On Complexity of a Set of Norms in Banach Spaces

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1. Let X be a separable Banach space and p an (equivalent) norm on X. An element x_0 of X is called *co-smooth* with respect to p if $p(x + tx_0)$ is a differentiable function of t for each x, except when $x + tx_0 = 0$. The set $\mathscr{G}_0$ then consists of those norms p admitting some co-smooth element $x_0 \neq 0$. What is the complexity of $\mathscr{G}_0$? (Here $\mathscr{G}$ = "Gateaux"). Compare [BGK].

The solution depends on the notion of a *Souslin scheme*: a system of sets E_s, where s ranges over the finite sequences of natural numbers. To any such scheme we can apply the *Souslin operation* $\mathscr{A}$ [K2, p. 198]. The class $\mathscr{A}(\Pi_1^1)$ of sets is then obtained when all the sets E_s are Π_1^1 (co-analytic). For more details on this class we refer to [K2, exercises 29.17, 37.4].

The set $\mathbb{N}(X)$ of all norms in X is endowed with a weak and a strong topology. The weak topology is the product topology of mappings of X into R. The strong (or metric) topology is that of uniform convergence on the unit ball of some fixed norm; this is too strong for certain applications. There is a small difficulty in using the weak topology in $\mathbb{N}(X)$: it is not a metric space. It is, however, a monotone union of compact, metrizable subsets; this allows us to define Borel and co-analytic subsets.

Theorem 1. *The set $\mathscr{G}_0$ is of class $\mathscr{A}(\Pi_1^1)$ in its product topology.*

Theorem 2. *A certain Banach space Z, contained in $L^2 \oplus c_0$, has this property:*

For each set E of type $\mathscr{A}(\Pi_1^1)$ in a Polish space M, there is a continuous map φ of M into $\mathbb{N}(Z)$ – provided with its strong (or metric) topology – such that $\varphi^{-1}(\mathscr{G}_0) = E$.

Following the usual practice in questions of smoothness of norms, we find dual norms with a related property of rotundity.

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The appearance of a weak and a strong topology, and their pleasant roles in our theorems, are a frequent occurrence in descriptive theory. The Souslin operation $\mathscr{A}$ commutes with formation of inverse images — without any measurability conditions of any kind — so that Theorem 2 is best possible in a certain sense.

Proof of Theorem 1. We define a function of three variables

$$H(p, x, y) = \lim_n n(p(x + n^{-1}y) + p(x - n^{-1}y) - 2p(x))$$

where $p \in \mathbb{N}(X)$, and $x, y \in X - (0)$. Then $p \in \mathscr{G}_0$ precisely when there is some y such that $H(p, x, y) = 0$ for all x. (That is, $y \neq 0$ and $x \neq 0$). Concerning the function H, we use only the facts that $H \geq 0$, H is Borel measurable and $H(p, x, y)$ is continuous in y when p and x are fixed (since p is Lipschitz-continuous on X). We can replace the variable y in $X - (0)$ by a variable σ in $\Sigma = N^N$ (Baire null space) by mapping Σ onto $X - (0)$ by a continuous mapping; after this substitution, we obtain a function $G(x, y, \sigma)$, with properties like those of H. To each finite sequence s we assign an open subset V_s of Σ: the set of all σ beginning with s. We define now a Borel function $G_s(p, x)$ by the formula

$$G_s(p, x) = \inf\{G(p, x, \sigma) : \sigma \in V_s\}.$$

For each s, this is a Borel function of p, x because the infimum can be evaluated over a countable subset of V_s. We define a Π_1^1 subset E_s of $\mathbb{N}(X)$:

$$p \in E_s \Leftrightarrow G_s(p, x) = 0 \quad \text{for all} \quad x \in X - (0).$$

We assert now that $\mathscr{G}_0 = \mathscr{A}(E_s)$; it is clear that $\mathscr{G}_0 \subseteq \mathscr{A}(E_s)$. Conversely, suppose that $p \in \mathscr{A}(E_s)$ so there is a σ such that $p \in E_s$ whenever σ extends s, i.e. $\sigma \in V_s$. We have to show that $G(p, x, \sigma) = 0$ for every $x \neq 0$; otherwise there would be a neighborhood V of σ such that the infimum of $G(p, x, \sigma)$ on V is positive. But then there is an open set V_s such that $V_s \subseteq V$ and σ extends s. For this s, $p \notin E_s$. This completes the proof that $\mathscr{G}_0 = \mathscr{A}(E_s)$.

