Petr Holický; Václav Komínek

On projection of nonseparable Souslin and Borel sets along separable spaces

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 42 (2001), No. 2, 33--41

Persistent URL: `http://dml.cz/dmlcz/702076`

On Projection of Nonseparable Souslin and Borel Sets along Separable Spaces

PETR HOLICKÝ and VÁCLAV KOMÍNEK

Praha

Received 11. March 2001

A method of reducing the study of subsets of products of a general (topological) space and a Polish space is used to derive several analogues to theorems of the classical descriptive set theory. Theorems on generalized projections, on bimeasurability, and on uniformization are obtained.

Introduction and some notation

The main aim of this remark is to point out which "classical" results on descriptive properties of subsets $S \subset X \times Y$ of products of two separable metric spaces can be, in a quite straightforward way, extended to more general non-separable spaces X from the known separable versions by a reduction of the respective nonseparable problem to its separable analogue. As examples we investigate some results on "generalized projections" (i.e. the sets of $x \in X$ such that the sections $S_x = \{y \in Y; (x, y) \in S\}$ are elements of some prescribed family of sets), on bimeasurability, and on uniformization.

It turns out that the mentioned attitude has some principal limitations. We are able to deduce in this way only those results that do not assume analyticity of X. Thence our considerations point out to some problems that probably need a deeper study in the nonseparable case.

Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Praha 8 - Karlín, Czech Republic

1980 *Mathematics Subject Clas ification* (1985 *Revision*). Primary 54H05; Secondary 28A05, 03E15.

Key words and phra es. Borel ets, Souslin sets, uniformization, nonseparable spaces, bimeasurable mapping.

The au hors were partial y supported by GAUK 160/1999; the first author also by GAČR 201/00/0767, CAČR 201/00/1466 and MSM 113200007.

We formulate explicitly the mentioned “separable reduction” in Section 1. The applications to (generalized) projections, bimeasurability, and uniformization are presented in the next sections. Let us notice that we obtain e.g. a result of [L] and an improvement of [RW, Theorem 17].

We begin now by recalling some notions and notation.

Given a family $\mathscr{H}$ of subsets of a set X, we use $\mathbf{S}(\mathscr{H})$ to denote the class of sets obtained from elements of the class $\mathscr{H}$ by the *Souslin* (or *Aleksandrov*) operation, i.e. the sets of the form $A = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{k \in \mathbb{N}} A_{\sigma_1 \dots \sigma_k}$, where $A_s \in \mathscr{H}$ for each finite sequence s of positive integers. We denote the set of all finite sequences of positive integers by $\mathbb{S}$. Later on we use the abbreviated notation $\sigma|k$ for $(\sigma_1, \dots, \sigma_k)$.

We use $\mathscr{H}_\sigma$ to denote the family of unions of all at most countable subfamilies of $\mathscr{H}$.

By $\mathscr{M} \times \mathscr{H}$ we understand the set $\{M \times H;\, M \in \mathscr{M}, H \in \mathscr{H}\}$ for families $\mathscr{M}$ and $\mathscr{H}$ of subsets of sets X and Y, respectively.

A mapping f of $(X, \mathscr{M})$ to a topological space Z is called $\mathscr{M}$ measurable if $f^{-1}(G) \in \mathscr{M}$ for every open subset G of Z.

If X is a topological space, we denote by $\mathbf{F}(X)$, $\mathbf{G}(X)$, $\mathbf{K}(X)$ and $\mathbf{B}(X)$ the classes of all closed, open, compact and Borel subsets of X, respectively. The symbol $(\mathbf{F} \wedge \mathbf{G})(X)$ stands for the family of sets of the form $F \cap G$ with $F \in \mathbf{F}(X)$ and $G \in \mathbf{G}(X)$.

The elements of $\mathbf{S}(\mathbf{B}(X))$ are called here Souslin subsets of X, their complements are called co-Souslin, and Souslin sets with Souslin complements are called bi-Souslin. Let us remark that $\mathbf{S}(\mathbf{F}(X))$ is in general smaller than $\mathbf{S}(\mathbf{B}(X))$ for non-metrizable spaces X, whence they coincide for metrizable X. Souslin and co-Souslin subsets of Polish spaces are also called *analytic* and *coanalytic*, respectively.

