Mad Families and Ultrafilters

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This expository note presents Shelah's proof of the consistency of $\mathfrak{u} < \mathfrak{a}$ assuming the consistency of the existence of a measurable cardinal where $\mathfrak{u}$ is the ultrafilter number and $\mathfrak{a}$ is the almost-disjointness number. We also discuss more recent related work of Shelah on characters of ultrafilters on ω.

Introduction

In a major breakthrough in iterated forcing theory [Sh2], Shelah proved the consistency of both $\mathfrak{d} < \mathfrak{a}$ and $\mathfrak{u} < \mathfrak{a}$ where $\mathfrak{d}$ is the *dominating number*, $\mathfrak{u}$ is the *ultrafilter number* and $\mathfrak{a}$ is the *almost-disjointness number.*

Recall that for functions $f, g \in \omega^\omega$, g *eventually dominates* f $(f \leq^* g$ in symbols) if $f(n) \leq g(n)$ holds for all but finitely many n. $\mathfrak{d}$ is the least size of a cofinal family in the ordering $(\omega^\omega, \leq^*)$. For sets $A, B \subseteq \omega$, A is *almost contained* in B ($A \subseteq^* B$ in symbols) if $A \setminus B$ is finite. $\mathscr{A}$ is a *base* of a free ultrafilter $\mathscr{U}$ on ω if $\mathscr{A} \subseteq \mathscr{U}$ and for all $U \in \mathscr{U}$ there is $A \in \mathscr{A}$ such that $A \subseteq^* U$. The *character* $\chi(\mathscr{U})$ of $\mathscr{U}$ is the least size of a base of $\mathscr{U}$ and $\mathfrak{u}$ is the smallest cardinal which is a character of an ultrafilter. $\mathscr{A} \subseteq [\omega]^\omega$ is an *almost disjoint family* (*a.d. family*, for short) if $|A \cap B| < \aleph_0$ for distinct members A, B of $\mathscr{A}$. An a.d. family $\mathscr{A}$ is *maximal* (a *mad family*, for short) if for all $C \in [\omega]^\omega$ there is $A \in \mathscr{A}$ such that $|A \cap C = \aleph_0$. $\mathfrak{a}$ is the least size of a mad family.

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Shelah [Sh2] produced two rather distinct models for his consistency results. The easier approach uses a measurable cardinal and yields the consistency of both $\mathfrak{d} < \mathfrak{a}$ and $\mathfrak{u} < \mathfrak{a}$. It is based on the simple and ingenious observation thhat if κ is measurable, $\mathcal{D}$ is a κ-complete ultrafilter on κ, and $\mathbb{P}$ is a ccc forcing notion which forces that $\mathfrak{a}$ has size at least κ, then the ultrapower $\mathbb{P}^\kappa/\mathcal{D}$ (which contains $\mathbb{P}$ as a complete subforcing) destroys all mad families of the intermediate extension $V^{\mathbb{P}}$ (see also Lemma 4 below). Thus iterating $\mathbb{P}$ by taking ultrapowers makes $\mathfrak{a}$ large. On the other hand, $\mathbb{P}$ can be chosen so as to add an ultrafilter base which still generates an ultrafilter in $V^{\mathbb{P}^\kappa/\mathcal{D}}$ so that $\mathfrak{u}$ stays small along the iteration.

The other, more complicated, approach is via *iterations along templates*, a powerful technique which generalizes traditional finite support iteration. So far it has been used only for the consistency of $\mathfrak{d} < \mathfrak{a}$, and it is unclear whether it can be modified to also yield the consistency of $\mathfrak{u} < \mathfrak{a}$ (see the discussion after Question 1 in Section 3). It provides a new, albeit more intricate, description of the proof of $CON(\mathfrak{d} < \mathfrak{a})$ from a measurable, but it also gives the consistency of, say, $\aleph_2 = \mathfrak{d} < \mathfrak{a} = \aleph_3$ on the basis of ZFC alone. The present author has since used the template technique to prove that $\mathfrak{a} = \aleph_\omega$ is consistent [Br2].

In this note, we present an account of Shelah's first approach, focusing on the consistency of $\mathfrak{u} < \mathfrak{a}$. In Section 1, we discuss the interplay between iterated forcing constructions and ultrapowers in some generality and part of this material may be useful for other purposes as well. Section 2 explains how Laver forcing with an ultrafilter fits into the framework of Section 1. In Section 3, then, we prove the consistency of $\mathfrak{u} < \mathfrak{a}$ (Theorem 1), present more recent work of Shelah on characters of ultrafilters [Sh3] which is methodologically closely related, and close with some open problems.

Let $\mathsf{Spec}(\chi) = \{\mu : \chi(\mathcal{U}) = \mu$ for some ultrafilter $\mathcal{U}\}$ denote the *spectrum of characters*. In [BrSh] it was proved that $\mathsf{Spec}(\chi)$ can be large, and it was asked whether it is consistent that $Spec(\chi)$ is not convex when restricted to regular cardinals [BrSh, Question (5) in Section 8], i.e., whether it is consistent that $\mu < \kappa < \lambda$ are regular, $\{\mu,\lambda\} \subseteq \mathsf{Spec}(\chi)$ and $\kappa \notin \mathsf{Spec}(\chi)$. Theorems 2 and 3 in Section 3 give a positive answer, assuming again the consistency of the existence of a measurable cardinal. Recall that $\mathcal{A} \subseteq [\omega]^\omega$ is a π-*base* of an ultrafilter $\mathcal{U}$ if for all $U \in \mathcal{U}$ there is $A \in \mathcal{A}$ such that $A \subseteq^* U$. It is well-known and easy to see (see, e.g., [Sh3]) that $\mathcal{A}$ is a π-base of some ultrafilter iff for any partition $\langle B_i : i < n\rangle$ of ω there are $A \in \mathcal{A}$ and $i < n$ such that $A \subseteq^* B_i$. The π-*character* $\pi\chi(\mathcal{U})$ of $\mathcal{U}$ is the least size of a π-base of $\mathcal{U}$ and the *reaping number* $\mathfrak{r}$ is the smallest cardinal which is a π-character of an ultrafilter. Clearly every base is a π-base so that $\pi\chi(\mathcal{U}) \leq \chi(\mathcal{U})$ and $\mathfrak{r} \leq \mathfrak{u}$. We also discuss the *spectrum of* π-*characters* $\mathsf{Spec}(\pi\chi) = \{\mu : \pi\chi(\mathcal{U}) = \mu$ for some ultrafilter $\mathcal{U}\}$ in our models.

Our notation is fairly standard. For cardinal invariants of the continuum see [Bl]. Apart from the cardinals defined above, we also need the *unbounding number*

$\mathfrak{b}$ which is defined as the least size of a family $\mathscr{F} \subseteq \omega^\omega$ which is unbounded in the eventual dominance ordering $(\omega^\omega, \leq^*)$. It is well-known that $\mathfrak{b} \leq \mathfrak{r}$ [Bl, Theorem 3.8] and $\mathfrak{b} \leq \mathfrak{a}$ [Bl, Proposition 8.4].

1. Iterations and ultrapowers

In this section, we introduce a few basic notions, and prove several results, which provide a general framework for iterating an iterated forcing construction and, thus, for building up a matrix of iterations. In the successor step, we take the ultrapower of the iterated forcing construction (see Lemma 5 and Corollary 6 below), while the limit step produces an iteration which is close, though not necessarily identical, to the direct limit of the earlier iterations (see Lemma 7 below). First we explain our (somewhat uncanonical) notion of iteration.

Let μ be an ordinal. A sequence of p.o.'s $\mathscr{P} = \langle \mathbb{P}_\gamma : \gamma \leq \mu \rangle$ is called an *iteration* if $\mathbb{P}_\gamma \lessdot \mathbb{P}_\delta$ for $\gamma < \delta$. Here $\lessdot$ denotes complete embeddability of forcing notions. Note that we do *not* require that $\mathbb{P}_\delta$ is any kind of limit of $\mathbb{P}_\gamma$, $\gamma < \delta$, for limit ordinals δ.

For simplicity, assume all $\mathbb{P}_\gamma$ are complete Boolean algebras (cBa's). For $\gamma < \delta$, let h^δ_γ be the natural projection from $\mathbb{P}_\delta$ to $\mathbb{P}_\gamma$, defined by $h^\delta_\gamma(p) = \prod \{q \in \mathbb{P}_\gamma : p \leq q\}$ for $p \in \mathbb{P}_\delta$. The *support* of $p \in \mathbb{P}_\mu$ is defined by

$$\operatorname{supp}(p) = \{\delta : \text{there is no } \gamma < \delta \text{ such that } h^\mu_\delta(p) = h^\mu_\gamma(p)\}.$$

Note that $\delta + 1 \in \operatorname{supp}(p)$ iff $h^\mu_{\delta+1}(p) < h^\mu_\delta(p)$. Similarly, for limit ordinals δ, $\delta \in \operatorname{supp}(p)$ iff $h^\mu_\delta(p) < h^\mu_\gamma(p)$ for all $\gamma < \delta$.

An iteration $\mathscr{P}$ has *finite supports* if $\operatorname{supp}(p)$ is finite for all $p \in \mathbb{P}_\mu$. While this is not the classical concept of a finite support iteration, it is easy to see an iteration $\mathscr{P}$ with finite supports is equivalent to a finite support iteration in the standard sense in a natural way:

Lemma 1. *Assume $\mathscr{P}$ has finite supports. Let $\delta \leq \mu$ be a limit ordinal. Then* $\operatorname{lim\,dir}_{\gamma<\delta}\mathbb{P}_\gamma \lessdot \mathbb{P}_\delta$.

Here, "lim dir" denotes the direct limit of forcing notions.

Proof. Let $p \in \mathbb{P}_\delta$. If $\delta \notin \operatorname{supp}(p)$, then $p = h^\mu_\delta(p) = h^\mu_{\gamma_0}(p)$ for some $\gamma_0 < \delta$. So $p \in \mathbb{P}_{\gamma_0} \subseteq \bigcup_{\gamma<\delta}\mathbb{P}_\gamma$, and there is nothing to prove.

If $\delta \in \operatorname{supp}(p)$, then $p = h^\mu_\delta(p)$ for all $\gamma < \delta$. However, since supports are finite, there is $\gamma_0 < \delta$ such that $h^\mu_{\gamma_0}(p) = h^\mu_\gamma(p)$ for all γ with $\gamma_0 \leq \gamma < \delta$. Set $p_0 = h^\mu_{\gamma_0}(p)$. So $p_0 \in \mathbb{P}_\gamma \subseteq \bigcup_{\gamma<\delta}\mathbb{P}_\gamma = \operatorname{lim\,dir}_{\gamma<\delta}\mathbb{P}_\gamma$. We claim p_0 is a reduction of p to $\operatorname{lim\,dir}_{\gamma<\delta}\mathbb{P}_\gamma$. Suppose $q \leq p_0$ belongs to $\operatorname{lim\,dir}_{\gamma<\delta}\mathbb{P}_\gamma$. There is $\gamma < \delta$, $\gamma \geq \gamma_0$, such that $q \in \mathbb{P}_\gamma$. Since $h^\mu_\gamma(p) = p_0 \geq q$ in $\mathbb{P}_\gamma$, p and q are indeed compatible in $\mathbb{P}_\delta$ (with common extension $p \cdot q$). $\square$

Assuming $\mathscr{P}$ has finitte supports, define $\mathbb{P}'_{\gamma+1} = \mathbb{P}_\gamma$ for $\gamma \leq \mu$, and let $\dot{\mathbb{Q}}'_{\gamma+1}$ be the $\mathbb{P}_\gamma$-name for the quotient forcing $\mathbb{P}_{\gamma+1}/\dot{G}_\gamma$ for $\gamma < \mu$ (where $\dot{G}_\gamma$ genotes as usual the canonical name for the $\mathbb{P}_\gamma$-generic filter). So $\mathbb{P}_{\gamma+1} = \mathbb{P}_\gamma \star \dot{\mathbb{Q}}'_{\gamma+1}$. Also set $\mathbb{P}'_0 = \{\mathbf{0,1}\}$, the trivial forcing, and $\mathbb{Q}'_0 = \mathbb{P}_0$. Finally, for limit ordinals $\delta \leq \mu$, define $\mathbb{P}'_\delta = \lim \operatorname{dir}_{\gamma<\delta}\mathbb{P}_\gamma$ and let $\dot{\mathbb{Q}}'_\delta$ be the $\lim \operatorname{dir}_{\gamma<\delta}\mathbb{P}_\gamma$-name for the quotient forcing $\mathbb{P}_\delta/\dot{G}_{\lim \operatorname{dir}_{\gamma<\delta}\mathbb{P}_\gamma}$. Then $\langle\langle \mathbb{P}'_\gamma : \gamma \leq \mu + 1\rangle, \langle \dot{\mathbb{Q}}'_\gamma : \gamma \leq \mu\rangle\rangle$ is naturally equivalent with $\mathscr{P} = \langle \mathbb{P}_\gamma : \gamma \leq \mu\rangle$.

