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Typical $\mathscr{F}_\sigma$ Sets and Typical Continuous Functions

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We state a theorem that connects typical $\mathscr{F}_\sigma$ sets and knot points of typical continuous functions, and give some remarks on related topics.

1. Introduction

This article is based upon the author's talk in the 34th Winter School in Abstract Analysis on his paper [Sa3], and deals with some related topics as well. The reader is reminded that the file used in the talk is available at http://www.ucl.ac.uk/~ucahssa/eng/maths/talks.html.

Let I denote the unit interval $[0, 1]$ and $C(I)$ the Banach space consisting of all realvalued continuous functions on I, equipped with the supremum norm $\|\cdot\|$. We say that a *typical* function $f \in C(I)$ satisfies a property P if the functions $f \in C(I)$ satisfying P form a residual subset of $C(I)$.

Many mathematicians have investigated *Dini derivatives* of typical continuous functions.

Definition 1.1. Let $f \in C(I)$. For $a \in [0, 1)$, we define

$$D^+f(a) = \limsup_{x \downarrow a} \frac{f(x) - f(a)}{x - a} \text{ and } D_+f(a) = \liminf_{x \downarrow a} \frac{f(x) - f(a)}{x - a},$$

and if they are equal to each other, the same value is denoted by $f'_+(a)$. We define $D^-f(a)$, $D_-f(a)$ and $f'_-(a)$ in a similar fashion for $a \in (0, 1]$. We call $D^\pm f(a)$ and $D_\pm f(a)$ *Dini derivatives* of f at a.

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Here we shall look at *knot points* of continuous functions.

Definition 1.2. Let $f \in C(I)$. A point $a \in (0,1)$ is called a *knot point* of f if $D^+f(a) = D^-f(a) = \infty$ and $D_+f(a) = D_-f(a) = -\infty$. We write $N(f)$ for the set of all points in (0, 1) that are not knot point of f.

Jarník [Ja] showed that $N(f)$ is Lebesgue null for a typical function $f \in C(I)$, and Preiss and Zajíček [PZ] completely determined how small $N(f)$ is for a typical function $f \in C(I)$. To state the theorem precisely, we denote by $\mathscr{K}$ the set of all closed subsets of I and equip it with the Vietoris topology.

Theorem 1.3 ([PZ]). *Let $\mathscr{N}$ be a σ-ideal on I. Then $N(f) \in \mathscr{N}$ for a typical function $f \in C(I)$ if and only if $\mathscr{N} \cap \mathscr{K}$ is residual in $\mathscr{K}$.*

The main theorem in [Sa3] is a generalisation of this result. Observing that $N(f)$ is an $\mathscr{F}_\sigma$ set for every $f \in C(I)$, we shall give a complete characterisation for a family $\mathscr{F}$ of $\mathscr{F}_\sigma$ subsets of I to have the property that $N(f) \in \mathscr{F}$ for typical functions $f \in C(I)$, by using the concept of residuality of families of $\mathscr{F}_\sigma$ sets introduced by the author [Sa1]. We denote by $\mathscr{K}^{\mathbb{N}}$ the set of all sequences of members of $\mathscr{K}$, and equip it with the product topology.

Theorem 1.4 ([Sa3, Main Theorem]). *Let $\mathscr{F}$ be a family of $\mathscr{F}_\sigma$ subsets of I. Then $N(f) \in \mathscr{F}$ for a typical function $f \in C(I)$ if and only if the set of all sequences $(K_n) \in \mathscr{K}^{\mathbb{N}}$ satisfying $\bigcup_{n=1}^{\infty} K_n \in \mathscr{F}$ is residual in $\mathscr{K}^N$.*

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2. Residuality of families of $\mathscr{F}_\sigma$ sets

We write $\mathscr{F}_\sigma$ for the family of all $\mathscr{F}_\sigma$ subsets of I. The author gave the following definition in [Sa1]:

Definition 2.1. A family $\mathscr{F} \subset \mathscr{F}_\sigma$ is said to be *residual* if $\{(K_n) \in \mathscr{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathscr{F}\}$ is residual in $\mathscr{K}^{\mathbb{N}}$.