1. A separable reduction

We formulate here explicitely statements that will help us to deduce the results on subsets of nonseparable products in the next sections. The reduction of the space X to a separable space $f(X)$ in Lemma 1 and its Corollary 2 was used more or less explicitly in other papers. One, in fact a bit more finer, application of such a reduction was used in the proof of [JR, Theorems 5.9.3 and 5.9.5] that are related to our Theorems 3(c) and 5(c).

Lemma 1. *Let X, Y be two sets, $\mathscr{M}$ be an algebra of subsets of X, $\mathscr{H}$ be an arbitrary family of subsets of Y. Let $S \subset X \times Y$ be in $\mathbf{S}(\mathscr{M} \times \mathscr{H})$.*

Then there exists an $\mathscr{M}_\sigma$-measurable mapping $f : X \to \{0,1\}^{\mathbb{N}}$ such that the set $T = \{(f(x), y);\, (x, y) \in S\} \subset f(X) \times Y$ is in $\mathbf{S}(\mathbf{F}(f(X)) \times \mathscr{H})$, and $S_x = T_{f(x)}$ for every $x \in X$.

Proof. Since S is in $\mathbf{S}(\mathscr{M} \times \mathscr{H})$, it may be written in the form

$$S = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} (M_{\sigma|n} \times H_{\sigma|n}),$$

where $M_s \in \mathscr{M}$ and $H_s \in \mathscr{H}$ for $s \in \mathbb{S}$.

For each $s \in \mathbb{S}$ let f_s be the characteristic function of M_s. Let us define $f : X \to \{0,1\}^{\mathbb{S}}$ by $f(x) = (f_s(x))_{s \in \mathbb{S}}$. We prove that it is $\mathscr{M}_\sigma$-measurable. The sets of the form $I_B^A = \{\tau \in \{0,1\}^{\mathbb{S}};\ \tau(A) = \{1\}, \tau(B) = \{0\}\}$, where A, B are disjoint finite subsets of $\mathbb{S}$, form a countable basis for the topology of $\{0,1\}^{\mathbb{S}}$. As the preimage $f^{-1}(I_B^A)$ of I_B^A equals to $\bigcup_{s \in A} M_s \setminus \bigcup_{t \in B} M_t$, it is in the algebra $\mathscr{M}$ and the $\mathscr{M}_\sigma$-measurability of f follows from the countability of the basis.

We define $\Phi : X \times Y \to \{0,1\}^{\mathbb{S}} \times Y$ by $\Phi(x, y) = (f(x), y)$ and we put $T = \Phi(S)$. Clearly $\Phi(X \times Y) = f(X) \times Y$. The mappings f and Φ have the following properties.

1) Obviously, $f(M_s) = f(X) \cap \{\tau \in \{0,1\}^{\mathbb{S}};\ \tau(s) = 1\}$ and thus $f(M_s)$ is clopen in $f(X)$ and $f^{-1}(f(M_s)) = M_s$ for each $s \in \mathbb{S}$.

2) $\Phi(S) = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} (f(M_{\sigma|n}) \times H_{\sigma|n})$.

The inclusion $\subset$ is clear as $\Phi(\bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} (M_{\sigma|n} \times H_{\sigma|n})) \subset \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} \Phi(M_{\sigma|n} \times H_{\sigma|n})$ and $\Phi(M_s \times H_s) = f(M_s) \times H_s$ for each $s \in \mathbb{S}$.

Conversely, let $z \in \bigcap_{n \in \mathbb{N}} (f(M_{\sigma|n}) \times H_{\sigma|n})$ for some $\sigma \in \mathbb{N}^{\mathbb{N}}$. So $z = (z_1, z_2)$, where $z_1 \in \bigcap_{n \in \mathbb{N}} f(M_{\sigma|n})$ and $z_2 \in \bigcap_{n \in \mathbb{N}} H_{\sigma|n}$. Cloose $x \in f^{-1}(z_1)$. Then $(x, z_2) \in \bigcap_{n \in \mathbb{N}} M_{\sigma|n} \times H_{\sigma|n} \subset S$ as $f^{-1}(f(M_s)) = M_s$ by 1), and hence $z = (z_1, z_2) = \Phi(x, z_2) \in \Phi(S)$.

3) It follows immediately from 1) and 2) that $\Phi(S) \in \mathbf{S}(\mathbf{F}(f(X)) \times \mathscr{H})$.