The point of our approach is that it admits an easier description of ultrapowers of iterations. Before discussing this, we need to review some basic facts about ultrapowers of p.o.'s. Much of the subsequent material (in particular, Lemmata 2 through 4) can be found in [Br1].

Let κ be measurable, let $\mathscr{D}$ be a κ-complete ultrafilter on κ, and let $\mathbb{P}$ be a p.o. The *ultrapower* $\mathbb{P}^\kappa/\mathscr{D} = \{[f] = f/\mathscr{D} : f \in \mathbb{P}^\kappa\}$ where $[f] = f/\mathscr{D} = \{g \in \mathbb{P}^\kappa : \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in \mathscr{D}\}$ is partially ordered by $[f] \leq [g]$ if $\{\alpha < \kappa : f(\alpha) < g(\alpha)\} \in \mathscr{D}$. As usual, we identify $p \in \mathbb{P}$ with $[f] \in \mathbb{P}^\kappa/\mathscr{D}$ given by $f(\alpha) = p$ for all $\alpha < \kappa$.

Lemma 2. (Shelah [Sh2], see also [Br1, Lemma 0.1]) *If* $\mathbb{P}$ *is* κ*-cc, then* $\mathbb{P} \lessdot \mathbb{P}^\kappa/\mathscr{D}$.

Proof. Let $\nu < \kappa$, and let $A = \{p_\gamma : \gamma < \nu\}$ be a maximal antichain in $\mathbb{P}$. Let $f \in \mathbb{P}^\kappa$ be arbitrary. Then for all $\alpha < \kappa$ there is $\gamma < \nu$ such that $f(\alpha)$ and p_γ are compatible. By κ-completeness of $\mathscr{D}$, there is $\gamma < \nu$ such that $\alpha : f(\alpha)$ and p_γ are compatible$\} \in \mathscr{D}$. This means, however, that $[f]$ and p_γ are compatible in $\mathbb{P}^\kappa/\mathscr{D}$. Thus A is still a maximal antichain in $\mathbb{P}^\kappa/\mathscr{D}$. $\square$

The converse also holds: if $\mathbb{P}$ is not κ-cc, then $\mathbb{P}$ does not completely embed into $\mathbb{P}^\kappa/\mathscr{D}$.

Lemma 3. (Shelah [Sh2], see also [Br1, Lemma 0.2]) *If* $\mathbb{P}$ *is* ν*-cc for some* $\nu < \kappa$*, then so is* $\mathbb{P}^\kappa/\mathscr{D}$.

Proof. Assume $f_\gamma \in \mathbb{P}^\kappa$, $\gamma < \nu$, are arbitrary. For each $\alpha < \kappa$ there are $\gamma_0 < \gamma_1 < \nu$ such that $f_{\gamma_0}(\alpha)$ and $f_{\gamma_1}(\alpha)$ are compatible. By κ-completeness of $\mathscr{D}$, there are $\gamma_0 < \gamma_1 < \nu$ such that $\{\alpha : f_{\gamma_0}(\alpha)$ and $f_{\gamma_1}(\alpha)$ are compatible$\} \in \mathscr{D}$. Thus $[f_{\gamma_0}]$ and $[f_{\gamma_1}]$ are compatible. Hence every antichain of $\mathbb{P}^\kappa/\mathscr{D}$ has size less than ν. $\square$

If $\mathbb{P}$ is not ν-cc for any $\nu < \kappa$, then $\mathbb{P}^\kappa/\mathscr{D}$ is not κ-cc.

Assume now $\mathbb{P}$ is ccc. Then $\mathbb{P} \lessdot \mathbb{P}^\kappa/\mathscr{D}$ and $\mathbb{P}^\kappa/\mathscr{D}$ is ccc as well by the previous lemmata.

Let $f_n \in \mathbb{P}^\kappa$, $n \in \omega$, be such that $\{[f_n] : n \in \omega\}$ is a maximal antichain of $\mathbb{P}^\kappa/\mathscr{D}$. Then the set $\{\alpha : \{f_n(\alpha) : n \in \omega\}$ is a maximal antichain of $\mathbb{P}\}$ belongs to $\mathscr{D}$. Hence, by changing each f_n on a set of coordinates which is small with respect to $\mathscr{D}$ (and thus not changing $[f_n]$ at all), we may assume without loss of generality that for all $\alpha < \kappa$, $\{f_n(\alpha) : n \in \omega\}$ is a maximal antichain of $\mathbb{P}$.

Recall that a $\mathbb{P}$-name $\dot{x}$ for a real (an element of ω^ω) is completely determined by maximal antichains $\{p_{n,i} : n \in \omega\}$ and numbers $\{k_{n,i} : n \in \omega\}$, $i \in \omega$, such that

$$p_{n,i} \Vdash \dot{x}(i) = k_{n,i}.$$

Thus a $\mathbb{P}^\kappa/\mathscr{D}$-name $\dot{y}$ for a real is determined by maximal antichains $\{p^\alpha_{n,i} : n \in \omega\}$ and numbers $\{k_{n,i} : n \in \omega\}$, $i \in \omega$, $\alpha < \kappa$, such that, letting $f_{n,i}(\alpha) = p^\alpha_{n,i}$,

$$[f_{n,i}] \Vdash \dot{y}(i) = k_{n,i}.$$

Since $\{p^\alpha_{n,i} : n \in \omega\}$ and $\{k_{n,i} : n \in \omega\}$, $i \in \omega$, determine a $\mathbb{P}$-name $\dot{x}^\alpha$ for a real by

$$p^\alpha_{n,i} \Vdash \dot{x}^\alpha(i) = k_{n,i},$$

we may think of $\dot{y}$ as the *mean* or *average* of the $\dot{x}^\alpha$ and write $\dot{y} = \langle \dot{x}^\alpha : \alpha < \kappa \rangle/\mathscr{D}$.

Lemma 4. (Shelah [Sh2], see also [Br1, Lemma 0.3]) *Let $\mathbb{P}$ be ccc. Assume $\nu \geq \kappa$ and $\dot{\mathscr{A}}$ is a $\mathbb{P}$-name for an a.d. family of size ν. Then $\mathbb{P}^\kappa/\mathscr{D}$ forces that $\dot{\mathscr{A}}$ is not maximal. In particular, if $\mathbb{P}$ forces $\mathfrak{a} \geq \kappa$, then no a.d. family of $V^{\mathbb{P}}$ is maximal in $V^{\mathbb{P}^\kappa}$.*

Proof. Assume $\dot{\mathscr{A}} = \{\dot{A}^\gamma : \gamma < \nu\}$ where all $\dot{A}^\gamma$ are $\mathbb{P}$-names for infinite subsets of ω. Then $\dot{A} = \langle \dot{A}^\alpha : \alpha < \kappa \rangle/\mathscr{D}$ is a $\mathbb{P}^\kappa/\mathscr{D}$-name for an infinite subset of ω by the preceding discussion.

(More explicitly, if the $\dot{A}^\gamma$ are determined by $\{p^\gamma_{n,i} : n \in \omega\}$ and $\{k^\gamma_{n,i} \in 2 : n \in \omega\}$, $i \in \omega$, such that $p^\gamma_{n,i} \Vdash i \in \dot{A}^\gamma$ if $k^\gamma_{n,i} = 1$ and $p^\gamma_{n,i} \Vdash i \notin \dot{A}^\gamma$ if $k^\gamma_{n,i} = 0$, then, letting $f_{n,i}(\alpha) = p^\alpha_{n,i}$ for $\alpha < \kappa$ and $k_{n,i}$ such that $\{\alpha : k^\alpha_{n,i} = k_{n,i}\} \in \mathscr{D}$, $\{[f_{n,i}] : n \in \omega\}$ and $\{k_{n,i} : n \in \omega\}$ determine $\dot{A}$.)

Fix $\gamma < \nu$. Since for all $\alpha < \kappa$ with $\alpha \neq \gamma$,

$$\Vdash_{\mathbb{P}} |\dot{A}^\gamma \cap \dot{A}^\alpha| < \aleph_0,$$

we see $\{\alpha < \kappa : \Vdash_{\mathbb{P}} |\dot{A}^\gamma \cap \dot{A}^\alpha| < \aleph_0\}$belongs to $\mathscr{D}$. Thus

$$\Vdash_{\mathbb{P}^\kappa \mathscr{D}} |\dot{A}^\gamma \cap \dot{A}| < \aleph_0$$

because $\dot{A}$ is the average of the $\dot{A}^\alpha$. $\square$

We next describe the connection between iterations and ultrapowers.

Lemma 5. *Assume $\mathbb{P} < \mathbb{Q}$. Then $\mathbb{P}^\kappa/\mathscr{D} \lessdot \mathbb{Q}^\kappa/\mathscr{D}$. Furthermore, the projection mapping is given by $h([f]) = \langle h(f(\alpha)) : \alpha < \kappa \rangle/\mathscr{D}$ for $f \in \mathbb{Q}^\kappa$.*

Proof. This is straightforward by elementary equivalence of a structure and its ultrapower, but we provide details for the sake of completeness.

If $f_\gamma \in \mathbb{P}^\kappa$, $\gamma < \nu$, are such that $\{[f_\gamma] : \gamma < \nu\}$ is a maximal antichain in $\mathbb{P}^\kappa/\mathscr{D}$, then $\{\alpha : \{f_\gamma(\alpha) : \gamma < \nu\}$ is a maximal antichain in $\mathbb{P}\} \in \mathscr{D}$ so that $\{\alpha : \{f_\gamma(\alpha) : \gamma < \nu\}$ is a maximal antichain in $\mathbb{Q}\} \in \mathscr{D}$ and $\{[f_\gamma] : \gamma < \nu\}$ is a maximal antichain in $\mathbb{Q}^\kappa/\mathscr{D}$.

If $g \in \mathbb{P}^\kappa/\mathscr{D}$ is such that $[g] \leq \langle h(f(\alpha)) : \alpha < \kappa\rangle/\mathscr{D}$, then $\{\alpha : g(\alpha) \leq h(f(\alpha))\} \in \mathscr{D}$. Thus $\{\alpha : g(\alpha)$ is compatible with $f(\alpha)\} \in \mathscr{D}$ and $[g]$ is compatible with $[f]$. Hence $\langle h(f(\alpha)) : \alpha < \kappa\rangle/\mathscr{D} \leq h([f])$.

On the other hand, if $[g]$ is incompatible with $\langle h(f(\alpha)) : \alpha < \kappa\rangle/\mathscr{D}$, then, by the same reasoning, $[g]$ is incompatible with $[f]$. Hence $\langle h(f(\alpha)) : \alpha < \kappa\rangle/\mathscr{D} \geq$ $\geq h([f])$. $\square$

Corollary 6. *Let $\mathscr{P} = \langle \mathbb{P}_\gamma : \gamma \leq \mu\rangle$ be an iteration. Then $\mathscr{P}^\kappa/\mathscr{D} = \langle (\mathbb{P}_\gamma)^\kappa/\mathscr{D} :$ $: \gamma \leq \mu\rangle$ is also an iteration. Furthermore, if $\mathscr{P}$ has finite supports, then so has $\mathscr{P}^\kappa/\mathscr{D}$.*

Proof. The first part is immediate by the previous lemma.

