exists a point $a \in X_1 \times \ldots \times X_n$ such that for each $i \leq n$;

$$f_i(a) = \sup_{x \in X_i} f_i(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n).$$

Proof. Let $X := X_1 \times \ldots \times X_n$ and define for each $i = 1, \ldots, n$;

$$\mu_i(x,y) := -f_i(N_i(x,y)) + \sup_{z \in X} f_i(N_i(z,y)),$$

where $N_i : X \times X \to X$ means the Nash projection;

$$N_i(x,y) := (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n).$$

Fix $i, y \in X$ and $r > 0$. Let $s := \sup_{x \in X} f_i(N_i(x,y))$. Since $B_i(y,r) := \{x \in X : : \mu_i(x,y) < r\} = \{x \in X : s - r < f_i(y_1, \ldots, y_{i\ 1}, x_i, y_{i+1}, \ldots, y_n)\}$, by the assumption, we infer that each pseudoball $B_i(y,r)$ is simplicially convex and therefore the set $A(y,r) := \{x \in X : \mu_i(x,y) < r$ for each $i = 1, \ldots, n\}$ is simplicially convex.

It remains to verify that the set $A(y,r)$ is nonempty. By compactness for each $i \leq n$ there is a point $a^i \in X$ such that $\mu_i(a^i, y) = 0$. Let $a = (a_1, \ldots, a_n) \in X$ be the unique point such that $a_i = a_i^i$ for each $i \leq n$. Since $N_i(a^i, y) = N_i(a,y)$ it is clear that $\mu_i(a,y) = 0$ for each $i \leq n$, and this implies $a \in A(y,r)$.

Applying Theorem on Signatures to the identity map $h : X \to X$, $h(x) = x$, we infer that there is a point $a \in X$ such that

$$\mu_i(a,a) = 0 \text{ for each } i = 1, \ldots, n.$$

But $N_i(a,a) = a$ and therefore we get

$$f_i(a) = \sup_{x \in X} f_i(N_i(x,a)) = \sup_{x \in X_i} f_i(a_1, \ldots, a_{i\ 1}, x, a_{i+1}, \ldots, a_i)$$

for each $i = 1, \ldots, n$. $\square$

A point $a \in X_1 \times \ldots \times X_n$, $a = (a_1, \ldots, a_n)$, is said to be *Nash's equilibrium point* for the family of functions $f_i : X_1 \times \ldots \times X_n \to R$, $i = 1, \ldots, n$, if $f_i(a_1, \ldots, a_{i_1}, x_i, a_{i+1}, \ldots, a_n) \leq f_i(a)$ for each $i \leq n$ and $x_i \in X_i$.

It is easy to check that if $a \in X$, $X := X_1 \times \ldots \times X_n$, is an ESS point for the function $f : X \times X \to R$;

$$f(x, y) := \sum_{\iota \ 0}^{n} f_i(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)$$

then a is Nash's equilibrium point for the family of functions f_i's.

In fact, in the proof of Nash Theorem and many others it is convenient to apply Infimum Principle [10];

Theorem (Infimum Principle). *Let $g : X \times Y \to R$, $g \in G$, be a family of continuous functions and quasi-simplicially convex with respect to the first variable x from a product of a compact metric simplicial space X and a compact metric space Y such that for each finite subcollection $G_0 \subset G$ and for each point $y \in Y$ there is a point $a \in X$ with*

$$g(a, y) = \inf_{x \in X} g(x, y) \textit{ for each } g \in G_0.$$

Then for each multivalued limit map $H : X \to 2^Y$ there is a point $a \in X$ and a point $b \in H(a)$ such that

$$g(a, b) = \inf_{x \in X} g(x, b) \text{ for each } g \in G.$$

Proof. Define $\mu_g(x, y) := g(x, y) - \inf_{x \in X} g(x, y)$, $M := \{\mu_g : g \in G\}$ and then check that the assumptions of Theorem on Signatures hold. □

Let $\Delta_n := [e_1, \ldots, e_n]$, where $e_i := (0, \ldots, , 0, 1, 0, \ldots, 0)$, $e_i(j) = 0$ for $j \neq i$ and $e_i(i) = 1$, denotes the $(n-1)$-dimensional standard simplex in the space R^n. The following theorem plays an important role in a proof of the existence of equilibrium points in economic models in the Walras sense (see [16]).

Gale-Nikaido Theorem. *Let $H : \Delta_n \to 2^C$ be an upper semicontinuous map from the standard simplex Δ_n such that for each $x \in \Delta_n$, $H(x)$ is nonempty compact convex subset of a compact convex set $C \subset R^n$. Suppose further that the Walras law in the general sense holds;*

$$< x, y > := \sum_{\iota - 1}^{n} x\, y_i \geq 0 \textit{ for each } x \in \Delta_n \textit{ and } y \in H(x).$$

Then there exist $a \in \Delta_n$ and $b \in H(a)$ such that $b_i \geq 0$ for each $i = 1, \ldots, , n$.

