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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 46 (2005), No. 2, 51--54

Persistent URL: http://dml.cz/dmlcz/702107

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Received 11. March 2005

We give a proof of the Kuratowski-Knaster-Mazurkiewicz-Shapley Theorem based on Kakutani's fixed point theorem. This theorem is a very important tool in the general equilibrium theory of economic analysis.

Introduction. In 1929 Knaster, Kuratowski and Mazurkiewicz [6] published a kind of an intersection theorem (the KKM theorem), where some conditions are given for a closed covering of a simplex has a non-empty intersection. In 1967 Scarf [11] proved that any non-transferable utility game whose characteristic function is balanced, has a non-empty core. His proof is based on an algorithm which approximates fixed points. Shapley [12] replaced the Scarf algorithm by a covering theorem (the KKMS theorem) being a generalization of the KKM theorem. Therefore the main difficulty to show the nonemptiness of the core lies in proofs of the KKMS theorem. Thus, Shapley's theorem as an extension of the KKM theorem became very useful to prove the existence of solutions in general equilibrium theory and game theory. In [9] the author considered some intersections theorems involving Helly's intersection theorem. In this paper we would like to present Shapley's theorem as a kind of a dual theorem on coverings. There are a number of papers (see e.g., [1], [7], [8]) containing elementary and simple proofs of the KKMS theorem. The proof which is given in this note is a direct consequence of well-known for economists, Kakutani's fixed point theorem [5].

Let us establish some terminology and notations. Denote the set $\{1, ..., n\}$ by N and the family of nonempty suubsets of N by $\mathcal{N}$. For each point $x \in R^n$ let

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1991 *Mathematics Subject Classification* 90 A14, 54 H25,55 M20.

Key words and phrases. KKM Theorem, fixed points, equilibrium theorems.

$$\sup x := \{i \in N : x_i > 0\} \quad \text{and} \quad \overline{\sup}\, x := \{i \in N : x_i \geq 0\}$$

Denote by Δ the unit simplex in R^n;

$$\Delta := \left\{x \in R^n : x = \sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad \overline{\sup}\, x = N\right\}$$

and for each $S \in \mathcal{N}$ let Δ^S be an S-face of Δ;

$$\Delta^S := \{x \in \Delta : \sup x \subset S\}$$

The symbol conv A stands for the convex hull of a set A.

Main Theorem. The following theorem is a covering version of the Shapley Theorem.

Theorem. *Let $\{C^S : S \in \mathcal{N}\}$ be a family of closed subsets of Δ such that $\Delta^T \subset \bigcup_{S \subset T} C^S$ for each $T \in \mathcal{N}$ and let $\{d^S : S \in \mathcal{N}\}$ be a family of points of Δ such that sup $d^S \subset S$ for each $S \in \mathcal{N}$.*
Then $\Delta = \bigcup_{x \in \Delta} conv\,\{d^S : x \in C^S\}$.

Proof. Let $X := \{x \in R^n : \sum_{i=1}^n x_i = 1$ and $x_i \geq -1$ for each $i \leq n\}$. Define a continuous map (retraction) $r : X \to \Delta$ such that $r(x) = x$ for each $x \in \Delta$;

$$r_i(x) := \frac{max\,\{0, x_i\}}{\sum_{j=1}^n max\,\{0, x_j\}} \text{ for each } i = 1, \ldots, n$$

Fix a point $m \in \Delta$ and define a continuous map $f : X \times \Delta \to X$;
(1) $f(x, p) := r(x) + m - p$.
Next, define set-valued maps $F : X \to 2^\Delta$ and $\phi : X \times \Delta \to 2^{X \times \Delta}$;
(2) $F(x) := \text{conv}\,\{d^S : r(x) \in C^S \text{ and } S \subset \overline{\sup}\, x\}$
(3) $\phi(x, p) := \{f(x, p)\} \times F(x)$.
Assume that $(\bar{x}, \bar{p})$ is a fixed point of the map ϕ, i.e., $(\bar{x}, \bar{p}) \in \phi(\bar{x}, \bar{p})$. Observe that $\bar{x} \in \Delta$. Indeed, if $\bar{x} \notin \Delta$ there exists j such that $\bar{x}_j < 0$. Since $j \notin \overline{\sup}\, \bar{x}$ and sup $d^S \subset S$, according to (2), we infer that $\bar{p}_j = 0$, and from (1) we obtain; $\bar{x}_j = f_j(\bar{x}, \bar{p}) = 0 + m_j \geq 0$, a contradiction to $\bar{x}_j < 0$.

Since $r(x) = x$ for each $x \in \Delta$, from (1) we obtain; $\bar{x} = f(\bar{x}, \bar{p}) = \bar{x} + m - p$, and this yields $m = p$.