So assume $\mathscr{P}$ has finite supports. Let $f \in (\mathbb{P}_\mu)^\kappa$. Then $\operatorname{supp}(f(\alpha))$ is finite for all α, say $\operatorname{supp}(f(\alpha)) = \{\gamma_0^\alpha < \gamma_1^\alpha < \ldots < \gamma_{n_\alpha - 1}^\alpha\}$ for all α. By ω_1-completeness of $\mathscr{D}$ there is an $n \in \omega$ such that $\{\alpha : n_\alpha = n\}$ belongs to $\mathscr{D}$. For each $i < n$ there is γ_i such that

- either $\{\alpha : \gamma_i^\alpha = \gamma_i\} \in \mathscr{D}$
- or $\{\alpha : \gamma_i^\alpha < \gamma_i\} \in \mathscr{D}$ while $\{\alpha : \gamma_i^\alpha \leq \delta\} \notin \mathscr{D}$ for all $\delta < \gamma$.

In the latter case, we necessarily have $cf(\gamma_i) = \kappa$ by the κ-completeness of $\mathscr{D}$.

We claim that $\operatorname{supp}([f]) = \{\gamma_i : i < n\}$. In particular, $|\operatorname{supp}([f])| \leq n$ and so $\operatorname{supp}([f])$ is finite. (Note that in the second case above $\gamma_i = \gamma_j$ for $i < j$ is possible so that $|\operatorname{supp}([f])| < n$ is possible.)

Indeed,

$$\begin{aligned}
\gamma \in \operatorname{supp}([f]) &\Leftrightarrow \forall \delta < \gamma : h_\gamma^\mu([f]) < h_\delta^\mu([f]) \\
&\Leftrightarrow \forall \delta < \gamma : \{\alpha : h_\gamma^\mu(f(\alpha)) < h_\delta^\mu(f(\alpha))\} \in \mathscr{D} && \text{(by Lemma 5)} \\
&\Leftrightarrow \text{either } \{\alpha : \gamma \in \operatorname{supp}(f(\alpha))\} \in \mathscr{D} && \text{(first case)} \\
& \text{ or } cf(\gamma) = \kappa \text{ and} \\
& \ \forall \delta < \gamma : \{\alpha : (\delta, \gamma) \cap \operatorname{supp}(f(\alpha)) \neq \emptyset\} \in \mathscr{D} && \text{(second case)} \\
&\Leftrightarrow \gamma = \gamma_i \text{ for some } i < n.
\end{aligned}$$

$\square$

Note that even if $\mathbb{P}_\delta = \lim \operatorname{dir}_{\gamma<\delta} \mathbb{P}_\gamma$ for all limits δ, this is not necessarily true for $(\mathbb{P}_\delta)^\kappa/\mathscr{D}$. Indeed, by the proof of Corollary 6, $\lim \operatorname{dir}_{\gamma<\delta} (\mathbb{P}_\gamma)^\kappa/\mathscr{D} \lneq\!\circ\, (\mathbb{P}_\delta)^\kappa/\mathscr{D}$ for δ with $cf(\delta) = \kappa$.

The previous corollary tells us we can iterate an iteration by taking ultrapowers. We next describe what we do with a sequence of iterations in the limit step.

Let $\mathbb{P}_{0\wedge 1} <\!\circ\, \mathbb{P}_i <\!\circ\, \mathbb{P}_{0\vee 1}$, $i \in \{0,1\}$, be cBa's. We say the projections are *correct* if $h_1^{0\vee 1}(p_0) = h_{0\wedge 1}^0(p_0)$ for all $p_0 \in \mathbb{P}_0$ iff $h_0^{0\vee 1}(p_1) = h_{0\wedge 1}^1(p_1)$ for all $p_1 \in \mathbb{P}_1$ iff whenever $h_{0\wedge 1}^0(p_1) = h_{0\wedge 1}^1(p_1)$ then p_0 and p_1 are compatible in $\mathbb{P}_{0\vee 1}$. For more on correctness see [Br3], [Br5] or [Br6]. The following lemma is a special case of the more general amalgamated limit construction (see [Br5] and [Br6]).

Lemma 7. *Let μ and λ be limit ordinals. Assume $\mathscr{P}^\alpha = \langle \mathbb{P}^\alpha_\gamma : \gamma \leq \mu \rangle$, $\alpha < \lambda$, are iterations such that $\mathbb{P}^\alpha_\gamma <\!\circ\, \mathbb{P}^\beta_\gamma$ for $\alpha < \beta < \lambda$ and $\gamma \leq \mu$. Also assume $\langle \mathbb{P}^\lambda_\gamma : \gamma < \mu \rangle$ is an iteration such that $\mathbb{P}^\alpha_\gamma <\!\circ\, \mathbb{P}^\lambda_\gamma$ for $\alpha < \lambda$ and $\gamma < \mu$. Further assume that for all $\alpha < \beta \leq \lambda$ and $\gamma < \delta \leq \mu$ with $(\beta, \delta) \neq (\lambda, \mu)$, the projections in $\mathbb{P}^\alpha_\gamma < \mathbb{P}^\beta_\gamma$, $\mathbb{P}^\alpha_\delta <\!\circ\, \mathbb{P}^\beta_\delta$ are correct.*

Then there is $\mathbb{P}^\lambda_\mu$ such that $\mathscr{P}^\lambda = \langle \mathbb{P}^\lambda_\gamma : \gamma \leq \mu \rangle$ is an iteration and $\mathbb{P}^\alpha_\mu < \mathbb{P}^\lambda_\mu$ for all $\alpha < \lambda$. Further, correctness is preserved.

Assume also all $\mathscr{P}^\alpha$ and $\langle \mathbb{P}^\lambda_\gamma : \gamma < \mu \rangle$ have finite supports. Then so does $\mathscr{P}^\lambda$.

Proof. Elements of $\mathbb{P}^\lambda_\mu$ are formal products of the form $p \cdot q$ where $p \in \mathbb{P}^\alpha_\mu$ for some $\alpha < \lambda$, $q \in \mathbb{P}^\lambda_\gamma$ for some $\gamma < \mu$, and $h^{\gamma,\lambda}_{\gamma,\alpha}(q) = h^{\mu,\alpha}_{\gamma,\alpha}(p)$ (where the h's denote the obvious projections). In this case, we say the pair α, γ witnesses $p \cdot q \in \mathbb{P}^\lambda_\mu$. For formal products $p_0 \cdot q_0$ with witness α, γ and $p_1 \cdot q_1$ with witness β, δ where $\beta \geq \alpha$ and $\delta \geq \gamma$, we define the partial order $\leq$ on $\mathbb{P}^\lambda_\mu$ by $p_1 \cdot q_1 \leq p_0 \cdot q_0$ if $p_1 \leq p_0$ in $\mathbb{P}^\beta_\mu$ and $q_1 \leq q_0$ in $\mathbb{P}^\lambda_\delta$.

Notice that if a pair α, γ as above witnesses $p \cdot q \in \mathbb{P}^\lambda_\mu$, then β, δ is also a witness for $\beta \geq \alpha$ and $\delta \geq \gamma$. For indeed, by correctness we have $h^{\gamma,\lambda}_{\gamma,\beta}(q) \leq h^{\gamma,\lambda}_{\gamma,\alpha}(q) = h^{\mu,\alpha}_{\gamma,\alpha}(p) = h^{\mu,\beta}_{\gamma,\beta}(p)$. Thus, letting $p' = p \cdot h^{\gamma,\lambda}_{\gamma,\beta}(q) \in \mathbb{P}^\beta_\mu$, we see $h^{\mu,\beta}_{\gamma,\beta}(p') = h^{\gamma,\lambda}_{\gamma,\beta}(q)$ so that the formal product $p' \cdot q$ belongs to $\mathbb{P}^\lambda$ as witnessed by β, γ. Obviously, $p' \cdot q \leq p \cdot q$, and it is easy to see that any $p'' \cdot q'' \leq p \cdot q$ is compatible with $p' \cdot q$ so that $p' \cdot q$ and $p \cdot q$ in fact describe the same condition. By symmetry of the situation, we can also move from γ to δ.

This fact provides us with an easier description of the ordering: give formal products $p_0 \cdot q_0$ and $p_1 \cdot q_1$ in $\mathbb{P}^\lambda_\mu$, we may assume they have the same witness α, γ, and we let $p_1 \cdot q_1 \leq p_0 \cdot q_0$ if $p_1 \leq p_0$ in $\mathbb{P}^\alpha_\mu$ and $q_1 \leq q_0$ in $\mathbb{P}^\lambda_\gamma$.

Next we prove completeness of the embeddings. By symetry it suffices to show, say, $\mathbb{P}^\alpha_\mu <\!\circ\, \mathbb{P}^\lambda_\mu$. Fix $p \cdot q \in \mathbb{P}^\lambda_\mu$ with witness β, γ. By the above discussion we may assume $\beta \geq \alpha$. Let $p_0 = h^{\mu,\beta}_{\mu,\alpha}(p) \in \mathbb{P}^\alpha_\mu$. We need to show any $p_1 \leq p_0$ with $p_1 \in \mathbb{P}^\alpha_\mu$ is compatible with $p \cdot q$. Let $p' = p_1 \in \mathbb{P}^\beta_\mu$. Then $p' \leq p$ and $h^{\mu,\beta}_{\gamma,\beta}(p') \leq h^{\mu,\beta}_{\mu,\beta}(p) = h^{\gamma,\lambda}_{\gamma,\beta}(q)$. Thus, letting $q' = q \cdot h^{\mu,\beta}_{\gamma,\beta}(p') \in \mathbb{P}^\lambda_\gamma$, we get $h^{\gamma,\lambda}_{\gamma,\beta}(q') = h^{\mu,\beta}_{\gamma,\beta}(p')$ and $p' \cdot q' \leq p \cdot q$ as required.

This argument shows, in fact, that $h^{\mu,\lambda}_{\mu,\beta}(p \cdot q) = p \cdot h^{\gamma,\lambda}_{\gamma,\beta}(q)$ for any $\beta \geq \alpha$ where α, γ witnesses $p \cdot q \in \mathbb{P}^\lambda_\mu$. Similarly $h^{\mu,\lambda}_{\delta,\lambda}(p \cdot q) = h^{\mu,\alpha}_{\delta,\alpha}(p) \cdot q$ for any $\delta \geq \gamma$ where α, γ witnesses $p \cdot q \in \mathbb{P}^\lambda_\mu (\bigstar)$.

Preservation of correctness, then, is straightforward.

Assume now all iterations have finite supports. Let $p \cdot q \in \mathbb{P}^\lambda_\mu$. By $(\bigstar)$, we see $\operatorname{supp}(p \cdot q) = \operatorname{supp}(p) \cup \operatorname{supp}(q)$ so that $\mathscr{P}^\lambda$ indeed has finite supports. □

This lemma is well-known and has been used often in the particular situation where the $\langle \langle \mathbb{P}^\alpha_\gamma : \alpha \leq \lambda \rangle, \langle \dot{\mathbb{Q}}^\alpha_\gamma : \gamma < \lambda \rangle \rangle$ are traditional finite support iterations (see, e.g., [BlSh] or [Br4, Section 2]). In this case correctness trivially holds. However, our iterations will be different: in successor steps we take ultrapowers, and in limit steps, not necessarily the direct limit but something larger.

2. Laver forcing with an ultrafilter

We review basic facts about Laver forcing with an ultrafilter, and then consider iterations which cofinally often add a Laver generic with respect to an appropriate ultrafilter. We shall see how we can iterate such iterations by taking ultrapowers in successor steps (Lemma 10) and a natural limit in limit steps (Lemma 11), and we prove such iterations are ccc (Lemma 12).

Let $\mathcal{U}$ be an ultrafilter on ω. *Laver forcing* $\mathbb{L}_{\mathcal{U}}$ with $\mathcal{U}$ consists of all subtrees $T \subseteq \omega^{<\omega}$ such that for all $\sigma \in T$ with $\mathrm{stem}(T) \subseteq \sigma$, the set of successor nodes $\mathrm{succ}_T(\sigma) = \{n : \sigma \frown n \in T\}$ belongs to $\mathcal{U}$. For $\sigma \in T$, the restriction of T to σ is $T_\sigma = \{\tau \in T : \sigma \subseteq \tau \text{ or } \tau \subseteq \sigma\}$. $\mathbb{L}_{\mathcal{U}}$ is a σ-centered forcing notion which generically adds a real $\ell_{\mathcal{U}} = \bigcup\{\mathrm{stem}(T) : T \in G\}$ (where G is the generic filter) which dominates the ground model reals and whose range diagonalizes $\mathcal{U}$. The latter means that $\mathrm{ran}\,(\ell_{\mathcal{U}}) \subseteq^* U$ for all $U \in \mathcal{U}$.

