On Scheepers' Conjecture

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The paper deals with some relationships between covering properties of a topological space X and convergence properties of its function space $C_p(X)$. We present partial solution of the conjecture stated by M. Scheepers in [Sc2].

We briefly introduce basic notation we deal with. For more detailed explanation we refer the reader for example to the survey article [Bu]. All considered topological spaces are supposed to be Hausdorff spaces. Let f_n, $n \in \mathbb{N}$, f be functions from topological space X into $\mathbb{R}$. Sequence $\{f_n\}_{n=0}^{\infty}$ **quasinormally converges** to f on X, we write

$$f_n \xrightarrow{QN} f \text{ on } X,$$

if there is a sequence $\{\varepsilon_n\}_{n=0}^{\infty}$ of positive reals (a control sequence witnessing the quasinormal convergence) converging to 0 such that

$$(\forall x \in X)(\forall^\infty n \in \mathbb{N})\, |f_n(x) - f(x)| < \varepsilon_n.$$

A space X is called a **QN-space** if each sequence $\{f_n\}_{n=0}^{\infty} \subseteq C_p(X)$ converging pointwise to 0 converges also quasinormally. A space is called a **wQN-space** if each sequence $\{f_n\}_{n=0}^{\infty} \subseteq C_p(X)$ converging pointwise to 0 contains quasinormally converging subsequence $\{f_{n_k}\}_{k=0}^{\infty}$. We say that X has **sequence selection property**, shortly **SSP**, if for any doubly-sequence $\{f_{n,m} : n,\ m \in \mathbb{N}\} \subseteq C_p(X)$ such that $\lim_{m\to} f_{n,m} = 0$ for any n there are m_n such that $\lim_{n\to\infty} f_{n,m_n} = 0$.

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An open cover $\mathcal{U}$ of a space X is an ω**-cover** if $X \notin \mathcal{U}$ and every finite $Y \subseteq X$ is contained in some member of $\mathcal{U}$. An open cover $\mathcal{U}$ of a space X is a γ**-cover** if it is infinite and each $x \in X$ is in all but finitely many members of $\mathcal{U}$. Obviously every infinite subcover of a γ-cover is again a γ-cover. Thus from every γ-cover one can choose a sequence $\mathcal{U} = \{U_n\}_{n=0}^{\infty}$ of open subsets of X such that $X \notin \{U_n : n \in \omega\}$ and every $x \in X$ is in all but finitely many members of $\mathcal{U}$.

We say that X is a γ**-space** if every ω-cover $\mathcal{V}$ of X contains subcover which is a γ-cover, i.e. for every $n \in \mathbb{N}$ there is $U_n \in \mathcal{V}$ such that $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ is a γ-cover of X.

A space X is called an $\mathbf{S_1}$ **(Γ, Γ)-space** if for every sequence $\mathcal{U}_n$ of γ-covers of X one can choose $U_n \in \mathcal{U}_n$ for every $n \in \mathbb{N}$ such that $\{U_n : n \in \mathbb{N}\}$ is a γ-cover of X.

Theorem ([Sc1]). *If X has property* $S_1(\Gamma, \Gamma)$ *then X possesses* SSP.

Theorem ([Sc2]). *If X has* SSP *then X is a* wQN*-space.*

Consequently in [Sc2] M. Scheepers conjectured

Conjecture. *Every perfectly normal* wQN*-space is an* $S_1(\Gamma,\Gamma)$*-space.*

Next, D. Fremlin proved the following

Theorem ([Fr]). *If X is a* wQN*-space then X possesses* SSP.

The main result of this paper is theorem 6 saying that this conjecture holds true for any σ-space X.We recall that a space X is a σ-space if every F_σ subset $F \subseteq X$ is also a G_δ subset of X.

Now we introduce another two notions suitable for further considerations. We say that X is a $\gamma\gamma_{co}$**-space** if for every γ-cover $\mathcal{U}$ of X there exists refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\mathcal{V}$ is a γ-cover consisting of clopen sets. We say that X is an **nCM-space** if X cannot be continuously mapped onto the unit interval $\mathbb{I}$. It was shown in [BRR] that every wQN-space has property nCM. It is easy to see that if X is a Tychonoff nCM-space then X has clopen basis thus X is zero-dimensional. As an example of $\gamma\gamma_{co}$-space we can take any perfectly normal γ-space (see theorem 3).

Lemma 1. *Let X be perfectly normal topological space with property $\gamma\gamma_{co}$. If X has* SSP *then X is an* $S_1(\Gamma, \Gamma)$*-space.*

Proof. Suppose we are given a sequence $\{\mathcal{U}_n\}_{n=0}^{\infty}$ of γ-covers. Take the corresponding sequence of clopen refinements $\{\mathcal{V}_n\}_{n=0}^{\infty}$ and enumerate bijectively each $\mathcal{V}_n = \{V_{n,m} : m \in \mathbb{N}\}$. Define for all $n, m \in \mathbb{N}$ continuous function $f_{n,m} : X \to \mathbb{I}$ as a characteristic function of $X \setminus V_{n,m}$.