Let Σ_1 be the set of pairs (σ, s), where $\sigma \in \Sigma$ and s is a finite sequence of natural numbers. The set of finite sequences is treated as a discrete metric space, and Σ_1 as the product of this space with Σ. Thus Σ_1 is homeomorphic to Σ. Given a scheme (E_s) of Π_1^1 sets in M, we choose closed sets F_s in $M \times \Sigma$, whose projection into M is $M \setminus E_s$. We define a closed set F of $M \times \Sigma_1$ consisting of elements (m, σ, s) such that $(m, \sigma) \in F_s$.

We define also a closed subset H of $\Sigma_1 \times \Sigma$: it consists of elements (σ, s, τ) such that τ doesn't extend s. An element m of M is *selected* by an element τ of Σ provided: for every element (σ, s) of Σ_1 either $(\sigma, s, \tau) \in H$ or $(m, \sigma, s) \notin F$. We claim that the selected elements of M are just the elements of $\mathscr{A}(E_s)$. Indeed m is selected by τ if and only if, for every initial segment s of τ, $m \notin M \setminus E_s$, i.e. $m \in E_s$.

Because of the demands of later details in Theorem 2, we want to replace the set Σ with S^1 the unit circle in the place, identified with $R/2\pi$. We'll use a similar definition of selected elements m, at the expense of a further complication. H^* will be a closed subset of $\Sigma_1 \times S^1$. Elements m of $\mathscr{A}(E_s)$ will be selected (at least) by a Cantor set $C(m)$ in S^1, whereas other elements of M will be selected by (at most) a countable set. The idea goes back to Mazurkiewicz and Sierpiński [MS, 1924].

The set Σ is homeomorphic to $\Sigma \times \Sigma$, whose elements we denote by (τ, t'). Let r be a homeomorphism of $\Sigma \times \Sigma$ onto the set of irrationals in $(0, 1)$; then H^* is the closure of the set of elements $(\sigma, s, r(\tau, t'))$ such that $(\sigma, s, \tau) \in H$. Using the closed set H^* we have a new method of selection: m is selected by a number t in $S^1 \equiv R/2\pi$ provided, for every element (σ, s) of Σ_1 either $(\sigma, s, t) \in H^*$ or $(m, \sigma, s) \notin F$. If m is selected by τ_0 (in the previous method of selection), then m is now selected by all the elements (τ_0, t'), and so by a Cantor set $C(m)$.

Conversely, suppose m is selected by an uncountable set of numbers t in $S^1 \equiv R/2\pi$. One of these numbers t_0 will then not be a rational in $[0, 1]$. If t_0 is not in $[0, 1]$, then (σ, s, t_0) is never in H^*, so that m belongs to all sets E_s. A more interesting argument is needed if t_0 is an irrational in $(0, 1)$. Then $t_0 = r(\tau, t')$ for some element (τ, t') of $\Sigma \times \Sigma$; since r is a homeomorphism, we see that $(\sigma, s, t_0) \in H^*$ if and only if $(\sigma, s, \tau) \in H$. (We recall that H is closed in $\Sigma_1 \times \Sigma$). Thus m is selected by τ, whence $m \in \mathscr{A}(E_s)$.

Using the first definition of selection, with selectors chosen from a compact metric space, we could only represent Π_1^1 sets. We need not introduce any more bizarre sets after this. What is accomplished by passing to S^1 is this: a certain Banach space has a separable dual.

The sets M, Σ_1 and S^1 have metrics — S^1 as a subset of R^2. In the sets $M \times \Sigma_1$ and $\Sigma_1 \times S^1$ we use a sum of the metric on the factors. We can find Lipschitz-continuous function u on $M \times \Sigma_1$ and v on $\Sigma_1 \times S^1$, both to $[0, 1]$, such that $u^{-1}(1) = F$ and $v^{-1}(0) = H^*$.

After this, the nature of Σ_1 isn't important, so we replace it by Σ. Thus u is defined on $M \times \Sigma$, and v on $\Sigma \times S^1$. The set Σ_1 doesn't appear again.