Thus we have proved that if the multivalued map ϕ has a fixed point then for each point $m \in \Delta$ there exist a point $\bar{x} \in \Delta$ such that $m \in F(\bar{x})$.

In order to complete the proof it suffices to verify that the map ϕ satisfies the assumptions of the Kakutani's fixed point theorem. It is clear that for each point $(x, p) \in X \times \Delta$, the set $\phi(x, p)$ is non-empty and convex. It remains to show that the graph $W(\phi) := \{(z, u) : z \in X \times \Delta,\ u \in \phi(z)\}$ is a closed subset of $(X \times \Delta)^2$.

Assume that $(z_m, u_m) \to (z, u)$ whenever $m \to \infty$, where $(z_m,\ u_m) \in W(\phi)$, $z_m = (x_m, p_m)$, $u_m = (f(z_m), y_m)$ and $y_m \in F(x_m)$. By continuity of f it follows that $z_m \to f(z)$ whenever $m \to \infty$. Now, we are reduced to proving that $y \in F(x)$.

For each $x \in X$ consider a subset of $\mathcal{N}$;

$$B(x) := \{S \subset N : r(x) \in C^S \text{ and } S \subset \overline{\sup}\, x\}.$$

The family $\{B(x_m) : m = 1, 2, \ldots\}$ consists of subsets of the finite set $\mathcal{N}$ and therefore there exists a set $B \subset \mathcal{N}$ and subsequence $\{n_k\}$ such that $B = B(x_{m_k})$ for each k.

Since the sets C^S are closed and $S \in B(x_m)$ implies $S \subset \overline{\sup}\, x$, we infer that $B \subset B(X)$ whenever $x_{m_k} \to x$.

Note that, $y_{m_k} \in \operatorname{conv} (d^S : S \in B\} = F(x_{m_k})$. Since $y_{m_k} \to y$ whenever $k \to \infty$, we infer that $y \in F(x_{m_k}) = \operatorname{conv} \{d^S : S \in B\} \subset \operatorname{conv} \{d^S : S \in B(x)\} = F(x)$. This completes the proof.

Statement of KKMS Theorem. For each $i \leq n$ let $e^i \in R^n$ be an n-vector whose i-th coordinate is 1 and 0 otherwise. Denote for each $S \in \mathcal{N}$, $e^S := \sum_{i \in S} e^i$. A subfamily $\mathcal{B}$ *of* $\mathcal{N}$ is said to be *balanced* if there are nonnegative weights λ^S, $S \in \mathcal{B}$, such that $e^N = \sum_{S \in \mathcal{B}} \lambda^S e^S$. One can prove that $\mathcal{B}$ is balanced if and only if $m^N \in \operatorname{conv} \{m^S : S \in \mathcal{B}\}$, where m^S is the center of gravity of the face Δ^S, that is, $m^S = \frac{e^S}{|S|}$, where $|S|$ denotes the cardinality of S. In fact we need to know that $m^N \in \operatorname{conv} \{m^S : S \in \mathcal{B}\}$ implies $e^N = \sum_{S \in \mathcal{B}} \lambda^S e^S$. But it is obvious, because if $m^N = \sum_{S \in \mathcal{B}} t^S m^S$, where $t^S \geq 0$, then $e^N = \sum_{S \in \mathcal{B}} \lambda^S e^S$,where $\lambda^S = \frac{nt^S}{|S|}$.

Replacing points d^S by the points m^S we immediately obtain a point $x \in \Delta$ such that $m^N \in \operatorname{conv} \{m^S : x \in C^S\}$ and this is exactly Shapley's theorem;

Theorem (KKMS). *Let* $\{C^S : S \in \mathcal{N}\}$ *be a family of closed subsets of* Δ *and assume that* $\Delta^T \subset \bigcup_{S \subset T} C^S$ *for each* $T \in \mathcal{N}$.

Then there exists a balanced family $\mathcal{B}$ *such that* $\bigcap_{S \in \mathcal{B}} C^S \neq \emptyset$.

KKMS Theorem is an extension of KKM Theorem. To see this, let us assume that $C^S = \emptyset$ for each set S of cardinality greater than 1. Under this assumption the family $\{\{i\}: i \in N\}$ is the only balanced family. Let $C^i := C^{\{i\}}$. Then, we immediately get

Theorem (KKM). *Let* $\{C^i : i \in N\}$ *be a family of closed subsets of* Δ *and assume that* $\Delta^T \subset \bigcup_{i \in T} C^i$ *for each non-empty subset* $T \subset N$.

Then $\bigcap_{i \in N} C^i \neq \emptyset$.

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