Lemma 8. (Shelah [Sh2], see also [Br4, Lemma 2.1]) *Assume $V \subseteq W$ are models of ZFC, $\mathcal{U}$ is an ultrafilter on ω in V, $\mathcal{V}$ is an ultrafilter in W such that $\mathcal{U} \subseteq \mathcal{V}$. Assume $\ell_{\mathcal{V}}$ is $\mathbb{L}_{\mathcal{V}}$-generic over W. Then $\ell_{\mathcal{V}}$ is also $\mathbb{L}_{\mathcal{U}}$-generic over V.*

Proof. Let $D \in V$ be an open dense subset of $\mathbb{L}_{\mathcal{U}}$. For $\sigma \in \omega^{<\omega}$ define the *rank* $\varrho_D(\sigma)$ by induction on the ordinals.

$$\begin{aligned} \varrho_D(\sigma) = 0 &\Leftrightarrow \exists T \in D \text{ such that } \mathrm{stem}(T) = \sigma \\ \text{for } \alpha \geq 1 : \varrho_D(\sigma) = \alpha &\Leftrightarrow \text{there is no } \beta < \alpha \text{ such that } \varrho_D(\sigma) = \beta \\ &\qquad \text{and } \{n : \varrho_D(\sigma \frown n) < \alpha\} \in \mathcal{U} \end{aligned}$$

We first claim that the rank σ_D is defined for all $\sigma \in \omega^{<\omega}$. For assume $\varrho_D(\sigma)$ is undefined. Then, clearly, $\{n : \varrho_D(\sigma \frown n) \text{ undefined}\} \in \mathcal{U}$. So we can recursively construct a tree $T \in \mathbb{L}_{\mathcal{U}}$ with $\mathrm{stem}(T) = \sigma$ and $\varrho_D(\tau)$ is undefined for all $\tau \in T$ with $\sigma \subseteq \tau$. Let $S \leq T$ be such that $S \in D$, and let $\tau = \mathrm{stems}(S)$. Then $\varrho_D(\tau) = 0$ by definition of ϱ_D and $\varrho_D(\tau)$ is undefined because $\sigma \subseteq \tau$ and $\tau \in T$, by construction of T, a contradiction.

We next prove that D is still predense in $\mathbb{L}_{\mathcal{V}}$ in the model W. For take $T \in \mathbb{L}_{\mathcal{V}}$. Let $\sigma = \mathrm{stem}(T)$. By induction on $\varrho_D(\sigma)$, we show that there is $S \in D$ such that T and S are compatible. If $\varrho_D(\sigma) = 0$, then there is $S \in D$ with $\mathrm{stems}(S) = \sigma$. Thus T and S are compatible. If $\varrho_D(\sigma) > 0$, there is $n \in \mathrm{succ}_T(\sigma)$ such that $\varrho_D(\sigma \frown n) < \varrho_D(\sigma)$. So, by induction hypothesis, there is $S \in D$ such that $T_{\sigma \frown n}$ and S are compatible. Since $T_{\sigma \frown n} \leq T$, T and S are compatible as well. □

Corollary 9. *Assume $\mathbb{P} \lessdot \mathbb{Q}$ are forcing notions, $\dot{\mathcal{U}}$ is a $\mathbb{P}$-name for an ultrafilter on ω and $\dot{\mathcal{V}}$ is a $\mathbb{Q}$-name for an ultrafilter on ω such that $\Vdash_{\mathbb{Q}} \dot{\mathcal{U}} \subseteq \dot{\mathcal{V}}$. Then $\mathbb{P} \star \mathbb{L}_{\dot{\mathcal{U}}} \lessdot \mathbb{Q} \star \mathbb{L}_{\dot{\mathcal{V}}}$.*

Proof. This is immediate from the previous lemma. □

Let us return to the situation where we look at the ultrapower $\mathbb{P}^\kappa/\mathscr{D}$ of a ccc p.o. $\mathbb{P}$. Assume $\dot{\mathscr{U}} = \{\dot{U}^\gamma : \gamma < \nu\}$ is $\mathbb{P}$-name for an ultrafilter on ω such that whenever $\Vdash \dot{V} \in \dot{\mathscr{U}}$ then there is $\gamma < \nu$ such that $\vdash \dot{V} = \dot{U}^\gamma$. The latter condition which roughly says the name is rich enough is used to make some technical arguments go through more smoothly. Let $\dot{\mathscr{U}}^\kappa/\mathscr{D} = \{\langle \dot{U}^{g(\alpha)} : \alpha < \kappa\rangle/\mathscr{D} : g : \kappa \to \nu\}$.

Lemma 10. *$\dot{\mathscr{U}}^\kappa\ \mathscr{D}$ is a $\mathbb{P}^\kappa$-name for an ultrafilter on ω which contains the ultrafilter $\dot{\mathscr{U}}$. Further, $\mathbb{P} \star \mathbb{L}_{\dot{\mathscr{U}}} <\!\circ\ \mathbb{P}^\kappa\ \mathscr{D} \star \mathbb{L}_{\dot{\mathscr{U}}^\kappa\ \mathscr{D}}$ and, letting $\mathbb{Q} = \mathbb{P} \star \mathbb{L}_{\dot{\mathscr{U}}}$, we have $\mathbb{Q}^\kappa/\mathscr{D} \cong \mathbb{P}^\kappa/\mathscr{D} \star \mathbb{L}_{\dot{\mathscr{U}}^\kappa\ \mathscr{D}}$.*

Proof. Again, this is straightforward by elementary equivalence of a structure and its ultrapower. We give some details for the sake of completeness.

For $g \cdot \kappa \to \nu$, set $\dot{U}^g = \langle \dot{U}^{g(\alpha)} : \alpha < \kappa\rangle/\mathscr{D}$. By the discussion in Section 1, $\dot{U}^g$ is a $\mathbb{P}^\kappa/\mathscr{D}$-name for a subset of ω. Since the $\mathbb{P}$-name $\dot{U}^\gamma$ is identified with the $\mathbb{P}^\kappa\ \mathscr{D}$-name $\dot{U}^g$ where g is the function with constant value γ, $\mathbb{P}^\kappa/\mathscr{D}$ forces that $\dot{\mathscr{U}} \subseteq \dot{\mathscr{U}}^\kappa/\mathscr{D}$.

We next check that $\dot{\mathscr{U}}^\kappa/\mathscr{D}$ is forced to be an ultrafilter.

Assume $\Vdash_{\mathbb{P}^\kappa\ \mathscr{D}} \dot{U}^g \subseteq \dot{V}$. Then $\dot{V} = \langle \dot{V}^\alpha : \alpha < \kappa\rangle/\mathscr{D}$ and $\{\alpha : \Vdash_{\mathbb{P}} \dot{U}^{g(\alpha)} \subseteq \dot{V}^\alpha\} \in \mathscr{D}$. Hence there is $k : \kappa \to \nu$ such that $\{\alpha : \Vdash \dot{V}^\alpha = \dot{U}^{k(\alpha)}\} \in \mathscr{D}$. Thus $\Vdash_{\mathbb{P}^\kappa\ \mathscr{D}} \dot{V} = \dot{U}^k \in$ $\in \dot{\mathscr{U}}^\kappa/\mathscr{D}$.

Next let $\dot{U}^{g_0}$ and $\dot{U}^{g_1} \in \dot{\mathscr{U}}^\kappa/\mathscr{D}$. Then $\Vdash_{\mathbb{P}} \dot{U}^{g_0(\alpha)} \cap \dot{U}^{g_1(\alpha)} \in \dot{\mathscr{U}}$ for all α. Thus there is k such that $\Vdash_{\mathbb{P}} \dot{U}^{g_0(\alpha)} \cap \dot{U}^{g_1(\alpha)} = \dot{U}^{k(\alpha)}$ for all α. Hence $\Vdash_{\mathbb{P}^\kappa\ \mathscr{U}} \dot{U}^{g_0} \cap \dot{U}^{g_1} = \dot{U}^k \in$ $\in \dot{\mathscr{U}}^\kappa/\mathscr{D}$.

Finally assume $\Vdash_{\mathbb{P}^\kappa\ \mathscr{D}} \dot{V} \subseteq \omega$. Then $\dot{V} = \langle \dot{V}^\alpha : \alpha < \kappa\rangle/\mathscr{D}$ and if we let $\dot{W}^\alpha$ be such that $[\![\dot{W}^\alpha = \dot{V}^\alpha]\!] = [\![\dot{V}^\alpha \in \dot{\mathscr{U}}]\!]$ and $[\![\dot{W}^\alpha = \omega\backslash\dot{V}^\alpha]\!] = [\![\dot{V}^\alpha \notin \dot{\mathscr{U}}]\!]$, we see that $\Vdash_{\mathbb{P}} W^\alpha \in \dot{\mathscr{U}}$ for all α. So there is k such that $\Vdash_{\mathbb{P}} \dot{W}^\alpha = \dot{U}^{k(\alpha)}$ and $\Vdash_{\mathbb{P}^\kappa\ \mathscr{D}}$ "either $\dot{V} = \dot{U}^k \in \dot{\mathscr{U}}^\kappa/\mathscr{D}$ or $\omega\backslash\dot{V} = \dot{U}^k \in \dot{\mathscr{U}}^\kappa/\mathscr{D}$".

Thus $\dot{\mathscr{U}}^\kappa/\mathscr{D}$ is indeed an ultrafilter.

$\mathbb{P} \star \mathbb{L}_{\dot{\mathscr{U}}} < \mathbb{P}^\kappa/\mathscr{D} \star \mathbb{L}_{\dot{\mathscr{U}}^\kappa\ \mathscr{D}}$ is immediate from Corollary 9.

We finally show $\mathbb{Q}^\kappa/\mathscr{D} \cong \mathbb{P}^\kappa/\mathscr{D} \star \mathbb{L}_{\dot{\mathscr{U}}^\kappa\ \mathscr{D}}$.

Conditions in $\mathbb{Q}$ are of the form $q = (p, \dot{T}) \in \mathbb{P} \star \mathbb{L}_{\dot{\mathscr{U}}}$ where $p \Vdash \dot{T} \in \mathbb{L}_{\dot{\mathscr{U}}}$. There is $g : \omega^{<\omega} \to \nu$ such that $p \Vdash$ "if $\tau \in \dot{T}$ and $\mathrm{stem}(\dot{T}) \subseteq \tau$ then $\mathrm{succ}_{\dot{T}}(\tau) = \dot{U}^{g(\tau)}$".

Hence conditions $[f]$ in $\mathbb{Q}^\kappa/\mathscr{D}$ are given by $f : \kappa \to \mathbb{Q}$ and $g : \kappa \times \omega^{<\omega} \to \nu$ where $f(\alpha) = (p^\alpha, \dot{T}^\alpha) \in \mathbb{P} \star \mathbb{L}_{\dot{\mathscr{U}}}$ is such that $p^\alpha \Vdash$ "$\dot{T}^\alpha \in \mathbb{L}_{\dot{\mathscr{U}}}$ and $\mathrm{stem}(\dot{T}^\alpha) \subseteq \tau$ then $\mathrm{succ}_{\dot{T}^\alpha}(\tau) = \dot{U}^{g(\alpha,\tau)}$".

On the other hand, conditions $(p, \dot{T})$ in $\mathbb{P}^\kappa/\mathscr{D} \star \mathbb{L}_{\dot{\mathscr{U}}^\kappa\ \mathscr{D}}$ are given by $\langle \dot{V}_\tau : \tau \in \omega^{<\omega}\rangle$ such that $p \in \mathbb{P}^\kappa/\mathscr{D}$ and $p \Vdash$ "$\dot{T} \in \mathbb{L}_{\dot{\mathscr{U}}^\kappa\ \mathscr{D}}$ and if $\tau \in \dot{T}$ and $\mathrm{stem}(\dot{T}) \subseteq \tau$ then $\mathrm{succ}_{\dot{T}}(\tau) =$ $= \dot{V}_\tau$". p is of the form $p = \langle p^\alpha : \alpha < \kappa\rangle/\mathscr{D}$ while each $\dot{V}_\tau$ is a $\mathbb{P}^\kappa/\mathscr{D}$-name for a member of $\dot{\mathscr{U}}^\kappa/\mathscr{D}$ and thus is of the form $\dot{V}_\tau = \langle \dot{U}^{g(\alpha,\tau)} : \alpha < \kappa\rangle/\mathscr{D}$.