3. Theorem 2 Technical matters (a) we give a simple example of a norm $|\cdot|$ on R^2, admitting no co-smooth vectors except 0. This will be true if there are two linearly independent vector in R^2, at which $|\cdot|$ isn't smooth. We define the unit ball of $|\cdot|$ by the inequalities $x^2 + y^2 \leq 1$, $|y| \leq 1/2$, so the norm isn't smooth at $(\pm\sqrt{3}/2, \pm 1/2)$. We denote $e = (1, 0)$, $f^* = (1, 0)$, so that e is a smooth point of the unit ball, f^* is a smooth point of the dual unit ball, and $|f^* + g^*| < |f^*| + |g^*|$ for all elements of the dual not proportional to f^*. The space $X = \ell^2(R^2, |\cdot|)$ using the norm $|\cdot|$ in R^2, is of course isomorphic to ℓ^2, and has no co-smooth vectors $\neq 0$. In the dual space, we define $f_1^* = (f^*, 0, 0, \ldots)$, $f_2^* = (0, f^*, 0, 0, \ldots)$, etc., and we use the sequence (f_n^*) to find a homeomorphism ψ of Σ into the sphere of the dual ball to $\ell^2(R^2)$.

Let $\sigma = (n_1, n_2, n_3, \ldots)$, $m_1 = n_1$, $m_2 = n_1 + n_2$, etc., and then $\psi(\sigma) = \sum_1^\infty 2^{-\kappa^2} f^*_{m_\kappa}$. This is our homeomorphism. Thus $\psi(\Sigma)$ isn't weakly closed, but it has a useful property which serves as a substitute: a sequence (y^*_κ) in $\psi(\Sigma)$ contains either a subsequence convergent in norm to an element of $\psi(\Sigma)$, or a subsequence convergent weakly to an element of norm < 1.

Technical matters (b) Let Y be the set of (formal) trigonometric series $\sum_{-\infty}^{\infty} c_n \, e^{in\theta}$ with complex coefficients c_n, such that $\sum_{-N}^{N} |c_n|^4 = o(N)$ as $N \to +\infty$ [K1]. A possible norm is defined by $|y|^4 = \sup\,(2N + 1)^{-1} \sum_{-N}^{N} |c_n|^4$, but later we choose an equivalent norm. Then Y is isomorphic to a subspace of c_0, whence Y^* is separable. The exponent 4 could be replaced by any number $p > 2$; the purpose of using such an exponent will appear presently. Elements of Y can be multiplied by trigonometric series $\sum a_n \, e^{in\theta}$ such that $\sum (1 + |n|^{1/4})\,|a_n| < +\infty$. Periodic functions of class $\wedge^1$ have Fourier coefficients a_n such that $\sum (1 + |n|^{1/3})\,|a_n| < +\infty$ (by Perseval's formula and Cauchy's inequality) so Y becomes a continuous module over $\wedge^1$. This enables us to define the support supp(y) of an element y in two ways. First, it is the common zero-set of the ideal of functions f in $\wedge^1$ such that $f \cdot y = 0$ (the annihilator of y). Second, it is the smallest closed set F, such that $f \cdot y = 0$ whenever $f = 0$ on a *neighborhood* of F. (We define $J(F)$ to be the ideal of such functions; it is the smallest ideal whose zero-set is F). We observe that supp(y) is a perfect, nonvoid set unless $y = 0$ [K1]. Clearly $f \cdot y = 0$ when f belongs to the norm closure J^- of $J(\text{supp}\, y)$. When $f = 0$ on a closed set F, then $f^2 \in J^-(F)$. From this we show that supp(y) must have at least two elements (unless $y \neq 0$) and then supp(y) can have no isolated points [K1]. Denoting the sum $\sum (1 + |n|^{1/4})\,|a_n|$ by $|f|^\#$, we obtain from Parseval's formula and Cauchy's inequality, $|f|^\# \le |a_0| + c\|f\|_{L^2}^{1/4} \|f'\|_{L^2}^{3/4}$. We observe that when F is an uncountable closed set in S^1, it carries a probability measure μ such that $y = \hat{\mu} \in Y$ (i.e. its Fourier-Stieltjes series), and in this case $f \cdot y = 0$ whenever $f = 0$ on F (there is no concern here about ideals, since μ is a set-function).

We choose a norm in Y so that Y^* is strictly convex; and a number c such that $|f \cdot y| \le c(\|f\|_\infty + \|f'\|_\infty)\,|y|$, when $f \in \wedge^1(S^1)$, $y \in Y$. (We don't need any further refinements of this norm). Later we use the constant $b = c^{-1}$. The space Z is now chosen to be $X \oplus Y$.

