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Solutions with Big Graph of Iterative Functional Equations of Higher Orders

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We prove the existence of very irregular solutions of the functional equation of the order N, in particular of the equations

$$\varphi(x) = \sum_{n-1}^{N} g_n(x)\varphi(f_n(x)) + h(x)$$

with given functions $g_1, \ldots, g_N, h\colon X \to Y$ and given commuting bijections $f_1, \ldots, f_N\colon X \to X$.

1. Introduction

Given sets X, Y and a family $\mathscr{R}$ of subsets of $X \times Y$, we say that $\varphi\colon X \to Y$ has a *big graph* with respect to $\mathscr{R}$ if its graph $\mathrm{Gr}\varphi$ meets every set of the family $\mathscr{R}$.

Following [5] and [7] (where iterative functional equations of the first and the second order was studied) we consider the equation of higher orders; i.e. the equation of the form

$$(1) \qquad F(x, \varphi(x), \varphi(f_1(x)), \ldots, \varphi(f_N(x))) = 0.$$

We are interested in finding conditions under which equation (1) has a solution with big graph with respect to a family $\mathscr{R}$ of subset of $X \times Y$ satisfying the following two conditions:

$$(2) \qquad \operatorname{card} \mathscr{R} \le \operatorname{card} X$$

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and

$$(3) \qquad \operatorname{card} \pi(R) = \operatorname{card} X \quad \text{for every} \quad R \in \mathscr{R}.$$

Observe that conditions (2) and (3) are satisfied by the family

$$(4) \qquad \{R \in \mathscr{B}(X \times Y) : \pi(R) \text{ is uncountable}\},$$

where X and Y are Polish spaces, X is uncountable, $\mathscr{B}(X \times Y)$ is the σ – algebra of all Borel subsets of $X \times Y$ and π: $X \times Y \to X$ is the projection.

Family (4) (in the case where $X = Y = \mathbb{R}$) appears in the paper by F. B. Jones [13]. F. B. Jones proved that there exist additive functions having a big graph with respect to family (4) (see [15, Ch. 12, §4] and [4]). A special case of the family $\mathscr{R}$ fulfilling (2) and (3), namely

$$\{B \times \{y\}: B \in \mathscr{B}(\mathbb{R}), \operatorname{card} B = \mathfrak{c}, y \in \mathbb{R}\}$$

occurs in the paper by P. Kahlig and J. Smítal [14]. They proved an analogous result to that given by F. B. Jones in the case of the Dhombres functional equation. Up to now several types of functional equations have been investigated in this direction (see [6] and references therein).

2. Main result

Let $\mathbb{N}_0$ denote the set of all nonnegative integers.

Let X be a nonempty set, let N be a positive integer and let $f_1, \ldots, f_N$: $X \to X$ be commuting bijections; i.e. one-to-one and onto functions such that $f_m \circ f_n = f_n \circ f_m$ for any $m, n \in \{1, \ldots, N\}$. For every $x \in X$ denote by $C_{f_1,\ldots,f_N}(x)$ the orbit of the point x generated by functions $f_1, \ldots, f_N$; i.e. the equivalence class, containing x, of the relation $\sim$ on X defined by

$$x \sim y \Leftrightarrow y = f_1^{m_1} \circ \ldots \circ f_N^{m_N}(x) \quad \text{for some} \quad m_1, \ldots, m_N \in \mathbb{Z}.$$

Clearly,

$$C_{f_1,\ldots,f_N}(x) = \{f_1^{m_1} \circ \ldots \circ f_N^{m_N}(x) : m_1, \ldots, m_N \in \mathbb{Z}\}.$$

From now on let

$$\mathscr{C} = \{C_{f_1,\ldots,f_N}(x) : x \in X\}$$

and let

$$\mathscr{C}_* = \{C_{f_1,\ldots,f_N}(x) \in \mathscr{C} : f_1^{m_1} \circ \ldots \circ f_N^{m_N}(x) = x \Rightarrow m_1 = \ldots = m_N = 0 \\ \text{for any } m_1, \ldots, m_N \in Z\}.$$

Our general hypothesis read as follows.

(H_0) The set X is uncountable, Z is a set with a distinguished element 0, Y is a set, N is a positive integer and $f_1, \ldots, f_N \colon X \to X$ are pairwise commuting bijections.