Thus $\lim_{m\to\infty} f_{n,m} = 0$ for every $n \in \mathbb{N}$. Since X has SSP we can choose $\{m_n\}_{n=0}^{\infty} \in {}^{\mathbb{N}}\mathbb{N}$ such that $\lim_{n\to\infty} f_{n,m_n} = 0$. Thus

$$(\forall x \in X)(\forall^\infty n \in \mathbb{N})\, f_{n,m_n}(x) < 1$$

and equivalently

$$(\forall x \in X)(\forall^\infty n \in \mathbb{N})\, x \in V_{n,m_n}.$$

Hence $\{V_{n,m_n} : n \in \mathbb{N}\}$ is a γ-cover of X and then also $\{U_n : n \in \mathbb{N}\}$ is a γ-cover of X, where $U_n \in \mathcal{U}_n$ is such that $V_{n,m_n} \subseteq U_n$. □

Lemma 2. *Let U be an open subset of perfectly normal* nCM-*space X. Then U is an union of an increasing sequence of clopen sets.*

Proof. According to perfect normality of the space X there are closed sets $F_0 \subseteq F_1 \subseteq \dots$ with $\bigcup_{n\in\mathbb{N}} F_n = U$. We define sequence $\{A_n : n \in \omega\}$ by induction. Put $A_{-1} = \emptyset$. At the n-th step, $n \in \mathbb{N}$ take continuous function $f_n : X \to \mathbb{I}$ such that

$$f_n(x) = \begin{cases} 0, & x \in F_n \cup A_{n-1} \\ 1, & x \in U^c \end{cases}.$$

Then there exists $a_n \in \mathbb{I}$ which is not in the range of the function f_n. Put $A_n = f_n^{-1}([0,a_n))$. Then A_n is evidently clopen and $F_n \subseteq A_n \subseteq U$. □

Theorem 3. *If X is a perfectly normal γ-space then X has property $\gamma\gamma_{co}$.*

Proof. By theorem 6.1 of [BRR] X is a wQN-space thus it has property nCM. Suppose we are given a γ-cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$. By lemma 2 there are clopen sets $A_{n,m}$, n, $m \in \mathbb{N}$ such that

$$A_{n,0} \subseteq A_{n,1} \subseteq \dots, \text{ and } U_n = \bigcup_{m\in\mathbb{N}} A_{n,m} \text{ for every } n \in \mathbb{N}.$$

For any finite set $B = \{x_1, \dots, x_k\} \subseteq X$ there are n, $m \in \mathbb{N}$ such that $B \subseteq A_{n,m}$ hence $\mathcal{U}' = \{A_{n,m} : n,\ m \in \mathbb{N}\}$ is an ω-cover of X. Thus there is a subcover $\mathcal{V}' \subseteq \mathcal{U}'$ which is a γ-cover. For every $n \in \mathbb{N}$ the set $\{m : A_{n,m} \in \mathcal{V}'\}$ is finite thus we can define

$$m_n = \max\{m : A_{n,m} \in \mathcal{V}'\}.$$

Then $\mathcal{V} = \{A_{n,m_n} : n \in \mathbb{N}\}$ is both a γ-cover and a refinement of $\mathcal{U}$. □

None of the properties we deal with here (except of QN, as we shall see later) is necessarily hereditary. However, it is not difficult to see that all properties are preserved by F_σ subsets.

Corollary 5.5 of [BRR] asserting that topological space X is hereditarily QN-space if and only if it is simultaneously a QN-space and a σ-space was an inspiration for next theorem.

Theorem 4. *Perfectly normal space X is hereditarily* wQN-*space if and only if X is simultaneously* wQN *and σ-space. Perfectly normal space X is hereditarily* $S_1(\Gamma,\Gamma)$-*space if and only if X is simultaneously* $S_1(\Gamma,\Gamma)$ *and σ-space.*

Proof. If X is hereditarily wQN then it has nCM. Take any G_δ set $A \subseteq X$. Let $\{U_n\}_{n=0}^\infty$ be decreasing sequence of open sets such that

$$A = \bigcap_{n\in\mathbb{N}} U_n .$$

By lemma 2 it follows that

$$A = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} A_{n,m}$$

where $A_{n,m}$ are clopen sets. Put $A'_{n,m} = A \cap A_{n,m}$ these are clopen in A. By theorem 5.8 of [BRR] family of open subsets of A is weakly distributive hence we can find $\varphi \in {}^{\mathbb{N}}\mathbb{N}$ such that

$$A = \bigcup_k \bigcap_{n\geqslant k} \bigcap_{m<\varphi(n)} A'_{n,m} = A \cap (\bigcup_k \bigcap_{n\geqslant k} \bigcup_{m<\varphi(n)} A_{n,m}).$$

Thus A is F_σ set in X.