Technical matters (c) Frequent use is made of this device: two norms have unit balls $\overline{co}(S_1)$ and $\overline{co}(S_2)$. When S_1 and S_2 are close, how close are the associated norms p_1 and p_2? We'll suppose S_1 and S_2 are symmetric, that the basic norm has unit ball B, and $\overline{co}S_2 \supseteq 2^{-1}B$. (This is a typical situation). The important inequality takes the form $S_1 \subseteq S_2 + aB$. Then $\overline{co}S_1 \subseteq \overline{co}S_2 + a'B$, for any $a' > a$. Hence $\overline{co}S_1 \subseteq (1 + 2a')\,\overline{co}S_2$, whence $p_2 \ge (1 + 2a)^{-1}\,p_1$. When $x \in B$ and $a < 1/4$, we conclude that $p_2(x) \ge p_1(x) - 4a$.

We can apply this to dual norms as well, replacing $\overline{co}(S_i)$ with w^*-convex closures.

Theorem 2, concluded We assigned to each element m of $\mathscr{A}(E_s)$, a Cantor set $C(m)$ on S^1, such that m is *selected* by all the elements t of $C(m)$, using the auxiliary functions u and v. We'll show that each continuous measure μ_m on $C(m)$ is a co-smooth vector for the norm $\varphi(m)$ in $\mathbb{N}(Z)$; of course μ_m belongs to Y through the Fourier expansion. We define $N^*(m)$ to be the linear subspace of Y^*, orthogonal to all of the measures μ_m.

Let p_1 be a norm defined by the unit ball of the dual space, which is the ω^*-closed convex hull of a set S; S is the union of two sets

(i) $B_1(X^*) \cup B_1(Y^*)$.

(ii) The set of all sums $\pm u(m, \sigma)\, \psi(\sigma) + v(\sigma, t) \cdot y^*$, where $\sigma \in \Sigma$, $y^* \in Y^*$ and $|y^*| \leq b$, and $v(\sigma, t)$ acts on Y^* as a Lipschitz function on S^1. Clearly the norm p_1 depends continuously on m, and $p_1(x^*) \equiv |x^*|$, $p_1(y^*) \equiv |y^*|$, $(x^* \in X^*, y^* \in Y^*)$.

The next lemma is a key step in the program outlined above.

Lemma A. *Suppose* $|x^*| = 1$, $y^* \neq 0$, *and* $p_1(x^* + y^*) = 1$, $(x^* \in X^*, y^* \in Y^*)$. *Then* $x^* = \pm\psi(\sigma)$ *for a certain* σ *in* Σ; *and* $y^* \in N^*(m)$, *provided* $m \in \mathscr{A}(E_s)$.

Proof. Since $y^* \neq 0$, there is some $y_0 \in Y$, such that $y^*(y_0) = 1$; will be convenient below to allow y_0 to be any solution of this equation; and there is some x_0 of norm 1 in X, such that $x^*(x_0) = 1$. For each $\kappa = 1, 2, 3, \ldots$, there is some $z_\kappa^* = x_\kappa^* + y_\kappa^*$ in S such that $z_\kappa^*(x_0 + \kappa^{-1} y_0) > 1 + \kappa^{-1}/2 > 1$, for κ large. Clearly z_κ^* must belong to the set listed under (ii), $z_\kappa^*(x_0) \geq 1 - 0(\kappa^{-1})$ and $y_\kappa^*(y_0) > 1/2$. The sequence (x_κ^*) has w^*-limits only on the unit sphere of X^*. Since $x_\kappa^* = \pm u(m, \sigma_\kappa)\, \psi(\sigma_\kappa)$, with σ_κ in Σ, we can apply our remarks on the mapping ψ, to conclude that the sequence $(\sigma_\kappa)_1^\infty$ has an accumulation point σ_∞, that $\lim u(m, \sigma_\kappa) = 1$, and finally $u(m, \sigma_\infty) = 1$. Thus $\langle \psi(\sigma_\infty), x_0 \rangle = \pm 1$. Now $\psi(\sigma_\infty)$ is a point of Fréchet-smoothness in X^*; we can read off x_0 from this and find that x_0 is an F-smooth point in X. (Thus we could conclude that the entire sequence (σ_κ) converges.) Now $y_\kappa^* = v(\sigma_\kappa, t)\, \tilde{y}_\kappa^*$, where $(\tilde{y}_\kappa^*)$ is a bounded sequence in Y^*. If (y_κ^*) doesn't belong to $N^*(m)$, we can choose y_0 to be a measure μ concentrated on the Cantor set $C(m)$. (Assuming, of course, that $m \in \mathscr{A}(E_s)$). Since $u(m, \sigma_\infty) = 1$, we see that $\lim v(\sigma_\kappa, t) = \lim v(\sigma_\infty, t) = 0$ uniformly on the set $C(m)$, so $v(\sigma_\kappa, t)\, \mu \to 0$ in variation (and thus in the norm of Y). This contradiction proves that $y^* \in N^*(m)$.