(H_1) The function $F : X \times Y^{N+1} \to Z$ is such that the equation

$$F(x, y_0, \ldots, y_N) = 0$$

is solvable under each variable $y_0, \ldots, y_N$, i.e.

$$\bigwedge_{x \in X} \bigwedge_{n \in \{0,\ldots,N\}} \bigwedge_{y_i \in Y, i \neq n} \bigvee_{y_n \in Y} F(x, y_0, \ldots, y_N) = 0 . \tag{5}$$

(H_2) The functions $f_1, \ldots, f_N$ are such that

$$\operatorname{card}(\mathscr{C} \backslash \mathscr{C}_*) < \operatorname{card} X.$$

(H_3) Equation (1) has a solution on the set $\bigcup(\mathscr{C} \backslash \mathscr{C}_*)$.

We are now in a position to formulate our main result.

Theorem. *Assume* (H_0) – (H_3). *Let* $\mathscr{R}$ *be a family of subset of* $X \times Y$ *satisfying* (2) *and* (3). *Then there exists a solution* $\varphi \colon X \to Y$ *of* (1) *which has a big graph with respect to the family* $\mathscr{R}$.

In the proof of the Theorem we need the following lemma.

Lemma. *Assume* (H_0) *and* (H_1), *and let* $x \in X$. *If* $C_{f_1,\ldots,f_N}(x) \in \mathscr{C}_*$ *then for every* $y \in Y$ *there exists a solution* $\varphi \colon C_{f_1,\ldots,f_N}(x) \to Y$ *of* (1) *such that* $\varphi(x) = y$.

Proof. Fix a $y \in Y$ and put $\varphi(x) = y$. Now we have to extend φ on the whole orbit $C_{f_1,\ldots,f_N}(x)$ in such a manner that

$$\begin{aligned} &F\big(f_1^{p_1} \circ \ldots \circ f_N^{p_N}(x), \varphi\big(f_N^{p_1} \circ \ldots \circ f_N^{p_N}(x)\big), \\ &\quad \varphi\big(f_1^{p_1+1} \circ \ldots \circ f_N^{p_N}(x)\big), \ldots, \varphi\big(f_1^{p_1} \circ \ldots \circ f_N^{p_N+1}(x)\big)\big) = 0 \end{aligned} \tag{6}$$

for any $(p_1, \ldots, p_N) \in \mathbb{Z}^N$.

We first define φ arbitrarily on the set

$$\left\{ f_1^{p_1} \circ \ldots \circ f_N^{p_N}(x) : (p_1, \ldots, p_N) \in \bigcup_{k=1}^{N} \mathbb{Z}^{N-k} \times \{0\} \times (-\mathbb{N})^{k-1} \backslash \{(0, \ldots, 0)\} \right\}.$$

Next using solvability condition (5) with $n = N$, for any $(p_1, \ldots, p_N) \in \mathbb{Z}^{N-1} \times \mathbb{N}$ we choose an element

$$\varphi\big(f_1^{p_1} \circ \ldots \circ f_N^{p_N}(x)\big) \in Y \tag{7}$$

in such a way that (6) is satisfied.

Then, in turn for $l = 1, \ldots, N-1$, using condition (5) with $n = N - l$, for any $(p_1, \ldots, p_N) \in \mathbb{Z}^{N-l-1} \times \mathbb{N} \times (-\mathbb{N})^l$ we choose an element (7) in such a way that (6) is satisfied.

Finally, using (5) with $n = 0$, for any $(p_1, \ldots, p_N) \in (-\mathbb{N})^N$ we choose an element (7) in such a way that (6) is satisfied. $\Diamond$

Proof of the Theorem. The family $\mathscr{C}$ of all the orbits is a partition of X and a function $\varphi\colon X \to Y$ is a solution of (1) if and only if for every $x \in X$ the function $\varphi|_{C_{f_1,\ldots,f_N}(x)}$ is a solution of (1). This allows us to define a solution φ of (1) by defining it on each orbit.

Using (H_3) we fix a solution φ_0 of (1) on the set $\bigcup(\mathscr{C}\setminus\mathscr{C}_*)$. Now we only need to define a solution φ of (1) on any orbit from $\mathscr{C}_*$.