Proof. Applying Infimum Principle to $X = \Delta_n$, $Y = C$, the given set-valued map H, the function g_1, given by $g_1(x, y) = < x, y >$ there is a point $(a, b) \in \Delta_n \times C$ such that $b \in H(a)$ and $< a, b > = \inf\{< x, b >: x \in \Delta_n\}$. By Walras law $< a, b > \geq 0$ and in consequence $0 \leq < a, b > \leq < x, b >$ for each $x \in \Delta_n$. Since $e_i \in \Delta_n$, $0 \leq < e_\iota, b > = b_\iota$ for each $i = 1, \ldots, n$. □

4. Shapley theorem

In 1929 Knaster, Kuratowski and Mazurkiewicz [5] published a kind of an intersection theorem (the KKM theorem), where some conditions are given for a closed covering of a simplex has a non-empty intersection. In 1967 Scarf [18] proved that any non-transferable utility game whose characteristic function is balanced, has a non-empty core. His proof is based on an algorithm which approximates fixed points. Shapley [19] replaced the Scarf algorithm by a covering theorem (the KKMS theorem) being a generalization of the KKM theorem. Therefore the main difficulty to show the nonemptiness of the core lies in proofs of the KKMS theorem. Shapley's theorem as an extension of the KKM theorem became very useful to prove the existence of solutions in general equilibrium theory and game theory. We would like to present Shapley's theorem as a kind of a dual theorem on coverings. There are a number of papers (see e.g., [2], [6], [7]) containing elementary and simple proofs of the KKMS theorem. The proof which is given in this note is a direct consequence of well-known for economists, Kakutani's fixed point theorem [4].

Let us establish some terminology and notation used by economists. Denote the set $\{1,\ldots, n\}$ by N and the family of nonempty subsets of N by $\mathcal{N}$. For each point $x \in R^n$ let

$$\sup x := \{i \in N : x_i > 0\} \quad \text{and} \quad \overline{\sup}\, x := \{i \in N : x_i \geq 0\}$$

Denote by Δ the unit simplex in R^n;

$$\Delta := \{x \in R^n : \sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad \overline{\sup}\, x = N\}$$

and for each $S \in \mathcal{N}$ let Δ^S be an S-face of Δ;

$$\Delta^S := \{x \in \Delta : \sup x \subset S\}.$$

The symbol conv A stands for the convex hull of a set A.

The following theorem is a covering version of the Shapley Theorem.

Theorem 3. *Let $\{C^S : S \in \mathcal{N}\}$ be a family of closed subsets of Δ such that $\Delta^T \subset \bigcup_{S \subset T} C^S$ for each $T \in \mathcal{N}$ and let $\{d^S : S \in \mathcal{N}\}$ be a family of points of Δ such that sup $d^S \subset S$ for each $S \in \mathcal{N}$.*

Then $\Delta = \bigcup_{x \in \Delta} conv\,\{d^S : x \in C^S\}$.

Proof. Let $X := \{x \in R^n : \sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad x_i \geq -1 \quad \text{for each} \quad i \leq n\}$. Define a continuous map (retraction) $r : X \to \Delta$ such that $r(x) = x$ for each $x \in \Delta$;

$$r_i(x) := \frac{max\,\{0, x_i\}}{\sum_{j=1}^{n} max\,\{0, x_j\}} \text{ for each } i = 1, \ldots, n$$

Fix a point $m \in \Delta$ and define a continuous map $f: X \times \Delta \to X$;

(1) $f(x,p) := r(x) + m - p$.

Next, define set-valued maps $F: X \to 2^{\Delta}$ and $\phi: X \times \Delta \to 2^{X \times \Delta}$;

(2) $F(x) := \operatorname{conv}\{d^S : r(x) \in C^S \text{ and } S \subset \overline{\sup}\, x\}$

(3) $\phi(x,p) := \{f(x,p)\} \times F(x)$.

Assume that $(\bar{x},\bar{p})$ is a fixed point of the map ϕ, i.e., $(\bar{x},\bar{p}) \in \phi(\bar{x},\bar{p})$. Observe that $\bar{x} \in \Delta$. Indeed, if $\bar{x} \notin \Delta$ there exists j such that $\bar{x}_j < 0$. Since $j \notin \overline{\sup}\, \bar{x}$ and $\sup d^s \subset S$, according to (2), we infer that $\bar{p}_j = 0$, and from (1) we obtain; $\bar{x}_j = f_j(\bar{x},\bar{p}) = \; = 0 + m_j \geq 0$, a contradiction to $\bar{x}_j < 0$.

Since $r(x) = x$ for each $x \in \Delta$, from (1) we obtain; $\bar{x} = f(\bar{x},\bar{p}) = \bar{x} + m - p$, and this yields $m = p$.

Thus we have proved that if the multivaled map ϕ has a fixed point then for each point $m \in \Delta$ there exist a point $\bar{x} \in \Delta$ such that $m \in F(\bar{x})$.

In order to complete the proof it suffices to verify that the map ϕ satisfies the assumptions of Kakutani's fixed point theorem. It is clear that for each point $(x,p) \in X \times \Delta$, the set $\phi(x,p)$ is non-empty and convex. It remains to show that the graph $W(\phi) := \{(z,u) : z \in X \times \Delta,\ u \in \phi(z)\}$ is a closed subset of $(X \times \Delta)^2$.