In a moment we shall define a sequence of norms such that $p_\kappa(x^* + y^*) \geq p_\kappa(x^*) = |x^*|$, $(x^* \in X^*, y^* \in Y^*)$, $p_1 \geq p_\kappa$ $(\kappa = 2, 3, 4, \ldots)$ and each depends continuously on m. We'll then set $p^2 = \sum_\kappa 2^{-\kappa} p_\kappa^2$ in Z^* and show that the norm $p = p(m)$, whose dual norm p is defined in Z^* by this process, works in Theorem 2. We show first that p isn't in $\mathscr{G}_0$ if $m \notin \mathscr{A}(E_s)$. We know that $p(x + y) \geq p(x)$, and the norm in X has no co-smooth vectors except 0; from this we find that all co-smooth vectors, in Z, must belong to Y. Let $y_0 \in Y$, and supp y_0

be its support, a non-empty perfect set. Since $m \notin \mathscr{A}(E_s)$, there is an element σ_0 of Σ, such that $u(m, \sigma_0) = 1$, while $v(\sigma_0, t)$ doesn't vanish for all $t \in \operatorname{supp} y_0$. Thus $v(\sigma_0, t) \cdot y_0 \neq 0$, whence we can choose y_0^*, of norm at most b, so that $\langle v(\sigma_0, t) \cdot y_0^*, y_0 \rangle = \delta > 0$. Thus $\psi(\sigma_0) \pm v(\sigma_0, t) \cdot y_0$ have norm 1; taking x_0 to be the solution of $\langle \psi(\sigma_0), x \rangle = 1 = |x_0|$, we find $p(x_0 + ry) \geq 1 + r|\delta|$ for all real r. Thus y_0 fails to be co-smooth at x_0 for the norm p, as required.

We now specify the norms $p_2, p_3, p_4, \ldots$, beginning with $p_2, p_4, p_6, \ldots$. Let (t_n) be a dense sequence in $(0, 1)$ and define $p_{2n}(x^* + y^*) = p_1(x^* + t_n y^*)$, $n \geq 1$. For the remaining norms, we choose a dense sequence $(g_n^*)_1^\infty$ in Y^* and define $p_{2n+1}(x^* + y^*) = \inf\{p_1(x^* + y^* - tg_n^*) + |t| : t \in R\}$. Each of these is a dual norm and each depends continuously on m.

Suppose that $z \in Z$, $z \neq 0$, and z_1^*, z_2^* are elements of the duality set $J(z)$. Then $p(z_1^*) = p(z_2^*) = p(z_1^* + z_2^*)/2$. We'll show that the last inequalities always imply that $z_1^* - z_2^*$ vanishes on all the measures μ_m, whence each μ_m is co-smooth for the norm p. (To repeat, $m \in \mathscr{A}(E_s)$).

The norms $p_3, p_5, p_7, \ldots$ all have the form $\inf p_1(z^* - tg_0^*) + |t|$, with varying choices of g_0^*. We want to examine how this changes if we replace g_0^* by g_1^*. The infimum is attained at some t in the interval $|t| \leq p_1(z^*)$; changing g_0^* to g_1^* yields an increase at most $p_1(z^*) \cdot p_1(g_0^* - g_1^*)$ (and hence a decrease of the same size).

Suppose, finally, that $p(x_1^* + y_1^*) = p(x_2^* + y_2^*) = p(x_1^* + y_1^* + x_2^* + y_2^*)/2$. Using the norms $p_2, p_4, p_6, \ldots$ we see that the norms $p_1(x^* + \lambda y^*)$, $0 \leq \lambda \leq 1$, all have the same property. We observe that if $x_1^* = 0$ or $x_2^* = 0$ then both are 0, and then $|y_1^*| = |y_2^*| = |y_1^* + y_2^*|/2$, whence $y_1^* = y_2^*$. Putting aside this trivial case, we can assume $|x_1^*| = |x_2^*| = 1 = |x_1^* + x_2^*|/2$.