Inverse implication for wQN can be proved as follows. Consider any $A \subseteq X$ and sequence $\{f_n\}_{n=0}^{\infty}, f_n : A \to \mathbb{I}$ such that $\lim_{n\to\infty} f_n = 0$. There is G_δ subset $G \supseteq A$ of X such that each f_n can be continuously extended to G. Denote this extensions by f'_n and define $B = \{x \in G : f'_n(x) \to 0,\ n \to \infty\}$, namely

$$B = \bigcap_k \bigcup_n \bigcap_{m\geqslant n} f'^{-1}_m([0, 2^{-k})).$$

Then $A \subseteq B \subseteq G$, B is borel hence F_σ subset of X so there exists an increasing sequence of natural nunmbers $\{n_k\}_{k=0}^{\infty}$ such that $f'_{n_k} \xrightarrow{QN} 0$ on B (B is a wQN-space). Then also $f_{n_k} \xrightarrow{QN} 0$ on A.

To prove this implication for $S_1(\Gamma,\Gamma)$ consider any subset A of X and $\{\mathcal{U}_n\}_{n=0}^{\infty}$ a sequence of γ-covers of . Enumerate bijectively $\mathcal{U}_n = \{U_{n,m} : n \in \mathbb{N}\}$, where $U_{n,m} = A \cap V_{n,m}$, $V_{n,m} \subseteq X$ open. Put

$$B = \bigcap_k \bigcup_n \bigcap_{m\geqslant n} V_{k,m}.$$

Then $A \subseteq B \subseteq X$ and $\{\mathcal{V}_n\}_{n=0}^{\infty}$, $\mathcal{V}_n = \{V_{n,m} : m \in \mathbb{N}\}$ is a sequence of γ-covers of B. But B is an F_σ subset of X so it is an $S_1(\Gamma,\Gamma)$-space. □

Lemma 5. *Let X be perfectly normal σ-space with property* nCM. *Then X has property $\gamma\gamma_{co}$.*

Proof. Take any γ-cover $\{U_n : n \in \mathbb{N}\}$ of the space X. Denote $G_n = \bigcap_{k>n} U_k$. Choose closed sets $F_{n,m}$, $n, m \in \mathbb{N}$ such that

$$F_{n,m} \subseteq F_{n,m+1}, \qquad \bigcup_{m\in\mathbb{N}} F_{n,m} = G_n, \qquad \text{for all } n, m \in \mathbb{N},$$

moreover we can assume that $F_{n,m} \subseteq F_{n+1,m}$, for all $n,m \in \mathbb{N}$. One can readily see that $\{F_{n,n} : n \in \mathbb{N}\}$ is increasing sequence of closed sets with

$$F_{n,n} \subseteq U_n \text{ and } \bigcup_{n\in\mathbb{N}} F_{n,n} = X .$$

Using a trick from the proof of the lemma 2 now we can find clopen sets A_n with $F_{n,n} \subseteq A_n \subseteq U_n$. So $\{A_n : n \in \mathbb{N}\}$ forms a γ-cover which is a refinement of $\{U_n : n \in \mathbb{N}\}$. □

So we have obtained the following result.

Theorem 6. *Let X be perfectly normal space. If X is hereditarily* wQN-*space then X is hereditarily* $S_1(\Gamma,\Gamma)$-*space.*

In [Re, theorem 1] author showed that every metric QN-space is σ-space. This result may be generalized for all perfectly normal QN-spaces.

Theorem 7. *Every perfectly normal* QN-*space X is σ-space.*

Proof. Let G be some G_δ subset of the space X, so there are open sets U_0, U_1, ... such that $G = \bigcap_{n\in\mathbb{N}} U_n$. Due to perfect normality of the space X there are continuous functions $f_n : X \to \mathbb{R}$ with $X \setminus U_n$ being a zero-set of f_n for every $n \in \mathbb{N}$.

Define $f : X \to {}^{\mathbb{N}}\mathbb{R}$ as follows

$$f(x)(n) = f_n(x), \quad \text{for} \quad x \in X, n \in \mathbb{N}.$$

Evidently f is continuous mapping, so $f(X)$ is QN-space and being a metric space it is also σ-space. Denote $S_n = \{y \in {}^{\mathbb{N}}\mathbb{R} : y(n) \neq 0\}$ and put $S = \bigcap_{n\in\mathbb{N}} S_n$ then S is G_δ subset of ${}^{\mathbb{N}}\mathbb{R}$. Now it is easy to see, that

$$(\forall x \in X)(x \in G \equiv f(x) \in S),$$

hence $G = f^{-1}(S \cap f(X))$ is an F_σ subset of X for $S \cap f(X)$ is F_σ subset of $f(X)$. □

Corollary 8. *Every perfectly normal* QN-*space is hereditarily* QN-*space. Every perfectly normal* QN-*space is hereditarily* $S_1(\Gamma,\Gamma)$-*space.*

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