4) It can be easily observed from the definition of f that

$$S_x = \bigcup_{\{\sigma; x \in \bigcap_{n \in \mathbb{N}} M_{\sigma|n}\}} \bigcap_{n \in \mathbb{N}} H_{\sigma|n} = \bigcup_{\{\sigma; \forall_n (f_{\sigma|n}(x) = 1)\}} \bigcap_{n \in \mathbb{N}} H_{\sigma|n} = (\Phi(S))_{f(x)}$$

for all $x \in X$. This concludes the proof (noticing that $\{0,1\}^{\mathbb{N}}$ is homeomorphic to $\{0,1\}^{\mathbb{S}}$). □

Corollary 2. *Let X, Y be two sets, $\mathscr{M}$ be an algebra of subsets of X, $\mathscr{H}$ be a family of subsets of Y. Let $S^i \subset X \times Y$, where $i \in N$ (for some $N \subset \mathbb{N}$), be in $\mathbf{S}(\mathscr{M} \times \mathscr{H})$.*

Then there exists an $\mathscr{M}_\sigma$-measurable mapping $f : X \to \{0,1\}^{\mathbb{N}}$ such that the sets $T^i = \{(f(x), y); (x, y) \in S^i\} \subset f(X) \times Y$ are in $\mathbf{S}(\mathbf{F}(f(X)) \times \mathscr{H})$ and $S_x^i = T_{f(x)}^i$ for every $x \in X$ and $i \in N$.

In particular, if S^i form a partition of $X \times Y$, then T^i form a partition of $f(X) \times Y$.

Proof. Let $f_i : X \to \{0,1\}^{\mathbb{N}}$ be the function whose existence is ensured by Lemma 1 such that the set $\hat{S}^i = \{(f_i(x), y); (x, y) \in S^i\}$ is in $\mathbf{S}(\mathbf{F}(f_i(X)) \times \mathscr{H})$ and such that $S_x^i = \hat{S}_{f_i(x}^i$ for each $x \in X$.

Let $f : X \to (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ be defined by the equality $f(x) = (f_i(x))_{i\in N}$. Put now $T^i = \{(f(x), y); (x, y) \in S^i\}$. Then $T^i = (f(X) \times Y) \cap \{(z, y); (z(i), y) \in \hat{S}^i\}$ is obviously in $\mathbf{S}(\mathbf{F}(f(X)) \times \mathscr{H})$ and $S^i_x = T^i_{f(x)}$ for $x \in X$.

The claim on partitions follows immediately from the last equalities of "vertical sections". □

2. Generalized projections along separable spaces

Here we generalize the theorems on sets $\{x\in X;\ S_x \in \mathscr{D}\}$, where X is non-separable in general and $\mathscr{D}$ stands for a family of subsets that are in some sense small. The separable counterparts can be found e.g. in [D1, Theorem 32] and [D2, Chap. 4, Theorems 50 and 51] and we also use the corresponding results to deduce our generalizations. We begin with a basic notion of hereditary coanalytic family used in this section.

Let X be a topological space. A class $\mathscr{C} \subset \mathbf{F}(X)$ is called *hereditary* if $H \in \mathscr{C}$ whenever $H \in \mathbf{F}(X)$, $H \subset F$, and $F \in \mathscr{C}$.

We denote by $\mathscr{C}^*$ the class of all sets whose closure is in $\mathscr{C}$.

If X is a separable metric space, we say that $\mathscr{C} \subset \mathbf{F}(X)$ is a *co-Souslin family* if $\bar{\mathscr{C}}$ is coanalytic (equivalently co-Souslin) subset of the Effros Borel structure on $\mathbf{F}(Y)$ for some Polish completion Y of X. Here $\bar{\mathscr{C}}$ means the family of closures of elements of $\mathscr{C}$ in Y (cf. [D2, Rem. (d), p. 218]).

One can find in [K, Chapter 35.G] or [D2, Chapter 4, Sect. 43] some examples of such hereditary coanalytic families $\mathscr{C}$ in Polish spaces, e.g. the families of all compact sets, of sets having at most one point, of finite sets, of closed nowhere dense sets, of compact sets that are null with respect to a given Radon measure can serve as such examples. Of course, the family consisting only of the empty set is hereditary coanalytic, too. This is to point out that the ordinary projections are covered by theorems on generalized projections.