Hence the mapping e which sends a condition $[f] \in \mathbb{Q}^\kappa/\mathscr{D}$ given by $p^\alpha, \dot{T}^\alpha$ and g to $(p, \dot{T}) \in \mathbb{P}^\kappa/\mathscr{D} \star \mathbb{L}_{\dot{\mathscr{U}}^\kappa\ \mathscr{D}}$ given by $\langle \dot{V}_\tau : \tau \in \omega^{<\omega}\rangle$, where $p = \langle p^\alpha : \alpha < \kappa\rangle/\mathscr{D}$ and $\dot{V}_\tau = \langle \dot{U}^{g(\alpha,\tau)} : \alpha < \kappa\rangle/\mathscr{D}$, is an embedding with dense range. We leave it to the reader to verify that $[f'] \leq [f]$ iff $e([f']) \leq e([f])$ for $[f], [f'] \in \mathbb{Q}^\kappa/\mathscr{D}$. $\square$

Note that, since $\mathbb{Q} \lessdot \mathbb{Q}^\kappa/\mathcal{D}$ by Lemma 2, the proof of the last part also shows that $\mathbb{P} \star \mathbb{L}_{\dot{\mathcal{U}}} \lessdot \mathbb{P}^\kappa/\mathcal{D} \star \mathbb{L}_{\dot{\mathcal{U}}^\kappa\mathcal{D}}$. In particular, Lemma 8 and Corollary 9 were not really needed for the latter. They are needed, however, to deal with the limit step.

Lemma 11. *Assume λ is a limit ordinal, $\mathcal{P}^\alpha = \langle \mathbb{P}^\alpha_\gamma : \gamma \leq \mu \rangle$ are iterations and $\langle \dot{\mathcal{U}}^\alpha_\gamma : \gamma \leq \mu \rangle$ are $\mathbb{P}^\alpha_\gamma$-names for ultrafilters for $\alpha < \lambda$ such that*

- $\mathbb{P}^\alpha_{\gamma+1} = \mathbb{P}^\alpha_\gamma \star \mathbb{L}_{\dot{\mathcal{U}}^\alpha_\gamma}$,
- $\mathbb{P}^\alpha_\gamma \lessdot \mathbb{P}^\beta_\gamma$ *for* $\alpha < \beta < \lambda$,
- $\Vdash_{\mathbb{P}^\beta_\gamma} \dot{\mathcal{U}}^\alpha_\gamma \subseteq \dot{\mathcal{U}}^\beta_\gamma$ *for* $\alpha < \beta < \lambda$.

Then there are an iteration $\mathcal{P}^\lambda = \langle \mathbb{P}^\lambda_\gamma : \gamma \leq \mu \rangle$ and $\mathbb{P}^\lambda_\gamma$-names for ultrafilters $\langle \dot{\mathcal{U}}^\lambda_\gamma : \gamma \leq \mu \rangle$ such that

- $\mathbb{P}^\lambda_{\gamma+1} = \mathbb{P}^\lambda_\gamma \star \mathbb{L}_{\dot{\mathcal{U}}^\lambda_\gamma}$,
- $\mathbb{P}^\alpha_\gamma \lessdot \mathbb{P}^\lambda_\gamma$ *for* $\alpha < \lambda$,
- $\Vdash_{\mathbb{P}^\lambda_\gamma} \dot{\mathcal{U}}^\alpha_\gamma \subseteq \dot{\mathcal{U}}^\lambda_\gamma$ *for* $\alpha < \lambda$.

Furthermore, if all $\mathcal{P}^\alpha$ have finite supports, then so does $\mathcal{P}^\lambda$.

Proof. We build $\mathbb{P}^\lambda_\gamma$ and $\dot{\mathcal{U}}^\lambda_\gamma$ by recursion on γ. Once $\mathbb{P}^\lambda_\gamma$ has been defined we let $\dot{\mathcal{U}}^\lambda_\gamma$ be a $\mathbb{P}^\lambda_\gamma$-name for an ultrafilter containing $\bigcup_{\alpha<\lambda} \dot{\mathcal{U}}^\alpha_\gamma$.

For $\gamma = 0$ let $\mathbb{P}^\lambda_0 = \lim \mathrm{dir}_{\alpha<\lambda} \mathbb{P}^\alpha_0$.

Assume $\mathbb{P}^\lambda_\gamma$ has been defined. Then let $\mathbb{P}^\lambda_{\gamma+1} = \mathbb{P}^\lambda_\gamma \star \mathbb{L}_{\dot{\mathcal{U}}^\lambda_\gamma}$. By Corollary 9, $\mathbb{P}^\alpha_{\gamma+1} \lessdot \mathbb{P}^\lambda_{\gamma+1}$.

Assume γ is a limit ordinal, and previous $\mathbb{P}^\lambda_\delta$ have been defined. Then we choose $\mathbb{P}^\lambda_\gamma$ according to Lemma 7. $\mathbb{P}^\alpha_\gamma \lessdot \mathbb{P}^\lambda_\gamma$ follows.

The furthermore-clause also follows from Lemma 7. $\square$

We next need to argue the iterations $\mathcal{P}^\alpha$ we want to build are ccc. Since we will take ultrapowers in successor steps, we know by Lemma 3 that if $\mathbb{P}^\alpha_\mu$ is ccc then so is $\mathbb{P}^{\alpha+1}_\mu$. But what about $\mathbb{P}^\lambda_\mu$ for limit λ? Since we iterate forcings of type $\mathbb{L}_{\mathcal{U}}$ with finite support, it is clear that if $\mathbb{P}^\lambda_\gamma$ is ccc then so is $\mathbb{P}^\lambda_{\gamma+1}$. Further, it δ is a limit ordinal and all $\mathbb{P}^\lambda_\gamma$ are ccc for $\gamma < \delta$, the so is $\lim \mathrm{dir}_{\gamma<\delta} \mathbb{P}^\lambda_\gamma$. But what about the perhaps larger $\mathbb{P}^\lambda_\delta$? This issue is addressed by

Lemma 12. *Assume $\mathcal{P}^\alpha = \langle \mathbb{P}^\alpha_\gamma : \gamma \leq \mu \rangle$ are iterations and $\langle \dot{\mathcal{U}}^\alpha_\gamma : \gamma \leq \mu \rangle$ are $\mathbb{P}^\alpha_\gamma$-names for ultrafilters, $\alpha \leq \lambda$, such that*

- $\mathbb{P}^0_0 = \{\mathbf{0},\mathbf{1}\}$ *and* $\mathbb{P}^0_\delta = \lim \mathrm{dir}_{\gamma<\delta} \mathbb{P}^0_\gamma$ *for limit* δ,
- $\mathbb{P}^\alpha_{\gamma+1} = \mathbb{P}^\alpha_\gamma \star \mathbb{L}_{\dot{\mathcal{U}}^\alpha_\gamma}$ *for all* α,
- $\mathbb{P}^{\alpha+1}_\gamma = (\mathbb{P}^\alpha_\gamma)^\kappa/\mathcal{D}$ *and* $\dot{\mathcal{U}}^{\alpha+1}_\gamma = (\dot{\mathcal{U}}^\alpha_\gamma)^\kappa/\mathcal{D}$,
- $\mathbb{P}^\beta_\gamma$ *is built according to Lemmata 7 and 11 for limit* β*, in particular* $\mathbb{P}^\alpha_\gamma \lessdot \mathbb{P}^\beta_\gamma$ *and* $\Vdash_{\mathbb{P}^\beta_\gamma} \dot{\mathcal{U}}^\alpha_\gamma \subseteq \dot{\mathcal{U}}^\beta_\gamma$ *for* $\alpha < \beta$.

Then all $\mathbb{P}^\alpha_\gamma$ satisfy property K.

Proof. By recursion on $\alpha \leq \lambda$, we define linear orders I^α, dense sets $D^\alpha_\gamma \subseteq \mathbb{P}^\alpha_\gamma$ and functions s^α_γ, $\gamma \leq \mu$, such that

(i) $\mu = I^0$, $I^\alpha \subseteq I^\beta$ for $\alpha \leq \beta$,

(ii) $D^\alpha_\gamma \subseteq D^\beta_\delta$ and $s^\alpha_\gamma \subseteq s^\beta_\delta$ for $\alpha \leq \beta$ and $\gamma \leq \delta$,

(iii) $\mathrm{dom}(s^\alpha_\gamma) = D^\alpha_\gamma$ and $s^\alpha_\gamma(p): I^\alpha \to \omega^{<\omega}$ is a finite partial functions for all $p \in D^\alpha_\gamma$,

(iv) if $\delta < \gamma$ and $s^\alpha_\gamma(p)(\delta) = \sigma$ then, letting $h^{\gamma,\alpha}_{\delta+1,\alpha}(p) = (h^{\gamma,\alpha}_{\delta,\alpha}(p), \dot{T}) \in \mathbb{P}^\alpha_\delta \star \mathbb{L}_{\dot{\mathscr{U}}^\alpha_\delta}$, $h^{\gamma,\alpha}_{\delta,\alpha}(p) \Vdash \mathrm{stem}(\dot{T}) = \sigma$,

(v) if $\delta < \gamma$, $s^\alpha_\gamma(p)$ and $s^\alpha_\gamma(q)$ agree on their common domain, and $r_0 \leq$ $\leq h^{\gamma,\alpha}_{\delta,\alpha}(p), h^{\gamma,\alpha}_{\delta,\alpha}(q)$, then there is $r \leq p, q$ with $r \leq p, q$ with $h^{\gamma,\alpha}_{\delta,\alpha}(r) = r_0$.

Note that for $\delta = 0$, this means in particular that p and q in $\mathbb{P}^\alpha_\gamma$ are compatible if $s^\alpha_\gamma(p)$ and $s^\alpha_\gamma(q\}$ agree on their common domain. By the Δ-system lemma it is then immediate that $\mathbb{P}^\alpha_\gamma$ has property K. So it suffices to carry out the recursion.

Basic step: $\alpha = 0$. Let $I^0 = \mu$. We define D^0_γ and s^0_γ with the required properties by recursion on $\gamma \leq \mu$.

If $\gamma = 0$, there is nothing to do.

Assume $\gamma = \delta + 1$ is successor, and D^0_δ and s^0_δ have been defined. $p \in \mathbb{P}^0_\gamma$ is of the form $(p_0, \dot{T}) \in \mathbb{P}^0_\delta \star \mathbb{L}_{\dot{\mathscr{U}}^0_\delta}$. There are $p'_0 \leq p_0$ and $\sigma \in \omega^{<\omega}$ such that $p'_0 \Vdash \mathrm{stem}(\dot{T}) = \sigma$. Thus, if we let $D^0_\gamma = D^0_\delta \cup \{(p_0, \dot{T}) \in \mathbb{P}^0_\gamma : p_0 \in D^0_\delta$ and $p_0 \Vdash \mathrm{stem}(\dot{T}) = \sigma$ for some $\sigma \in \omega^{<\omega}\}$, then D^0_γ is dense in $\mathbb{P}^0_\gamma$. Also, for $p = (p_0, \dot{T}) \in D^0_\gamma \setminus D^0_\delta$, we define s^0_γ by $\mathrm{dom}(s^0_q(p)) = \mathrm{dom}(s^0_\delta(p_0)) \cup \{\delta\}$, $s^0_\delta(p_0) \subseteq s^0_\gamma(p)$ and $s^0_\gamma(p)(\delta) = \sigma$ where σ is such that $p_0 \Vdash \mathrm{stem}(\dot{T}) = \sigma$. Then (iv) is satisfied and we need to show (v).

If $p = (p_0, T), q = (q_0, S) \in D^0_\gamma$, $s^0_\gamma(q)$ agree on their common domain, and $r_0 \leq p_0, q_0$, then, letting $\dot{R} = \dot{T} \cap \dot{S}$, we see that $r_0 \Vdash$ "$\dot{R} \in \mathbb{L}_{\dot{\mathscr{U}}^0_\delta}$, $\dot{R} \leq \dot{T}, \dot{S}$, and $\mathrm{stem}(\dot{R}) = \sigma$" where $\sigma = s^0_\gamma(p) = s^0_\gamma(q)$. Thus $r = (r_0, \dot{R}) \leq p, q$ is as required. The rest follows by induction hypothesis (v).

