Petr Simon

An upper bound for countably splitting number

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 45 (2004), No. 2, 81--82

Persistent URL: http://dml.cz/dmlcz/702100

2004 ACTA UNIVERSITATIS CAROLINAE – MATHEMATICA ET PHYSICA VOL. 45, NO. 2

An Upper Bound for Countably Splitting Number

PETR SIMON

Praha

Received 15. March 2004

Countably splitting number cannot exceed the maximum of boundedness number and splitting number.

Let us recall three well-known cardinal invariants of continuum:

A family $\mathscr{S} \subseteq [\omega]^\omega$ is called *splitting*, if for every $X \in [\omega]^\omega$ there is some $S \in \mathscr{S}$ such that $|X \cap S| = |X \setminus S| = \omega$. Define then

$$\mathfrak{s} = \min\{|\mathscr{S}|: \mathscr{S} \subseteq [\omega]^\omega \text{ is splitting}\}.$$

Order ${}^\omega\omega$ by $f \leq^* g$ iff the set $\{n \in \omega : f(n) > g(n)\}$ is finite, and call a set $F \subseteq {}^\omega\omega$ *unbounded*, if for every $g \in {}^\omega\omega$ there is some $f \in F$ with $\neg(f \leq^* g)$. A set $D \subseteq {}^\omega\omega$ is called *dominating*, if for every $g \in {}^\omega\omega$ there is some $f \in D$ satisfying $g \leq^* f$. Define then

$$\mathfrak{b} = \min\{|F|: F \subseteq {}^\omega\omega \text{ is unbounded}\}$$

$$\mathfrak{d} = \min\{|D|: D \subseteq {}^\omega\omega \text{ is dominating}\}.$$

The next definition is, up to our knowledge, due to Bogdan Węglorz. A family $\mathscr{T} \subseteq [\omega]^\omega$ is called *countably splitting*, if for every countable $\mathscr{X} \subseteq [\omega]^\omega$ there is some $T \in \mathscr{T}$ such that T splits all members of $\mathscr{X}$, i.e., for every $X \in \mathscr{X}$, $|X \cap T| = |X \setminus T| = \omega$ holds. Define then

$$\aleph_0\text{-}\mathfrak{s} = \min\{|\mathscr{T}|: \mathscr{T} \subseteq [\omega]^\omega \text{ is countably splitting}\}.$$

Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic

Supported by grant GAČR 201/03/0933

It is well-known (cf. [Va]) that $\mathfrak{s} \leq \mathfrak{d}$ and $\mathfrak{b} \leq \mathfrak{d}$. Also, it is easy to show that $\mathfrak{s} \leq \aleph_0\text{-}\mathfrak{s} \leq \mathfrak{d}$. In an attempt to give a sharper bound, we prove in this short note the following.

Theorem. $\aleph_0\text{-}\mathfrak{s} \leq \max\{\mathfrak{s},\mathfrak{b}\}$.

Proof. Fix a splitting family $\mathscr{S} \subseteq [\omega]^\omega$ of size $\mathfrak{s}$ and and unbounded set $F \subseteq {}^\omega\omega$ of size $\mathfrak{b}$. We may and shall assume that for every $f \in F, f(0) = 0$ and the mapping f is strictly increasing. For $S \in \mathscr{S}$ and $f \in F$, put

$$T(S,f) = \bigcup\{[f(n),f(n+1)) : n \in \mathscr{S}\}$$

Clearly, $|\mathscr{T}| \leq \mathfrak{s} \cdot \mathfrak{b}$, so it remains to show that the family $\mathscr{T}$ is countably splitting. To this end, fix a countable family $\mathscr{X} = \{X_n : n \in \omega\}$ of infinite subsets of ω. Define a strictly increasing mapping $g \in {}^\omega\omega$ by putting $g(0) = 0$ and next, by induction, $g(k+1) = \min\{\ell \in \omega : (\forall i \leq k) X_i \cap [g(k),\ell) \neq \emptyset\}$. The set F is unbounded and so for a mapping h, defined by $h(n) = g(2n)$, there is some $f \in F$ with $\{n \in \omega : h(n) \leq f(n)\}$ infinite.

Let n be such that $f(n) \geq g(2n)$. The initial segment $[0,g(2n))$ is covered by $2n$ intervals $[g(k),g(k+1))$ and contains at most n points $f(i)$. Consequently, the number of intervals $[g(k),g(k+1))$ such that $[g(k),g(k+1))$ is not a subset of any $[f(i), f(i+1))$ is less or equal to n. All the remaining intervals $[g(k),g(k+1))$ must be contained in some $[f(i),f(i+1))$. So, $|\{k \in \omega : (\exists i < n)[g(k),g(k+1)) \subseteq$ $\subseteq [f(i),f(i+1))\}| \geq n$.

Since the set of those n's which satisfy $f(n) \geq g(2n)$ is infinite, we conclude that the set $\{k \in \omega : (\exists i \in \omega)[g(k),g(k+1)) \subseteq [f(i),f(i+1))\}$ is infinite. Therefore, also the set $Y = \{n \in \omega : (\exists k \in \omega)[g(k),g(k+1)) \subseteq [f(n),f(n+1))\}$ is infinite.

The family $\mathscr{S}$ is splitting, thus there is some $S \in \mathscr{S}$ such that $|Y \cap S| = |Y \setminus S| = \omega$.

Let us conclude the proof by showing that for this f and S, the set $T(S,f)$ splits all $X_n \in \mathscr{X}$. Whenever $i \in Y$ is such that $|Y \cap i| \geq n$, then for $k \in \omega$ with $[f(i), f(i+1)) \supseteq [g(k),g(k+1))$ we have $k \geq n$ and so, using the definition of the mapping g,

$$[f(i),f(i+1)) \cap X_n \supseteq [g(k),g(k+1)) \cap X_n \neq \emptyset.$$

But if $i \in Y \setminus S$, then $[f(i),f(i+1)) \subseteq \omega \setminus T(S,f)$, while if $i \in Y \cap S$, then $[f(i),f(i+1)) \subseteq T(S,f)$. So $|T(S,f) \cap X_n| = |X_n \setminus T(S,f)| = \omega$. □

References

[Va] VAUGHAN, Jerry, E., *Small uncountable cardinals and topology*, Open Problems in Topology, (ed. by J. van Mill and G. M. Reed), Elsevier 1990, 195–218.