Theorem 3. *Let X be a set and $\mathscr{M}$ be an algebra of subsets of X, Y be an analytic space, $S \subset X \times Y$ be in $\mathbf{S}(\mathscr{M} \times \mathbf{F}(Y))$. Let $\mathscr{C}$ be a hereditary co-Souslin family of subsets of Y.*

(a) Then the sets

$$C_1 = \{x\in X; S_x \in \mathscr{C}^*\} \quad \text{and} \quad C_2 = \{x\in X; S_x \in \mathscr{C}^*_\sigma\}$$

are complements of sets from $\mathbf{S}(\mathscr{M})$.

(b) If moreover the complement of S is also in $\mathbf{S}(\mathscr{M} \times \mathbf{F}(Y))$, then also

$$C_3 = \{x\in X; S_x \in \mathscr{C}\} \quad \text{and} \quad C_4 = \{x\in X; S_x \in \mathscr{C}_\sigma\}$$

are complements of sets in $\mathbf{S}(\mathscr{M})$.

(c) If moreover Y is Polish, the complement of S is also in $\mathbf{S}(\mathscr{M} \times \mathbf{F}(Y))$, and every element of $\mathscr{C}$ is σ-compact, then the sets

$$C_5 = \{x \in X; \emptyset \neq S_x \in \mathscr{C}\} \quad and \quad C_6\{x \in X; \emptyset \neq S_x \in \mathscr{C}_\sigma\}$$

are complements of sets from $\mathbf{S}(\mathscr{M})$

Proof. (a) Let $S \subset X \times Y$ be in $\mathbf{S}(\mathscr{M} \times \mathbf{F}(Y))$, f and T be as in Lemma 1, and let $D_1 = \{z \in f(X); T_z \in \mathscr{C}^*\}$. It follows from Lemma 1 that $C_1 = f^{-1}(D_1)$. As f is $\mathscr{M}_\sigma$-measurable, it suffices to verify that D_1 is a complement of a set from $\mathbf{S}(\mathbf{F}(f(X)))$. Let Z be a Polish completion of $f(X)$ and T' be an analytic subset of $Z \times Y$ with $T' \cap (f(X) \times Y) = T$. Applying [D2, Chapter 4, Theorem 50] we get that $\{z \in Z; T'_z \notin \mathscr{C}^*\}$ is analytic, thus Souslin in Z, and so $D_1 = \{z \in Z; T'_z \in \mathscr{C}^*\} \cap f(X)$ is co-Souslin in $f(X)$.

The claim for C_2 follows similarly using [D2, Chapter 4, Theorem 51].

(b) Now let S and its complement be in $\mathbf{S}(\mathscr{M} \times \mathbf{F}(Y))$, $S^1 = S$ and $S^2 = S^C$. Let f, T^1, and T^2 be as in Corollary 2 with $N = \{1,2\}$. Then T^1 and T^2 are in $\mathbf{S}(\mathbf{F}(f(X)) \times \mathbf{F}(Y))$, where $f(X)$ and Y are separable metric spaces. Since $T^2 = T^{1C}$, it follows that T^1 is bi-Souslin in $f(X) \times Y$.

To prove the claim on C_3 and C_4 it is enough to show that the sets

$$D_3 = \{z \in f(X); T^1_z \in \mathscr{C}\} \quad \text{and} \quad D_4 = \{z \in f(X); T^1_z \in \mathscr{C}_\sigma\}$$

are co-Souslin subsets of $f(X)$. Let Z be a Polish completion of $f(X)$ again.

Since T^1 is bi-Souslin in $f(X) \times Y$, we can find $S_A, S_C \subset Z \times Y$, the first being analytic (equivalently Souslin), and the second co-Souslin, subset of the analytic space $Z \times Y$ such that $T^1 = S_A \cap (f(X) \times Y) = S_C \cap (f(X) \times Y)$.

Let us consider a continuous $h: \mathbb{N}^\mathbb{N} \to Z \times Y$ such that $h(\mathbb{N}^\mathbb{N}) = S_A$. Then define a closed set $F \subset Z \times \mathbb{N}^\mathbb{N}$ by $(z, \sigma) \in F \Leftrightarrow \Pi_Z(h(\sigma)) = z$, and a family $\hat{\mathscr{C}} \subset \mathbf{F}(\mathbb{N}^\mathbb{N})$ of all K such that $\overline{\Pi_Y(h(K))}^Y \in \mathscr{C}$ and $\overline{h(K)}^{Z \times Y} \subset S_C$.