Finally assume γ is a limit ordinal. Since $\mathbb{P}^0_\gamma = \lim \mathrm{dir}_{\delta<\gamma} \mathbb{P}^0_\delta$, $D^0_\delta = \bigcup_{\delta<\gamma} D^0_\delta$ is dense in $\mathbb{P}^0_\gamma$, and we let $s^0_\gamma = \bigcup_{\delta<\gamma} s^0_\delta$.

Successor step: $\alpha = \beta + 1$. Then $\mathbb{P}^\alpha_\gamma = (\mathbb{P}^\beta_\gamma)^\kappa/\mathscr{D}$ for all $\gamma \leq \mu$. We let $I^\alpha = (I^\beta)^\kappa/\mathscr{D}$. I^α is linearly ordered by $[x] \leq [y]$ if $\{\bar{\alpha} : x(\bar{\alpha}) \leq y(\bar{\alpha})\} \in \mathscr{D}$ for $x, y : \kappa \to I^\beta$. Clearly $\mu \subseteq I^\beta \subseteq I^\alpha$. Next let $D^\alpha_\gamma = \{[f] \in \mathbb{P}^\alpha_\gamma : f : \kappa \to \mathbb{P}^\beta_\gamma$ and $\{\bar{\alpha} : f(\bar{\alpha}) \in D^\beta_\gamma\} \in \mathscr{D}\}$. Clearly D^α_γ is dense and $D^\beta_\delta \subseteq D^\alpha_\gamma$ for $\delta \leq \gamma$. For such f, and for $\bar{\alpha}$ with $f(\bar{\alpha}) \in D^\beta_\gamma$, $\mathrm{dom}(s^\beta_\gamma(f(\bar{\alpha}))) \subseteq I^\beta$ is finite, say $\mathrm{dom}(s^\beta_\gamma(f(\bar{\alpha}))) = \{i^{\bar{\alpha}}_0 <$ $< i^{\bar{\alpha}}_1 < \ldots < i^{\bar{\alpha}}_{n_\alpha - 1}\}$, By ω_1-completeness of $\mathscr{D}$, there is n such that $\{\bar{\alpha} : n_{\bar{\alpha}} = n\} \in \mathscr{D}$. Define $x_j : \kappa \to I^\beta$ by

$$x_j(\bar{\alpha}) = \begin{cases} i^{\bar{\alpha}}_j & \text{if this is defined} \\ 0 & \text{otherwise} \end{cases}$$

for $j < n$. (Note the first case occurs $\mathscr{D}$-almost everywhere.) Then $\{[x_0] <$ $< [x_1] < \ldots < [x_{n-1}]\} \subseteq I^\alpha$ and we let $\mathrm{dom}(s^\alpha_\gamma([f])) = \{[x_j] : j < n\}$. Applying once more the ω_1-completeness of $\mathscr{D}$, we see there are $\sigma_j \in \omega^{<\omega}$, $j < n$, such that

$\{\bar\alpha : s^{\alpha}_{\gamma}(f(\bar\alpha))(i^{\bar\alpha}_j) = \sigma_j\} \in \mathscr{D}$. Thus we let $s^{\alpha}_{\gamma}([f])([x_j]) = \sigma_j$ for $j < n$. Clearly $s^{\beta}_{\delta} \subseteq s^{\alpha}_{\gamma}$ for $\delta \le \gamma$.

For (iv), if $\delta < \gamma$ and $s^{\alpha}_{\gamma}([f])(\delta) = \sigma$, then $\{\bar\alpha : s^{\beta}_{\gamma}(f(\bar\alpha))(\delta) = \sigma\} \in \mathscr{D}$. Also, $h^{\gamma,\alpha}_{\delta+1,\alpha}([f]) = (h^{\gamma,\alpha}_{\delta,\alpha}([f]), \dot T) \in (\mathbb{P}^{\beta}_{\delta})^{\kappa}/\mathscr{D} \star \mathbb{L}_{(\mathscr{U}^{\beta}_{\delta})^{\kappa}/\mathscr{D}}$ is identified with $\langle h^{\gamma,\beta}_{\delta+1,\beta}(f(\bar\alpha)) : : \bar\alpha < \kappa\rangle/\mathscr{D} = \langle (h^{\gamma,\beta}_{\delta,\beta}(f(\bar\alpha)), \dot T^{\bar\alpha}) : \bar\alpha < \kappa\rangle/\mathscr{D} \in (\mathbb{P}^{\beta}_{\delta} \star \mathbb{L}_{\mathscr{U}^{\beta}_{\delta}})^{\kappa}/\mathscr{D}$ where $\dot T^{\bar\alpha} = \langle \dot T : \bar\alpha < \kappa\rangle\ \mathscr{D}$ (see Lemmata 5 and 10). By induction hypothesis (iv), we know $\{\bar\alpha : h^{\gamma,\beta}_{\delta,\beta}(f(\bar\alpha)) \Vdash \mathrm{stem}(\dot T^{\bar\alpha}) = \sigma\} \in \mathscr{D}$. Therefore, $h^{\gamma,\alpha}_{\delta,\alpha}([f]) \Vdash \mathrm{stem}(\dot T) = \sigma$.

To show (v), assume $\delta < \gamma$, $[f], [g] \in D^{\alpha}_{\gamma}$, $s^{\alpha}_{\gamma}([f])$ and $s^{\alpha}_{\gamma}([g])$ agree on their common domain, and $[h_0] \le h^{\gamma,\alpha}_{\delta,\alpha}([f]), h^{\gamma,\alpha}_{\delta,\alpha}([g])$. So $[h_0] \in \mathbb{P}^{\alpha}_{\delta}$. Let $\{[x_0] < [x_1] < < \ldots < [x_{n-1}]\}$ list the common domain of $s^{\alpha}_{\gamma}([f])$ and $s^{\alpha}_{\gamma}([g])$. Then $\{\bar\alpha : x_0(\bar\alpha) < \ldots < x_{n-1}(\bar\alpha)\}$ is the common domain of $s^{\beta}_{\gamma}(f(\bar\alpha))$ and $s^{\beta}_{\gamma}(g(\bar\alpha))$ and $s^{\beta}_{\gamma}(f(\bar\alpha))$ and $s^{\beta}_{\gamma}(g(\bar\alpha))$ agree on this common domain$\} \in \mathscr{D}$. Also $\{\bar\alpha : h_0(\bar\alpha) \le \le h^{\gamma,\beta}_{\delta,\beta}(f(\bar\alpha)), h^{\gamma,\beta}_{\delta,\beta}(g(\bar\alpha))\} \in \mathscr{D}$ (see Lemma 5). For $\bar\alpha$ which belong to both sets we find, by induction hypothesis (v), $h(\bar\alpha) \le f(\bar\alpha)$, $g(\bar\alpha)$ with $h^{\gamma,\beta}_{\delta,\beta}(h(\bar\alpha)) = h_0(\bar\alpha)$. So $[h] \le [f]$, $[g]$ and $h^{\gamma,\alpha}_{\delta,\alpha}([h]) = [h_0]$, as required.

Limit step: α is a limit ordinal. Let $I^{\alpha} = \bigcup_{\beta<\alpha} I^{\beta}$, equipped with the obvious ordering. As in the basic step, we define D^{α}_{γ} and s^{α}_{γ} by recursion on $\gamma \le \mu$.

The cases $\gamma = 0$ and $\gamma = \delta + 1$ are identical to the basic step. The only difference is that, this time, D^{α}_{γ} must contain all D^{β}_{γ}, and that s^{α}_{γ}, and that s^{α}_{γ} must extend all s^{β}_{γ}, for $\beta < \alpha$. We then use the induction hypothesis for (iv) to see (v) still holds (this is the only place where we actually need (iv)).

So assume γ is a limit ordinal, and D^{α}_{δ} and s^{α}_{δ} have been defined for $\delta < \gamma$. Since supports are finite (see Lemmata 7 and 9), we know that $\bar{\mathbb{P}}^{\alpha}_{\gamma} <\!\circ\ \mathbb{P}^{\alpha}_{\gamma}$ where $\bar{\mathbb{P}}^{\alpha}_{\gamma} := \lim \mathrm{dir}_{\delta<\gamma}\mathbb{P}^{\alpha}_{\delta}$ (see Lemma 1). By the proof of Lemma 7, elements of $\mathbb{P}^{\alpha}_{\gamma}$ are formal products $p \cdot \bar p$ with $p \in \mathbb{P}^{\beta}_{\gamma}$ for some $\beta < \alpha$ and $\bar p \in \bar{\mathbb{P}}^{\alpha}_{\gamma}$, and $h^{\gamma,\alpha}_{\bar\gamma,\beta}(\bar p) = h^{\gamma,\beta}_{\bar\gamma,\beta}(p)$ (where we use $\bar\gamma$ as an index for the direct limit $\bar{\mathbb{P}}^{*}_{\gamma}$ of the $\mathbb{P}^{*}_{\delta}$, $\delta < \gamma$, which completely embeds into $\mathbb{P}^{*}_{\gamma}$). By strengthening p and $\bar p$, if necessary, we may assume $p \in D^{\beta}_{\gamma}$. By further strengthening $\bar p$, we may assume $\bar p \in \bar D^{\alpha}_{\gamma} := \bigcup_{\delta<\gamma} D^{\alpha}_{\delta}$. In general, we will then only have $h^{\bar\gamma,\alpha}_{\bar\gamma,\beta}(\bar p) \le h^{\gamma,\beta}_{\bar\gamma\beta}(p)$, but this does not concern us because the collection of formal products satisfying this weaker condition is obviously forcing equivalent with the original $\mathbb{P}^{\alpha}_{\gamma}$. Hence, if we let D^{α}_{γ} consist of formal products $p \cdot \bar p$ with $p \in D^{\beta}_{\gamma}$ for some $\beta < \alpha$, $\bar p \in \bar D^{\alpha}_{\gamma}$ and $h^{\bar\gamma,\alpha}_{\bar\gamma,\beta}(\bar p) \le h^{\gamma,\beta}_{\bar\gamma,\beta}(p)$, thhen D^{α}_{γ} is dense in $\mathbb{P}^{\alpha}_{\gamma}$. Clearly $\bar D^{\alpha}_{\gamma} \subseteq D^{\alpha}_{\gamma}$ and $D^{\beta}_{\gamma} \subseteq D^{\alpha}_{\gamma}$ for $\beta < \alpha$. For such $p \cdot \bar p \in D^{\alpha}_{\gamma}$, we define s^{α}_{γ} by $\mathrm{dom}(s^{\alpha}_{\gamma}(p \cdot \bar p)) = \mathrm{dom}(\bar s^{\alpha}_{\gamma}(\bar p)) \cup \{i \in \mathrm{dom}(s^{\beta}_{\gamma}(p)) : i > \delta$ for all $\delta < \gamma\}$ where $\bar s^{\alpha}_{\gamma} := \bigcup_{\gamma<\delta} s^{\alpha}_{\delta}$, and

$$s^{\alpha}_{\gamma}(p \cdot \bar p)(i) = \begin{cases} \bar s^{\alpha}_{\gamma}(\bar p)(i) & \text{for } i \in \mathrm{dom}(\bar s^{\alpha}_{\gamma}(\bar p)) \\ s^{\beta}_{\gamma}(p)(i) & \text{for } i \in \mathrm{dom}(s^{\beta}_{\gamma}(p)) \text{ with } i > \delta \text{ for all } \delta < \gamma. \end{cases}$$

Clause (iv) is immediate using the induction hypothesis (iv) for $\bar s^{\alpha}_{\gamma}(\bar p)$.