As in the case of continuous functions, we say that a *typical* $\mathscr{F}_\sigma$ subset of I satisfies a property P if the $\mathscr{F}_\sigma$ sets satisfying P form a residual subset of $\mathscr{F}_\sigma$. It is natural to ask whether this residuality is induced by some topology. The answer is, rather surprisingly, yes.

Proposition 2.2. *Let $\mathscr{F}$ be a σ-filter on a nonempty set X. Then $\mathscr{F} \cup \{\emptyset\}$ fulfills the axioms of open sets, and $\mathscr{F}$ is equal to the family of all residual subsets of X with respect to this topology.*

Proof. It is obvious that $\mathscr{F} \cup \{\emptyset\}$ fulfills the axioms of open sets. We may readily verify that belonging to $\mathscr{F}$ is equivalent to being open dense, and so to being residual as well. □

Since the collection of residual families of $\mathscr{F}_\sigma$ subsets of I is a σ-filter on $\mathscr{F}_\sigma$, it is true that this proposition gives us a topology that yields our residuality. However this topology is 'bad' (for example, it is not Hausdorff), and the author does not know whether there exists a 'good' topology on $\mathscr{F}_\sigma$ that induced our residuality.

3. Banach-Mazur game

The Banach-Mazur game is of great use in the study of residuality.

Definition 3.1. Let X be a topological space and S a subset of X. The (X,S)-*Banach-Mazur game* is described as follows. Two players, called Player I and Player II, alternately choose a nonempty open subsets of X with the restriction that they must choose a subset of the set chosen in the previous turn. Player II will win if the intersection of all the sets chosen by the players is contained in S; otherwise Player I will win.

Fact 3.2 ([Ox, Theorem 1]). *The (X,S)-Banach-Mazur game has a winning strategy for Player II if and only if S is residual in X.*

4. $\mathscr{N}$-game

Zajíček [Za] introduced a new game called $\mathscr{N}$-game to investigate knot points (and points defined similarly). A *figure* is a finite union of (at least one) nondegenerate closed intervals in I. The *norm* of a figure $F = \bigcup_{j=1}^{n} [a_j,b_j]$, where $0 \leqq a_1 < b_1 < \ldots < a_n < b_n \leqq 1$, is defined as

$$\max\{a_1, b_1 - a_1, a_2 - b_1, \ldots, b_n - a_n, 1 - b_n\}$$

and denoted by $\nu(F)$.

Definition 4.1 ($\mathscr{N}$-game). Far a family $\mathscr{N}$ of subsets of I, the *$\mathscr{N}$-game* is described as follows. The players, called the ε-player and the F-player, move alternately. For each positive integer n, the nth round consists of the ε-player choosing a positive number ε_n and the F-player choosing a figure F_n with $\nu(F_n) \leqq \varepsilon_n$. The F-player will win if $\liminf_{n\to\infty} F_n = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} F_n \in \mathscr{N}$; otherwise the ε-player will win.

Remark 4.2. Note that $\liminf_{n\to\infty} F_n$ is always $\mathscr{F}_\sigma$.

Theorem 4.3 ([PZ]). *Let $\mathscr{N}$ be a family of subsets of I.*

(1) If *$\mathscr{N}$ is hereditary (i.e. $A \in \mathscr{N}$ and $B \subset A$ always imply $B \in \mathscr{N}$) and $\mathscr{N} \cap \mathscr{K}$ is meagre in some nonempty open subset of $\mathscr{K}$, then the ε-player has a winning strategy in the $\mathscr{N}$-game.*

(2) *If $\mathscr{N}$ is a σ-ideal and $\mathscr{N} \cap \mathscr{K}$ is residual in $\mathscr{K}$, then the F-player has a winning strategy in the $\mathscr{N}$-game.*

However the following proposition seems to suggest that the $\mathscr{N}$-game might be of little use for a family $\mathscr{N}$ which is not hereditary:

Proposition 4.4. *Let δ be a positive number less than 1, and write $\mathscr{N}$ for the family of all $\mathscr{F}_\sigma$ subsets of I of measure at least δ. Then $\mathscr{F}_\sigma \setminus \mathscr{N}$ is residual in $\mathscr{F}_\sigma$, but the F-player has a winning strategy in the $\mathscr{N}$-game.*

Proof. It is easy to see that the null $\mathscr{F}_\sigma$ sets form a residual family in $\mathscr{F}_\sigma$ (see [Sa2] for the proof), which implies that $\mathscr{F}_\sigma \setminus \mathscr{N}$ is residual.

We shall describe a winning strategy for the F-player. Take a sequence (δ_n) of positive numbers less than 1 satisfying $\Pi_{n=1}^{\infty} \delta_n = \delta$ (set $\delta_n = \delta^{2^{-n}}$ for instance). In the nth turn, dividing each component of $\bigcap_{j=1}^{n-1} F_j$ (which is assumed to be I if $n = 1$) into so many intervals that each of them is of length at most $\varepsilon_n/3$, *the* F-player chooses from each of these tiny intervals J a subinterval contained in J of length $\delta_n \mu(J)$, where μ denotes the Lebesgue measure, and takes a figure F_n with $\nu(F_n) \leqq \varepsilon_n$ as the nth move so that $\bigcap_{j=1}^{n} F_j$ is the union of these subintervals.

The since $\mu\left(\bigcap_{j=1}^{n} F_j\right) = \Pi_{j=1}^{n} \delta_j$, we obtain

$$\mu\left(\liminf_{n\to\infty} F_n\right) \geqq \mu\left(\bigcap_{n=1}^{\infty} F_n\right) = \prod_{n=1}^{\infty} \delta_n = \delta. \qquad \square$$

For a figure F, put $\mathscr{U}(F) = \{K \in \mathscr{K} \mid K \subset \operatorname{Int} F\}$, which is an open subset of $\mathscr{K}$.

Lemma 4.5. *Let $\mathscr{U}$ be a nonempty open subset of $\mathscr{K}$. Then there exists a positive number ε such that if F is a figure with $\nu(F) \leqq \varepsilon$, then $\mathscr{U}(F) \cap \mathscr{U} \neq \emptyset$.*

Proof. Take a finite subset K_0 of I and a positive number r so that $B(K_0, r) \subset \mathscr{U}$. We shall show that any $\varepsilon > 0$ less than r will do. Let F be a figure with $\nu(F) \leqq \varepsilon$. For each $x \in I$, we may choose $y_x \in B(x, r) \cap \operatorname{Int} F$. Then $\{y_x \mid x \in K_0\} \in \mathscr{U}(F) \cap B(K_0, r) \subset \mathscr{U}(F) \cap \mathscr{U}$. $\square$

The following proposition gives a relation between the F-player having a winning strategy and our residuality of families of $\mathscr{F}_\sigma$ sets:

Proposition 4.6. *Let $\mathscr{N}$ be a hereditary family of subsets of I such that the F-player has a winning strategy in the $\mathscr{N}$-game. Then $\mathscr{N} \cap \mathscr{F}_\sigma$ is residual in $\mathscr{F}_\sigma$.*

Proof. By fact 3.2, it is enough to prove that Player II has a winning strategy in the $(\mathscr{K}^{\mathbb{N}}, \mathscr{K}^{\mathbb{N}}_{\mathscr{N}})$-Banach-Mazur game, where $\mathscr{K}^{\mathbb{N}}_{\mathscr{N}} = \{(\mathscr{K}_n) \in \mathscr{K}^{\mathbb{N}} \mid \bigcup_{n=1}^{\infty} K_n \in \mathscr{N}\}$.

To obtain the mth move $\mathscr{V}_m$, Player II defines a positive number ε_m by using the mth move $\mathscr{U}_m$ of Player I, and consider the figure F_m that in the $\mathscr{N}$-game, the winning strategy tells the F-player to reply wen the moves ε_1, F_1, ε_2, F_2, ..., ε_m are given.