Let γ be the smallest ordinal such that its cardinal $|\gamma|$ equals that of $\mathscr{R}$ and let $(R_\alpha : \alpha < \gamma)$ be a one-to-one transfinite sequence of all the elements of $\mathscr{R}$. Using the transfinite induction we will define a sequence $((x_\alpha, y_\alpha) : \alpha < \gamma)$ of elements of $X \times Y$ such that for every $\alpha < \gamma$ we have

$$(x_\alpha, y_\alpha) \in R_\alpha \tag{8}$$

and

$$x_\alpha \in \pi(R_\alpha) \setminus \bigcup \{C \in \mathscr{C}_* : x_\beta \in C \text{ for some } \beta < \alpha\}. \tag{9}$$

Fix $\alpha < \gamma$ and suppose that we have already defined suitable (x_β, y_β) for $\beta < \alpha$. It follows from (H_0, (H_1), (H_2) and (2) that

$$\operatorname{card} \bigcup \{C \in \mathscr{C}_* : x_\beta \in C \text{ for some } \beta < \alpha\} \le \aleph_0 \cdot |\alpha| = \max\{\aleph_0, |\alpha|\} < \operatorname{card} X$$

which jointly with (3) ensures that the set occurring in (9) is nonempty and we can choose a point x_α from it. In particular, $x_\alpha \in \pi(R_\alpha)$ and so there exists a y_α such that (8) holds.

Fix now an orbit $C \in \mathscr{C}_*$. If the set

$$C \cap \{x_\alpha : \alpha < \gamma\} \tag{10}$$

is nonempty, then, according to (9), it consists of exactly one point x_α and we put

$$(x, y) = (x_\alpha, y_\alpha).$$

If the set (10) is empty, then we choose $(x, y) \in C \times Y$ arbitrarily. In both these cases $C = C_{f_1,\ldots,f_N}(x)$. Applying now Lemma 1 we get a solution $\varphi_C\colon C \to Y$ of (1) such that

$$\varphi_C(x) = y.$$

Putting

$$\varphi = \bigcup_{C \in \mathscr{C}_*} \varphi_C \cup \varphi_0$$

we obtain a solution of (1) satisfying $\varphi(x_\alpha) = y_\alpha$ for every $\alpha < \gamma$, which jointly with (8) shows that φ has a big graph with respect to the family $\mathscr{R}$ $\Diamond$

3. Corollaries

Let us mention two consequences of the Theorem. Both concern special cases of equation (11) and was studied by many authors.

Given a field $\mathbb{K}$ with 0 as the neutral element of addition we denote by $\mathbb{K}^*$ the set $\mathbb{K}\setminus\{0\}$.

Let V be an uncountable linear space over a field $\mathbb{K}$ and let $g_1, \dots, g_N \colon X \to \mathbb{K}^*$ and $h\colon X \to V$ be arbitrary functions. Define a function $F\colon X \times V^{N+1} \to V$ by

$$F(x, y_0, \dots, y_N) = -y_0 + g_1(x)y_1 + \dots + g_N(g)y_N + h(x).$$

Then (H_1) holds and (1) takes the form (see [16, Ch. 6] and [3])

$$\varphi(x) = \sum_{n-1}^{N} g_n(x)\varphi(f_n(x)) + h(x). \tag{11}$$

Corollary 1. Assume that V is an uncountable linear space over a field $\mathbb{K}$ and $\alpha_1, \dots, \alpha_N \in \mathbb{K}^*$ fulfill

$$\prod_{n\ 1}^{N} \alpha_n^{m_n} = 1 \Rightarrow m_1 = \dots = m_N = 0 \quad \text{for any} \quad m_1, \dots, m_N \in \mathbb{Z}.$$

Let $\mathscr{R}$ be a family of subsets of $V \times V$ such hat (2) and (3) hold. Then for every function $g_1, \dots, g_N\colon V \to \mathbb{K}^*$ and $h\colon V \to V$ such that $\sum_{n=1}^{N} g_n(\theta) \neq 1$ or $h(\theta) = \theta$ there exists a solution $\varphi\colon V \to V$ of equation

$$\varphi(x) = \sum_{n-1}^{N} g_N(x)\varphi(\alpha_n x) + h(x) \tag{12}$$

which has a big graph with respect to the family $\mathscr{R}$.