Assume that $(z_m, u_m) \to (z,u)$ whenever $m \to \infty$, where $(z_m, u_m) \in W(\phi)$, $z_m = (x_m, p_m)$, $u_m = (f(z_m), y_m)$, $y_m \in F(x_m)$, $z = (x,p)$ and $u = (f(z), y)$.

By continuity of f it follows that $z_m \to f(z)$ whenever $m \to \infty$. Now, we are reduced to proving that $y \in F(x)$.

For each $x \in X$ consider a subset of $\mathscr{N}$;

$$B(x) := \{S \subset N : r(x) \in C^S \text{ and } S \subset \overline{\sup}\, x\}.$$

The family $\{B(x_m) : m = 1,2,\ldots\}$ consists of subsets of the finite set $\mathscr{N}$ and therefore there exists a set $B \subset \mathscr{N}$ and subsequence $\{m_k\}$ such that $B = B(x_{m_k})$ for each k.

Since the sets C^S are closed and $S \in B(x_m)$ implies $S \subset \overline{\sup}\, x$, we infer that $B \subset B(x)$ whenever $x_{m_k} \to x$.

Note that $y_{m_k} \in \operatorname{conv}\{d^S : S \in B\} = F(x_{m_k})$. Since $y_{m_k} \to y$ whenever $k \to \infty$, we infer that $y \in F(x_{m_k}) = \operatorname{conv}\{d^S : S \in B\} \subset \operatorname{conv}\{d^S : S \in B(x)\} = F(x)$. This completes the proof. □

Statement of KKMS Theorem. For each $i \leq n$ let $e^i \in R^n$ be an n-vector whose i-th coordinate is 1 and 0 otherwise. Denote for each $S \in \mathscr{N}$, $e^S := \sum_{i \in S} e^i$. A subfamily $\mathscr{B}$ of $\mathscr{N}$ is said to be *balanced* if there are nonnegative weights λ^S, $S \in \mathscr{B}$, such that $e^N = \sum_{S \in \mathscr{B}} \lambda^S e^S$. One can prove that $\mathscr{B}$ is balanced if and only if $m^N \in \operatorname{conv}\{m^S : S \in \mathscr{B}\}$, were m^S is the center of gravity of the face Δ^S, that is, $m^S = \frac{e^S}{|S|}$, where $|S|$ denotes the cardinality of S. In fact we need to know that $m^N \in \operatorname{conv}\{m^S : S \in \mathscr{B}\}$ implies $e^N = \sum_{S \in \mathscr{B}} \lambda^S e^S$. But it is obvious, because if $m^N = \sum_{S \in \mathscr{B}} t^S m^S$, where $t^S \geq 0$, then $e^N = \sum_{S \in \mathscr{B}} \lambda^S e^S$, where $\lambda^S = \frac{nt^S}{|S|}$.

Conversely, assume that $\mathscr{B}$ is balanced i.e., (1) $e^N = \sum_{S\in\mathscr{B}} \lambda^S e^S$, where $\lambda^S \geq 0$. Then (2) $m^N = \frac{e^N}{n} = \sum_{S\in B} \frac{\lambda^S|S|}{n}\frac{e^S}{|S|} = \sum_{S\in B} t^S m^S$. We shall show that $1 = \sum_{S\ B} \frac{S\ S}{n}$. Let $B_i = \{S \in B : i \in S\}$. The condition (1) means that $1 = \sum_{S\in B_i} \lambda^S$. Hence $n = \sum_{i=1}^{n} \sum_{S\in B_i} \lambda^S = \sum_{S\in B} \lambda^S |S|$. The last equality is a consequence of the fact that each set $S \in B$ appears in the sum $\sum_{i=1}^{n} \sum_{S\in B_i}$ $|S|$-times.

Replacing points d^S by the points m^S we immediately obtain a point $x \in \Delta$ such that $m^N \in \operatorname{conv}\{m^S : x \in C^S\}$ and this is exactly Shapley's theorem.

Theorem (KKMS). *Let $\{C^S : S \in \mathscr{N}\}$ be a family of closed subsets of Δ and assume that $\Delta^T \subset \bigcup_{S\subset T} C^S$ for each $T \in \mathscr{N}$.*

Then there exists a balanced family $\mathscr{B}$ such that $\bigcap_{S\ \mathscr{B}} C^S \notin \emptyset$.

KKMS Theorem is an extension of KKM Theorem. To see this, let us assume that $C^S = \emptyset$ for each set S of cardinality greater that 1. Under this assumption the family $\{\{i\}: i \in N\}$ is the only balanced family. Let $C^i := C^{\{i\}}$. Then we immediately get

Theorem (KKM). *Let $\{C^i : i \in N\}$ be a family of closed subsets of Δ and assume that $\Delta^T \subset \bigcup_{i\in T} C^i$ for each non-empty subset $T \subset N$.*

Then $\bigcap_{i\in N} C^i \neq \emptyset$.

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