Let us look at the mth round, so that we already know $\mathscr{U}_j$, ε_j, F_j, and $\mathscr{V}_j$ for $j = 1$, ..., $m - 1$. Given the mth move $\mathscr{U}_m$ of Player I, take a positive integer n_m and nonempty open subsets $\mathscr{U}_{m1}, \ldots, \mathscr{U}_{mn_m}$ of $\mathscr{K}$ so that $\mathscr{U}_{m1} \times \ldots \times \mathscr{U}_{mn_m} \times \mathscr{K} \times \times \mathscr{K} \times \ldots \subset \mathscr{U}_m$. We may assume that $n_1 < n_2 < \ldots$. In view of Lemma 4.5, choose $\varepsilon_m > 0$ so that if F is a figure with $v(F) \leqq \varepsilon_m$, then $\mathscr{U}(F) \cap \mathscr{U}_{mn} \neq \emptyset$ for $n = 1, \ldots, n_m$. Let F_m be the figure that the F-player, following the winning strategy, replies in the mth round of the $\mathscr{N}$-game where the jth move of the ε-player is ε_j for $j = 1, \ldots, m$ and the jth move of the F-player is F_j for $j = 1, \ldots, m - 1$. Set

$$\mathscr{V}_m = (\mathscr{U}(F_m) \cap \mathscr{U}_{m1}) \times \ldots \times (\mathscr{U}(F_m) \cap \mathscr{U}_{mn_m}) \times \mathscr{K} \times \mathscr{K} \times \ldots,$$

which is the mth move of Player II.

Now we need to verify that this is a winning strategy for Player II. Let $(K_n) \in \bigcap_{m=1}^{\infty} \mathscr{V}_m$. Note that if $n \leqq n_m$, then $K_n \in \mathscr{U}(F_m)$, that is $K_n \subset \operatorname{Int} F_m$. Consequently for every $n \in \mathbb{N}$ we have

$$K_n \subset \bigcap_{m \text{ with } n_m \geqq n} \operatorname{Int} F_m \subset \bigcup_{k=1}^{\infty} \bigcap_{m=k}^{\infty} \operatorname{Int} F_m \subset \liminf_{m \to \infty} F_m$$

since $n_1 < n_2 < \ldots$. Therefore $\bigcup_{n=1}^{\infty} K_n \subset \liminf_{m \to \infty} F_m$, and so $\bigcup_{n=1}^{\infty} K_n \in \mathscr{N}$ because $\mathscr{N}$ is hereditary and $\liminf_{m \to \infty} \in \mathscr{N}$. □

5. Knot Points of Typical Continuous Functions

As we defined in Section 1, *a knot point* of $f \in C(I)$ is a point a in $(0, 1)$ at which $D^+f(a) = D^-f(a) = \infty$ and $D_+f(a) = D_-f(a) = -\infty$, and we write $N(f)$ for the set of all points in $(0, 1)$ that are knot points of f.

5.1 Basic Propositions

Proposition 5.1. *If $a \in [0, 1)$, then $D^+f(a) = \infty$ for a typical function $f \in C(I)$.*

Proof. For each positive integer n, let A_n denote the set of all functions $f \in C(I)$ such that $f(x) - f(a) > n(x - a)$ for some $x \in (a, a + 1/n) \cap I$. Since all functions $f \in \bigcap_{n=1}^{\infty} A_n$ satisfy $D^+f(a) = \infty$, it suffices to show that A_n is open dense for every $n \in \mathbb{N}$.

Let n be any positive integer.

We first prove that A_n is open. Take any $f \in A_n$. We may find a point $x \in (a, a + 1/n) \cap I$ with $f(x) - f(a) > n(x - a)$, and then a positive number ε with $f(x) - f(a) > n(x - a) + 2\varepsilon$. If $g \in B(f, \varepsilon)$, then we have

$$g(x) - g(a) \geqq f(x) - f(a) - 2\varepsilon > n(x - a),$$

which shows that $g \in A_n$. Therefore A_n is open.