Proof. Define functions $f_1, \dots, f_N\colon V \to V$ by

$$f_n(x) = \alpha_n x. \tag{13}$$

Then $f_1, \dots, f_N$ are pairwise commuting bijections and $C_{f_1,\dots,f_N}(x) \in \mathscr{C}_*$ if and only if $x \neq \theta$. Hence (H_0) and (H_2) hold. Since $\bigcup(\mathscr{C}\setminus\mathscr{C}_*) = \{\theta\}$, we conclude that (H_3) holds if and only if $h(\theta) = \theta$ or $\sum_{n=1}^{N} g_n(\theta) \neq 1$. ◇

Equation (12) was studied, among others, in [1], [8], [9] and [12].

Corollary 2. Assume that X is an uncountable set, V is a linear space over a field $\mathbb{K}$, $\psi\colon X \to V$ is a bijection and $a_1, \dots, a_N \in V\setminus\{\theta\}$ fulfill

$$\sum_{n\ 1}^{N} m_n a_n = \theta \Rightarrow m_1 = \dots = m_N = 0 \quad \text{for any} \quad m_1, \dots, m_N \in \mathbb{Z}.$$

Let $\mathscr{R}$ be a family of subsets of $X \times V$ such that (2) and (3) hold. Then for every functions $g_1, \dots, g_N\colon X \to \mathbb{K}^*$ and $h\colon X \to V$ there exists a solution $\varphi\colon X \to V$ of equation

$$\varphi(x) = \sum_{n\ 1}^{N} g_n(x)\varphi(\psi^{-1}(\psi(x) + a_n)) + h(x) \tag{14}$$

which has a big graph with respect to the family $\mathscr{R}$.

Proof. Define functions $f_1, \ldots, f_N$: $X \to X$ by

$$f_n(x) = \psi^{-1}(\psi(x) + a_n).$$

Then $f_1, \ldots, f_N$ are pairwise commuting bijections and $\mathscr{C}_* = \mathscr{C}$. Hence ($H_0$) and ($H_2$) hold. Since $\bigcup(\mathscr{C} \setminus \mathscr{C}_*) = \emptyset$, we see that ($H_3$) holds. D

Equation (14) was studied, for example in [2], [10], [11] and [17].

4. Properties of functions having a big graph

For the convenience of the reader we repeat from [5] topological and measure-theoretical properties of functions with big graph with respect to family (4).

Proposition 1. *Assume X is a* T_1 – space *and has no isolated point. If φ: $X \to Y$ has a big graph with respect to family* (4), *then the set $(X \times Y) \setminus \mathrm{Gr}\varphi$ contains no subset of $X \times Y$ of second category having the property of Baire.*

Proposition 2. *Assume X is a* T_1 – space *and λ is a measure on $\mathscr{B}(X \times Y)$ vanishing on all the vertical lines $\{x\} \times Y$, $x \in X$. If φ: $X \to Y$ has a big graph with respect to family* (4), *then the set $(X \times Y) \setminus \mathrm{Gr}\varphi$ contains no Borel subset of $X \times Y$ of positive measure λ.*

Proposition 3. *Assumme that X and Y are connected spaces and every nonempty open subset of X is uncountable. If φ: $X \to Y$ has a big graph with respect to family* (4), *then* $\mathrm{Gr}\varphi$ *is dense and connected in $X \times Y$.*

5. Remarks

It is evident that assumption (H_3) is needed. We will show that any of the assumptions (H_1) and (H_2) cannot be omitted, too.

1. Let $p_1, \ldots, p_N$ be pairwise different prime numbers and consider solutions φ: $\mathbb{R} \to \mathbb{R}$ of the following functional equation

$$\sum_{n=1}^{N} |\varphi(p_n x)| = 0.$$

It is easy to see that the zero function is the only solution of the above equation. Hence there is no solution with big graph. Here (H_1) does not hold but both (H_2) and (H_3) are satisfied.