To prove (v), let $\delta < \gamma$, $p \cdot \bar p$, $q \cdot \bar q \in D^{\alpha}_{\gamma}$, $s^{\alpha}_{\gamma}(p \cdot \bar p)$ and $s^{\alpha}_{\gamma}(q \cdot \bar q)$ agree on their common domain, and $r_0 \le h^{\gamma,\alpha}_{\delta,\alpha}(p \cdot \bar p)$, $h^{\gamma,\alpha}_{\delta,\alpha}(q \cdot \bar q)$. It is immediate (see also ($\bigstar$) in the proof of Lemma 7) that $h^{\gamma,\alpha}_{\delta,\alpha}(p \cdot \bar{p}) = h^{\gamma,\alpha}_{\delta,\alpha}(\bar{p})$ and similarly for $q \cdot \bar{q}$. Thus, by induction hypothesis (v), there is $\bar{r} \leq \bar{p}, \bar{q}$ in $\bar{\mathbb{P}}^{\alpha}_{\gamma}$ such that $h^{\bar{\gamma},\alpha}_{\delta,\alpha}(\bar{r}) = r_0$. Let $\beta_p, \beta_q < \alpha$ be such that $p \in D^{\beta_p}_{\gamma}$ and $q \in D^{\beta_q}_{\gamma}$. Without loss of generality $\beta_p \leq \beta_q$. Let $\beta = \beta_p$. By correctness $h^{\gamma,\beta}_{\gamma,\beta}(p) = h^{\gamma,\beta_p}_{\gamma,\beta_p}(p)$. So we know that $h^{\bar{\gamma},\alpha}_{\bar{\gamma},\beta}(\bar{r}) \leq h^{\bar{\gamma},\alpha}_{\gamma,\beta}(\bar{p}) \leq$ $\leq h^{\bar{\gamma},\alpha}_{\gamma,\beta_p}(\bar{p}) \leq h^{\gamma,\beta_p}_{\gamma,\beta_p}(p) = h^{\gamma,\beta}_{\gamma,\beta}(p)$ and $h^{\bar{\gamma},\alpha}_{\bar{\gamma},\beta}(\bar{r}) \leq h^{\bar{\gamma},\alpha}_{\bar{\gamma},\beta}(\bar{q}) \leq h^{\gamma,\beta}_{\bar{\gamma},\beta}(q)$. Thus by induction hypothesis (v) there is $r \leq p, q$ in $\mathbb{P}^{\beta}_{\gamma}$ such that $h^{\gamma,\beta}_{\bar{\gamma},\beta}(r) = h^{\bar{\gamma},\alpha}_{\bar{\gamma},\beta}(\bar{r})$. Hence $h^{\gamma,\alpha}_{\delta,\alpha}(r \cdot \bar{r}) = h^{\bar{\gamma},\alpha}_{\delta,\alpha}(\bar{r}) = r_0$, and $r \cdot \bar{r} \leq p \cdot \bar{p}$, $q \cdot \bar{q}$ is as required. □

3. Theorems and problems

We are ready to prove the main results.

Theorem 1. (Shelah [Sh2]) *Assume CON (ZFC + there is a measurable cardinal). Then* $CON(ZFC + \mathfrak{u} < \mathfrak{a})$. *More explicitly, if GCH holds, κ is measurable and $\lambda > \mu > \kappa$ are regular, then there is a ccc forcing extension which satisfies* $\mathfrak{u} = \mathfrak{b} = \mathfrak{d} = \mu$ *and* $\mathfrak{a} = \mathfrak{c} = \lambda$.

Proof. By recursion on $\alpha \leq \lambda$ we construct iterations $\mathscr{P}^{\alpha} = \langle \mathbb{P}^{\alpha}_{\gamma} : \gamma \leq \mu \rangle$ and $\mathbb{P}^{\alpha}_{\gamma}$-names for ultrafilters $\langle \dot{\mathscr{U}}^{\alpha}_{\gamma} : \gamma \leq \mu \rangle$ such that

(i) $\mathbb{P}^0_0 = \{\mathbf{0},\mathbf{1}\}$ and $\mathbb{P}^0_{\delta} = \lim \mathrm{dir}_{\gamma<\delta} \mathbb{P}^0_{\gamma}$ for limit δ,

(ii) $\mathbb{P}^{\alpha}_{\gamma+1} = \mathbb{P}^{\alpha}_{\gamma} \star \mathbb{L}_{\dot{\mathscr{U}}^{\alpha}_{\gamma}}$ for all α,

(iii) $\mathbb{P}^{\alpha+1}_{\gamma} = (\mathbb{P}^{\alpha}_{\gamma})^{\kappa}/\mathscr{D}$ and $\dot{\mathscr{U}}^{\alpha+1}_{\gamma} = (\dot{\mathscr{U}}^{\alpha}_{\gamma})^{\kappa}/\mathscr{D}$,

(iv) $\mathbb{P}^{\beta}_{\gamma}$ is built according to Lemmata 7 and 11 for limit β,

(v) if $\dot{\ell}^{\alpha}_{\gamma}$ denotes the canonical name for the $\mathbb{L}_{\dot{\mathscr{U}}^{\alpha}_{\gamma}}$-generic, then $\Vdash_{\mathbb{P}^{\alpha}_{\gamma+1}} \mathrm{ran}(\dot{\ell}^{\alpha}_{\gamma}) \in$ $\in \dot{\mathscr{U}}^{\alpha}_{\gamma+1}$,

(vi) $\Vdash_{\mathbb{P}^{\alpha}_{\delta}} \bigcup_{\gamma<\delta} \dot{\mathscr{U}}^{\alpha}_{\gamma} \subseteq \dot{\mathscr{U}}^{\alpha}_{\delta}$ for limit δ.

Note that by (iii) and (iv) we have $\mathbb{P}^{\alpha}_{\gamma} \lessdot \mathbb{P}^{\beta}_{\gamma}$ and $\Vdash_{\mathbb{P}^{\beta}_{\gamma}} \dot{\mathscr{U}}^{\alpha}_{\beta} \subseteq \dot{\mathscr{U}}^{\beta}_{\gamma}$ for $\alpha < \beta$ and all γ (see Lemmata 2, 7, 10 and 11). Also the name $\dot{\ell}^{\alpha}_{\gamma}$ of (v) does not really depend on α: by Lemma 8 we know that if $\ell_{\gamma} = \ell^{\lambda}_{\gamma}$ is $\mathbb{L}_{\mathscr{U}^{\lambda}_{\gamma}}$-generic over $V[G^{\lambda}_{\gamma}]$, then ℓ_{γ} is also $\mathbb{L}_{\mathscr{U}^{\alpha}_{\gamma}}$-generic over $V[G^{\alpha}_{\gamma}]$ for all $\alpha < \lambda$. So we may simply write ℓ_{γ}. Since $\mathrm{ran}(\dot{\ell}^{\alpha}_{\gamma}) = \mathrm{ran}(\dot{\ell}_{\gamma})$ is forced to diagonalize the ultrafilter $\dot{\mathscr{U}}^{\alpha}_{\gamma}$, it follows from (v) that $\Vdash_{\mathbb{P}^{\alpha}_{\gamma}} \dot{\mathscr{U}}^{\alpha}_{\gamma} \subseteq \dot{\mathscr{U}}^{\alpha}_{\gamma+1}$. Taken togethher with (vi) this means that at each stage α, we build a tower of ultrafilters $\langle \dot{\mathscr{U}}^{\alpha}_{\gamma} : \gamma \leq \mu \rangle$.

By our work in Section 1 and 2, it is immediate that we can carry out the construction satisfying (i) through (iv). So it suffices to argue we can make (v) and (vi) hold as well. Fix α and let $\gamma = \delta + 1$ be a successor ordinal. By induction hypothesis (v), we know that $\Vdash_{\mathbb{P}^{\beta}_{\gamma}} \mathrm{ran}(\dot{\ell}_{\delta}) \in \dot{\mathscr{U}}^{\beta}_{\gamma}$ for $\beta < \alpha$. Hence, if we choose any $\dot{\mathscr{U}}^{\alpha}_{\gamma}$ such that $\Vdash_{\mathbb{P}^{\alpha}_{\gamma}} \dot{\mathscr{U}}^{\beta}_{\gamma} \subseteq \dot{\mathscr{U}}^{\alpha}_{\gamma}$ for all $\beta < \alpha$, (v) will be satisfied automatically. Similarly, if γ is a limit ordinal, the $\mathrm{ran}(\dot{\ell}_{\delta})$, $\delta < \gamma$, are forced to diagonalize the $\dot{\mathscr{U}}^{\alpha}_{\delta}$. By induction hypothesis (v) and (vi), $\Vdash_{\mathbb{P}^{\beta}_{\gamma}}$ "$\mathrm{ran}(\dot{\ell}_{\delta}) \in \dot{\mathscr{U}}^{\beta}_{\gamma}$ for all $\delta < \gamma$" for $\beta < \alpha$. So, choosing $\dot{\mathscr{U}}^{\alpha}_{\gamma}$ such that $\Vdash_{\mathbb{P}^{\alpha}_{\gamma}} \dot{\mathscr{U}}^{\beta}_{\gamma} \subseteq \dot{\mathscr{U}}^{\alpha}_{\gamma}$ for all $\beta < \alpha$, (vi) will automatically be true.

By Lemma 12, $\mathbb{P} = \mathbb{P}_\mu^\lambda$ is ccc.

Since $|\mathbb{P}| = \lambda$ and GCH holds, we see that $\mathfrak{c} \leq \lambda$ in the extension. By construction, $\mathscr{P}^\lambda = \langle \mathbb{P}_\gamma^\lambda : \gamma \leq \mu \rangle$ is a μ-stage iteration which cofinally often adds a dominating real $\ell_\gamma^\lambda = \ell_\gamma$. Thus $\mathfrak{b} = \mathfrak{d} = \mu$ holds in the extension. $\mathbb{P}$ also adds an ultrafilter $\mathscr{U}_\mu^\lambda$ which is the union of the $\mathscr{U}_\gamma^\lambda$ for $\gamma < \mu$ because μ has uncountable cofinality and $\mathbb{P}$ is ccc. However $\operatorname{ran}(\ell_\gamma)$ diagonalizes $\mathscr{U}_\gamma^\lambda$ and is contained in $\mathscr{U}_{\gamma+1}^\lambda$ by clause (v). Thus $\{\operatorname{ran}(\ell_\gamma) : \gamma < \mu\}$ is a base for $\mathscr{U}_\mu^\lambda$ and $\mathfrak{u} \leq \mu$ follows. Since $\mathfrak{b} \leq \mathfrak{u}$ in ZFC, $\mathfrak{u} \geq \mu$ is immediate.

To see $\mathfrak{a} \geq \lambda$, we use Lemma 4: assume $\mathscr{A}$ is an a.d. family of size ν for some $\nu < \lambda$. If $\nu < \mu$, then $\mathscr{A}$ is not maximal because $\mathfrak{b} \leq \mathfrak{a}$ in ZFC. So assume $\nu \geq \mu > \kappa$. By the ccc and the regularity of λ, there is an $\alpha < \lambda$ such that $\dot{\mathscr{A}}$ is a $\mathbb{P}_\mu^\alpha$-name. By Lemma 4, we then see that $\mathbb{P}_\mu^{\alpha+1}$ forces that $\dot{\mathscr{A}}$ is not maximal. Hence we are done. □

We next deal with characters of ultrafilters.

A π-base $\mathscr{A}$ is called *strict* [Sh3] if no subset of $\mathscr{A}$ of size less than $|\mathscr{A}|$ is a π-base. Let $\operatorname{Spec}(\text{strict} - \pi\chi) = \{\nu : \text{there is a strict } \pi\text{-base of size } \nu\}$. It is easy to see that $\mathfrak{r} = \min \operatorname{Spec}(\text{strict} - \pi\chi)$.

Theorem 2. (Shelah [Sh3]) *Assume $CON(ZFC$ + there is a measurable cardinal$)$. Then $CON(ZFC + \operatorname{Spec}(\chi)$ and $\operatorname{Spec}(\text{strict} - \pi\chi)$ are not convex for regulars$)$. More explicitly, if GCH holds, κ is measurable, and $\lambda > \kappa > \mu \geq \aleph_1$ are regular, then there is a ccc forcing extension in which $\mathfrak{u} = \mathfrak{r} = \mathfrak{b} = \mathfrak{d} = \mu$, $\mathfrak{c} = \lambda$, $\{\mu,\lambda\} \subseteq \operatorname{Spec}(\chi) \cap \operatorname{Spec}(\text{strict} - \pi\chi)$ and $\kappa \notin \operatorname{Spec}(\chi) \cup \operatorname{Spec}(\text{strict} - \pi\chi)$.*

Proof. As in the proof of Theorem 1, construct, by recursion on $\alpha \leq \lambda$, $\mathscr{P}^\alpha = \langle \mathbb{P}_\gamma^\alpha : \gamma \leq \mu \rangle$ and $\langle \dot{\mathscr{U}}_\gamma^\alpha : \gamma \leq \mu \rangle$ such that (ii) through (vi) are satisfied and (i) is replaced by

(i') $\mathbb{P}_0^0$ adds κ Cohen reals and $\mathbb{P}_\delta^0 = \lim \operatorname{dir}_{\gamma<\delta} \mathbb{P}_\gamma^0$ for limit δ.