Now we prove that A_n is dense. Let $g \in C(I)$ be any piecewise linear function and ε any positive number. Take a piecewise linear function $h \in C(I)$ satisfying $\|h\| < \varepsilon$ and $h'_{\times}(a) > n - g'_{\times}(a)$. Then $(g + h)'_{+}(a) = g'_{+}(a) + h'_{+}(a) > n$ and so $g + h \in A_n$ because $g + h$ is piecewise linear. Since $g + h \in B(g, \varepsilon)$, this implies that A_n is dense, and the proof is complete. □

Corollary 5.2. (1) *For typical functions $f \in C(I)$, we have $D^{+}f(0) = = D^{-}f(1) = \infty$ and $D_{+}f(0) = D_{-}f(1) = -\infty$.*

(2) *If $a \in (0, 1)$, then a is a knot point of f for typical functions $f \in C(I)$.*

Proof. Immediate from Proposition 5.1 by symmetry. □

Remark 5.3. It goes without saying that (2) in the above corollary does NOT imply that, for typical functions $f \in C(I)$, every point in (0, 1) is a knot point of f.

Proposition 5.4. *For every $f \in C(I)$, the set $N(f)$ is $\mathscr{F}_{\sigma}$.*

Proof. For positive integers m and n, let A_{mn} denote the set of all points $x \in [0, 1 - 1/m]$ such that $f(x + h) - f(x) \leqq nh$ for all $h \in (0, 1/m)$. It is easy to see that A_{mn} is closed for any m and n. Accordingly the set of all points $x \in [0, 1)$ with $D^{+}f(x) < \infty$ is $\mathscr{F}_{\sigma}$ because it is equal to the union $\bigcup_{m,n=1}^{\infty} A_{mn}$. Since

$$N(f) = \{x \in [0,1) \mid D^{+}f(x) < \infty\} \cup \{x \in [0,1) \mid D_{+}f(x) > -\infty\} \\ \cup \{x \in (0,1] \mid D^{-}f(x) < \infty\} \cup x \in (0,1] \mid D_{-}f(x) > -\infty\},$$

we obtain the conclusion by symmetry. □

5.2 Main Theorem

Jarník [Ja] showed that $N(f)$ is Lebesgue null for a typical function $f = C(I)$. Accordingly we see from Proposition 5.4 that $N(f)$ is meagre for a typical function $f \in C(I)$, bearing in mind that any null $\mathscr{F}_{\sigma}$ set is meagre because any null closed set is nowhere dense. Then a natural question is how small $N(f)$ is for a typical function $f \in C(I)$, namely for which notion of smallness it is true that $N(f)$ is small for a typical function $f \in C(I)$. Preiss and Zajíček [PZ] gave a complete answer to this question:

Theorem 5.5 ([PZ]). *Let $\mathscr{N}$ be a σ-ideal on I. Then $N(f) \in \mathscr{N}$ for typical functions $f \in C(I)$ if and only if $\mathscr{N} \cap \mathscr{K}$ is residual in $\mathscr{K}$.*

Then it may well be asked for which family of subsets of I, not necessarily a σ-ideal, it is true that $N(f)$ belongs to the family for a typical function $f \in C(I)$. Proposition 5.4 allows us to consider only families of $\mathscr{F}_\sigma$ subsets of I without loss of generality. The main theorem in [Sa3] solves this problem completely:

Theorem 5.6 ([Sa3, Main Theorem]). *Let $\mathscr{F}$ be a family of $\mathscr{F}_\sigma$ subsets of I. Then $N(f) \in \mathscr{F}$ for typical functions $f \in C(I)$ if and only if $\mathscr{F}$ is residual.*

Remark 5.7. The conclusion can be rephrased as follows: $N(f) \in \mathscr{F}$ for typical functions $f \in C(I)$ if and only if $F \in \mathscr{F}$ for typical $\mathscr{F}_\sigma$ subset F of I.

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