2. Let $p_1, \ldots, p_N \in \mathbb{C}^*$, let $p \in \mathbb{C}$, let $\alpha_n = \cos\frac{2\pi n}{N+1} + i\sin\frac{2\pi n}{N+1}$ for every $n \in \{1, \ldots, N\}$ and let $f_1, \ldots, f_N\colon \mathbb{C}^* \to \mathbb{C}^*$ be functions defined by (13). Consider solutions $\varphi\colon \mathbb{C}^* \to \mathbb{C}$ of the following functional equation

$$\varphi(z) = \sum_{n=1}^{N} p_n \varphi(\alpha_n z) + p. \tag{15}$$

Observe that for every $z \in \mathbb{C}^*$ we have $C_{f_N,\ldots,f_N}(z) = \{z, \alpha_1 z, \ldots, \alpha_N z\}$. Thus the problem of finding a solution $\varphi\colon \mathbb{C}^* \to C$ of (15) reduces to the problem of finding a solution $\varphi\colon C_{f_1,\ldots,f_N}(z) \to \mathbb{C}$ of (15) for given $z \in \mathbb{C}^*$ of the following matrix equation

$$\begin{bmatrix} 1 & -p_1 & -p_2 & \cdots & -p_N \\ -p_N & 1 & -p_1 & \cdots & -p_{N-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -p_1 \\ -p_1 & -p_2 & \cdots & -p_N & 1 \end{bmatrix} \begin{bmatrix} \varphi(z) \\ \varphi(\alpha_1 z) \\ \vdots \\ \varphi(\alpha_N z) \end{bmatrix} = \begin{bmatrix} p \\ \vdots \\ p \end{bmatrix}.$$

Now take p_n's in such a manner that the determinant of the above matrix of p_n's is nonzero then equation (15) has exactly one solution and this solution takes at most $N + 1$ values. Hence there is no solution with big graph. Here (H_2) does not hold but both (H_1) and (H_3) are satisfied.

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References

[1] BAKER, J. A., *A functional equation from probability theory*, Proc. Amer. Math. Soc. 121 (1994), 767 – 773.

[2] BARON, K., JARCZYK, W., *On a way of division of segments*, Aequationes Math. 34 (1987), 195 – 205.

[3] BARON, K., JARCZYK, W., *Recent results on functional equations in a single variable, perspectives and open problems*, Aequationes Math. 61 (2001), 1 – 48.

[4] BARON, K., VOLKMANN, P., *Dense sets of additive functions*, Seminar LV, No. 16 (2003), 4 pp., http://www.mathematik.uni-karlsruhe.de/~semlv.

[5] BARTŁOMIEJCZYK, L., *Solutions with big graph of iterative functional equations of the first order*, Colloq. Math. 82 (1999), 223 – 230.

[6] BARTŁOMIEJCZYK, L., *Solutions with big graph of iterative functional equations*, Real Analysis Exch., 24th Summer Symposium Conference Reports, May 2000, 133 – 135.

[7] BARTŁOMIEJCZYK, L., MORAWIEC, J., *Solutions with big graph of iterative functional equations of the second order*, Math. Pannon. 15 (2004), 141 – 148.

[8] DERFEL, G. A., *The behavior of solutions of functional and functional difference equations with certain transformations of the argument* [Russian], Ukrain. Math. Zh. 34 (1982), 350 – 356.

[9] GER, J., SABLIK, M., *The law of large numbers and a functional equation,* Ann. Polon. Math. 68 (1998), 165 – 175.

[10] GRINČ, M., *On non-negative measurable solutions of a difference functional equation*, Positivity 2 (1998), 221 – 228.

[11] JARCZYK, W., *A recurrent method of solving iterative functional equations*, Scientific Publications of the University of Silesia, Katowice, 1991.

[12] JARCZYK, W., SABLIK, M., *Duplicating the cube and functional equations,* Results Math. 26 (1994), 324 – 335.

[13] JONES, F. B., *Connected and disconnected plane sets and the functional equation* $f(x) + f(y) = f(x + y)$, Bull. Amer. Math. Soc. 48 (1942), 115 – 120.

[14] KAHLIG, P. AND SMÍTAL, J., *On the solutions of a functional equation of Dhombres*, Results Math. 27 (1995), 362 – 367.

[15] KUCZMA, M., *An Introduction to the Theory of Functional Equations and Inequalities. Cauchy's Equation and Jensen's Inequality*, Polish Scientific Publishers, Warsaw-Cracow-Katowice, 1985.

[16] KUCZMA, M., CHOCZEWSKI, B. AND GER, R., *Iterative Functional Equations*, Encyclopedia of Mathematics and its Applications, v. 32, Cambridge University Press, 1990.

[17] LACZKOVICH, M., *Nonnegative measurable solutions of difference equations*, J. London Math. Soc. 34 (1986), 139 – 147.