(The reason for doing this is that the forcing has to have size at least κ to guarantee the ultrapower is nontrivial.)

As in the proof of Theorem 1, it is clear this construction can be carried out.

Clearly, $\mathbb{P} = \mathbb{P}_\mu^\lambda$ is ccc. (Lemma 12 is not needed for this because we will automatically have $\mathbb{P}_\delta^\alpha = \lim \operatorname{dir}_{\gamma<\delta} \mathbb{P}_\gamma^\alpha$ for limit δ and all α because $\mu < \kappa$ (cf. the proof of Corollary 6).)

The proof of $\mathfrak{u} = \mathfrak{b} = \mathfrak{d} = \mu$ and $\mathfrak{c} = \lambda$ is as for Theorem 1. Since $\mathfrak{b} \leq \mathfrak{r} \leq \mathfrak{u}$ in ZFC, $\mathfrak{r} = \mu$ is immediate, and $\mu \in \operatorname{Spec}(\chi) \cap \operatorname{Spec}(\text{strict} - \pi\chi)$ follows. In fact, the generic ultrafilter $\mathscr{U}_\mu^\lambda$ satisfies $\pi\chi(\mathscr{U}_\mu^\lambda) = \chi(\mathscr{U}_\mu^\lambda) = \mu$ (see the proof of Theorem 1). It is well-known there is always an ultrafilter $\mathscr{U}$ with $\chi(\mathscr{U}) = \mathfrak{c}$ [vM, Theorem 4.4.2] and, if $\mathfrak{c}$ is regular, with $\pi\chi(\mathscr{U}) = \mathfrak{c}$ [vM, Theorem 4.4.3]. Hence $\lambda \in \operatorname{Spec}(\chi) \cap \operatorname{Spec}(\pi\chi)$. To see $\lambda \in \operatorname{Spec}(\text{strict} - \pi\chi)$, simply take a maximal independent family $\mathscr{I}$ of size $\mathfrak{c}$ and let $\mathscr{A} = \{\bigcap\{I^{f(I)} : I \in \operatorname{dom}(f)\} : f : I \to$ $\to \{1, -1\}$ is a finite partial function$\}$ where $I^1 = I$ and $I^{-1} = \omega \backslash I$. $\mathscr{A}$ is a π-base, but no subfamily of smaller size is a π-base by independence.

To see that $\kappa \notin \mathsf{Spec}(\chi)$, assume $\mathscr{U}$ is an ultrafilter generated by $\mathscr{A} =$ $= \{A_\gamma : \gamma < \kappa\} \subseteq \mathscr{U}$. We may assume that $A_\gamma \not\subseteq^* A_\delta$ for any $\gamma < \delta$ and that this is forced by the trivial condition. By the ccc and the regularity of λ, there is $\alpha < \lambda$ such that $\dot{\mathscr{A}}$ is a $\mathbb{P}^\alpha_\mu$-name. Let $\dot{A} = \langle \dot{A}_\gamma : \gamma < \kappa \rangle / \mathscr{D}$ be the $\mathbb{P}^{\alpha+1}_\mu$-name which is the average of the $\dot{A}_\gamma$ (see the discussion in Section 1). Since

$$\Vdash_{\mathbb{P}^\alpha_\mu} \dot{A}_\gamma \not\subseteq^* \dot{A}_\delta$$

for all $\gamma < \delta$, we see that

$$\Vdash_{\mathbb{P}^{\alpha+1}_\mu} \dot{A}_\gamma \not\subseteq^* \dot{A}.$$

But clearly

$$\Vdash_{\mathbb{P}^\alpha_\mu} \dot{A}_\gamma \not\subseteq^* \omega \backslash \dot{A}_\delta$$

for all δ so that

$$\Vdash_{\mathbb{P}^{\alpha+1}_\mu} \dot{A}_\gamma \not\subseteq^* \omega \backslash \dot{A}.$$

Since either of A and $\omega \backslash A$must belong to $\mathscr{U}$, $\mathscr{A}$ is not a base of $\mathscr{U}$, a contradiction.

Similarly, to show that $\kappa \notin \mathsf{Spec}(\text{strict} - \pi\chi)$, assume $\mathscr{A} = \{A_\gamma : \gamma < \kappa\}$ is a strict π-base. Since $\mathscr{A}$ is strict we may find partitions $\{B^\gamma_i : i < n^\gamma\}$ of ω such that for all $\gamma < \delta$, there is no $i < n^\delta$ with $A_\gamma \subseteq^* B^\delta_i$. Again there is $\alpha < \lambda$ such that $\dot{\mathscr{A}}$ and $\dot{\mathscr{B}} = \{\{\dot{B}^\gamma_i : i < n^\gamma\} : \gamma < \kappa\}$ are $\mathbb{P}^\alpha_\mu$-names. Let $\{\dot{B}_i : i < n\} =$ $= \langle \{\dot{B}^\gamma_i : i < n^\gamma\} \rangle : \gamma < \kappa \rangle / \mathscr{D}$. Fix γ. Since

$$\Vdash_{\mathbb{P}^\alpha_\mu} \forall i < n^\delta \quad \dot{A}_\gamma \not\subseteq^* \dot{B}^\delta_i$$

for all $\gamma < \delta$, we see that

$$\Vdash_{\mathbb{P}^\alpha_\mu} \forall i < n \quad \dot{A}_\gamma \not\subseteq^* \dot{B}_i\,.$$

Thus $\mathscr{A}$ is not a π-base, a contradiction. □

Mixing Theorems 1 and 2 gives

Theorem 3. (Shelah [Sh3]) *If GCH holds, $\kappa_0 < \kappa_1$ are measurable, λ, μ are regular with $\lambda > \kappa_1 > \mu > \kappa_0$, then there is a ccc forcing extension in which $\mathfrak{u} = \mathfrak{r} = \mathfrak{b} = \mathfrak{d} = \mu$, $\mathfrak{a} = \mathfrak{c} = \lambda$, $\{\mu, \lambda\} \subseteq \mathsf{Spec}(\chi) \cap \mathsf{Spec}(\text{strict} - \pi\chi)$ and $\kappa_1 \notin \mathsf{Spec}(\chi) \cup \mathsf{Spec}(\text{strict} - \pi\chi)$.*

Proof. Replace condition (iii) in the proof of Theorem 1 by

(iii') if $\alpha \equiv i \bmod 2$ then $\mathbb{P}^{\alpha+1}_\gamma = (\mathbb{P}^\alpha_\gamma)^{\kappa_i} / \mathscr{D}_i$ and $\dot{\mathscr{U}}^{\alpha+1}_\gamma = (\dot{\mathscr{U}}^\alpha_\gamma)^{\kappa_i} / \mathscr{D}_i$ where $\mathscr{D}_i$ denotes the κ_i-complete ultrafilter on the measurable cardinal κ_i.

This works by the proofs of Theorem 1 and 2. □

A number of problems are left open.

Question 1. (Shelah [Sh1, Question 10.1(2)]) *Does* $CON(ZFC)$ *imply* $CON(ZFC + \mathfrak{u} < \mathfrak{a})$?

As mentioned in the Introduction, the consistency of $\aleph_2 = \mathfrak{d} < \mathfrak{a} = \aleph_3$ has been established on the basis of ZFC alone, by the technique of iteration along templates ([Sh2], see also [Br1]). If $\mathfrak{u} < \mathfrak{a}$ could be done in the template framework, this would give a positive answer to Question 1. While this framework typically works only for nicely definable forcing notions (e.g., Suslin ccc forcing), the embeddability results Lemma 8 and Corollary 9 suggest it should also apply to Laver forcing with an ultrafilter. The main problem, then, is the following: suppose we construct fragments of the iteration and corresponding names for ultrafilters as in the template framework. Can we then extend these names to a name for an ultrafilter in a larger fragment? The simplest instance of this is:

Question 2. *Assume* $\mathbb{P}_{0\wedge 1} \lessdot \mathbb{P}_i \lessdot \mathbb{P}_{0\vee 1}$, $i \in \{0,1\}$, *are forcing notions with correct projections and* $\dot{\mathcal{U}}_i$ *are* $\mathbb{P}_i$*-names for ultrafilters,* $i \in \{0 \wedge 1, 0, 1\}$, *such that* $\Vdash_{\mathbb{P}_i} \dot{\mathcal{U}}_{0\wedge 1} \subseteq \dot{\mathcal{U}}_i$, $i \in \{0,1\}$. *Is there a* $\mathbb{P}_{0\vee 1}$*-name* $\dot{\mathcal{U}}_{0\vee 1}$ *for an ultrafilter such that* $\Vdash_{\mathbb{P}_{0\vee 1}} \dot{\mathcal{U}}_0, \dot{\mathcal{U}}_1 \subseteq \dot{\mathcal{U}}_{0\vee 1}$?

This may be false in general. The real question, then, would be whether it is true for the forcing notions which occur in the iteration.

In all models for $\mathfrak{d} < \mathfrak{a}$. $\mathfrak{d}$ is at least $\aleph_2$ and it is an old open problem of Roitman whether $\aleph_1 = \mathfrak{d} < \mathfrak{a}$ is consistent (see also [Sh1, Question 10.1.(1)]). The same problem is open for $\mathfrak{u}$.

Question 3. *Is* $\aleph_1 = \mathfrak{u} < \mathfrak{a}$ *consistent?*

Concerning Theorems 2 and 3 we may ask

Question 4. *Does* $CON(ZFC)$ *imply* $CON(\mathrm{Spec}(\chi)$ *is not convex for regulars*$)$?

Ideally one would like to have, say, $CON(\mathfrak{c} = \aleph_3$ and $\mathrm{Spec}(\chi) = \{\aleph_1, \aleph_3\})$ (see also [BrSh, Question (5) in Section 8]).

Question 5. *Is it consistent that* $\mathrm{Spec}(\pi\chi)$ *is not convex for regulars?*

The proof of Lemma 12 is rather technical.

Question 6. *Is there a shorter and/or more general proof for the ccc-ness of the forcing?*

References

[Bl] Blass, A., *Combinatorial cardinal characteristics of the continuum,* in: Handbook of Set Theory (A. Kanamori et al., eds.), to appear.

[BlSh] Blass, A. and Shelah, S., *Ultrafilters with small generating sets,* Israel J. Math. 65 (1989), 259–271.

[Br1] Brendle, J., *Mad families and iteration theory,* in: Logic and Algebra (Y. Zhang, ed.), Contemp. Math. 302 (2002), Amer. Math. Soc., Providence, 1 – 31.

[Br2] Brendle, J., *The almost disjointness number may have countable cofinality,* Trans. Amer. Math. Soc. 355 (2003), 2633 – 2649.

[Br3] Brendle, J., *Templates and iterations: Luminy 2002 lecture notes,* RIMS Kōkyūroku 1423 (2005), 1 – 12.

[Br4] Brendle, J., *Van Douwen's diagram for dense sets of rationals,* Ann. Pure Appl. Logic 143 (2006), 54 – 69.

[Br5] Brendle, J., *Measure, category, and forcing theory,* preprint.

[Br6] Brendle, J., *Shattered iterations,* in preparation.

[BrSh] Brendle, J. and Shelah, S., *Ultrafilters on ω — their ideals and their cardinal characteristics,* Trans. Amer. Math. Soc. 351 (1999), 2643 – 2647.

[Sh1] Shelah, S., *On what I do not understand (and have something to say),* Fund. Math. 166 (2000), 1 – 82. (publication number 666)

[Sh2] Shelah, S., *Two cardinal invariants of the continuum $\mathfrak{d} < \mathfrak{a}$ and FS linearly ordered iterated forcing,* Acta Math. 192 (2004), 187 – 223. (publication number 700)

[Sh3] Shelah, S., *The spectrum of characteristics of ultrafilters on ω*, preprint. (publication number 846)

[vM] van Mill, J., *An introduction to $\beta\omega$,* in: Handbook of Set-Theoretic Topology (K. Kunen and J. Vaughan, eds.), North-Holland, Amsterdam (1984), 503 – 567.