We first deal with the case of linearly independent functionals y_1^* and y_2^*, and the norms $q(z^*) = \inf\{p_1(z^* - try_1^*) + |t| : t \in R\}$, depending on a real number $r > 0$. As $r \to \infty$ the limit is 1 on $x_1^* + y_1^*$, whence the same is true for $x_2^* + y_2^*$; here we take limits of the norms $p_3, p_5, p_7, \ldots$. Hence $\inf p_1(x_2^* + y_2^* - ty_1^*) = 1$, and the infimum is attained at some t_0, since we can assume that $p_1(y_2^* - ty_2^*) \leq 2$ in taking the infimum. Since y_1^* and y_2^* are linearly independent, we find by Lemma A that $x_1^* = \pm\psi(\sigma)$ for some σ, and by the properties of the mapping ψ, we see that $x_2^* = x_1^*$. (Here we refer to the properties of the norm in X^*, as well.) In the case $t_0 = 1$, Lemma A implies that $y_2^* - y_1^*$ belongs to $N^*(m)$. We can assume $t_0 \neq 1$.

Let $0 < \lambda \leq \min(1, |1 - t_0|)$. We'll show that $p_1(x_1^* + \lambda y_1^*) = p_1(x_1^* + \lambda y_2^*) = 1$, whence $y_1^*, y_2^* \in N^*(m)$. The norm p_1 is constant on the segment joining $x_1^* + \lambda y_1^*$ to $x_1^* + \lambda y_2^*$, taking there a value $e \geq 1$. Its value is at least e at $x_1^* + u\lambda y_1^* + (1 - u)\lambda y_2^*$, for any real u, by convexity. We can choose u so that $u\lambda y_1^* + (1 - u)\lambda y_2^*$ is a multiple $\alpha(y_2^* - t_0 y_1^*)$; this occurs when $\alpha = \lambda(1 - t_0)^{-1}$, so $|\alpha| \leq 1$. We see that $p_1(x_1^* + y_2^* - t_0 y_1^*) \geq e$, whence $e = 1$, and y_1^*, y_2^* belong to $N^*(m)$.

The remaining case, of different but dependent functionals y_1^* and y_2^*, is more difficult. We can assume that $y_2^* = cy_1^*$, with $|c| \leq 1$. In case $c \leq 0$, the segment joining $x_1^* + y_1^*$ to $x_2^* + cy_1^*$ traverses a points at which $p_1 = 1$. Then we would have $p_1(x_1^* + y_1^*) = 1$, and could apply Lemma A. Hence we can assume $0 < c < 1$, and $p_1(x_1^* + y_1^*) > 1$, to obtain a contradiction. We consider a norm depending on a parameter $r > 0$:

$$q(z^*, r) = \inf p_1(z^* - try_1^*) + |t|, \qquad t \in R.$$

When $z^* = x_1^* + y_1^*$, we make a substitution $s = 1 - tr$ and obtain

$$q(x_1^* + y_1^*, r) = \inf p_1(x_1^* + sy_1^*) + r^{-1}|1 - s|, \qquad s \in R.$$

Clearly, the infimum is obtained only on the set $0 \leq s \leq 1$, i.e. $0 \leq tr \leq 1$. When r is small enough, the infimum cannot be attained at $s = 0$; we fix such an r, and a value s in $(0, 1]$ at which the infimum is attained. This means that $0 \leq tr < 1$.

We can majorize the norm $q(z_2^*)$ by using $t' = ct$ in the infimum, obtaining

$$q(x_2^* + cy_1^*) \leq p_1(x_2^* + c(1 - tr)\, y_1^*) + c|t|.$$

Since $q(z_2^*) = q(z_1^*)$ we obtain

$$p_1(x_2^* + c(1 - tr)\, y_1^*) \geq p_1(x_1^* + (1 - tr)\, y_1^*).$$

But $p_1(x_2^* + \lambda cy_1^*) = p_1(x_1^* + \lambda y_1^*)$ for all λ in $[0, 1]$, so

$$p_1(x_1^* + (1 - tr)\, y_1^*) \leq p_1(x_1^* + c(1 - tr)\, y_1^*).$$

Now $0 < c < 1$ and $0 < 1 - tr \leq 1$, and so $x_1^* = \pm\psi(\sigma)$ for some σ, whence $x_1^* = x_2^*$ and finally $p_1(x_1^* + y_1^*) = 1$. Thus y_1^* and $y_2^* \in N^*(m)$.

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