The mappings $K \mapsto \overline{\Pi_Y(h(K))}^Y$ and $K \mapsto \overline{h(K)}^{Z \times Y}$ are Borel measurable as h and Π_Y are continuous by [D2, Chapter II, Theorem 10]. Thus the family $\hat{\mathscr{C}}$ is coanalytic in the Effros structure of $\mathbf{F}(\mathbb{N}^\mathbb{N})$ using the fact that $\{F \in \mathbf{F}(Z \times Y); F \subset S_C\}$ is co-Souslin due to [D2, Chapter II, Rem. 13c].

Using [D2, Chapter 4, Theorem 50 and Theorem 51] we get that

$$D'_3 = \{z \in Z; F_z \in \hat{\mathscr{C}}\} \quad \text{and} \quad D'_4 = \{z \in Z; F_z \in \hat{\mathscr{C}}_\sigma\}$$

are complements of sets from $\mathbf{S}(\mathbf{F}(Z))$. As $T^1_z \in \mathscr{C}$ if and only if $F_z \in \hat{\mathscr{C}}$ and $T^1_z \in \mathscr{C}_\sigma$ if and only if $F_z \in \hat{\mathscr{C}}_\sigma$ for $z \in f(X)$, our claim easily follows from the equalities $D_i = f(X) \cap D'_i$ for $i = 3\ 4$.

(c) *Let* $S^1 = S$, $S^2 = S^C$ and f, T^1, and T^2 be as in Corollary 2 with $N = \{1,2\}$. Since $T^2 = T^{1C}$, we have that T^1 is bi-Souslin in $f(X) \times Y$. The set $D = (z \in f(X);\ \emptyset \neq T_z \in \mathbf{K}_\sigma(Y)\}$ is a complement of a set in $\mathbf{S}(\mathbf{F}(f(X)))$ due to [D1, Theorem 31] and [D2, Remark (b), p. 255].

So $C = f^{-1}(D)$ is a complement of a set from $\mathbf{D}(\mathscr{M})$ because of the $\mathscr{M}_\sigma$-measurability of f. We have $C = \{x \in X; \emptyset \neq S_x \in \mathbf{K}_\sigma(Y)\}$, $C_5 = C \cap \{x \in X; S_x \in \mathscr{C}\}$, and $C_6 = C \cap \{x \in X;\ S\ \in \mathscr{C}_\sigma$. The sets $\{x \in X;\ S_x \in \mathscr{C}\}$ and $\{x \in X;$

$S_x \in \mathscr{C}_\sigma\}$ are complements of sets from $\mathbf{S}(\mathscr{M})$ due to part (b) and this concludes the proof. □

To get some corollaries of Theorem 3 for topological spaces X, we need the following elementary lemma.

Lemma 4. *Let X and Y be Hausdorff spaces, Y having a countable basis. Then*

$$\mathbf{S}(\mathbf{F}(X \times Y)) = \mathbf{S}(\mathbf{F}(X) \times \mathbf{F}(Y)) \quad \textit{and} \quad \mathbf{S}(\mathbf{B}(X \times Y)) = \mathbf{S}(B(X) \times \mathbf{B}(Y)).$$

Proof. Let $\{B_n\}_{n \in \mathbb{N}}$ be a countable basis of the topology of Y. Let $G \subset X \times Y$ be open. For every $x \in G$ we find $n = n(x) \in \mathbb{N}$ and an open $H(x) \subset X$ so that $x \in H(x) \times B_n \subset G$. The sets $G_k = \bigcup\{H(x); x \in G, k = n(x)\}$ are open in X and $G = \bigcup_k G_k \times B_k$. Now $G^{\mathrm{C}} = \bigcap_{k \in \mathbb{N}} (G_k \times B_k)^{\mathrm{C}}$ and each of the sets $(G_k \times B_k)^{\mathrm{C}}$ is a finite union of sets from $\mathbf{F}(X) \times \mathbf{F}(Y)$. So all open subsets of $X \times Y$ are in $(\mathbf{G}(X) \times \mathbf{G}(Y))_\sigma$ and all closed subsets of $X \times Y$ are in $(\mathbf{F}(X) \times \mathbf{F}(Y))_{\sigma\delta}$.

Since $\mathbf{S}(\mathbf{S}(\mathscr{S})) = \mathbf{S}(\mathscr{S})$ for any class of sets $\mathscr{S}$ (see [JR, Theorem 2.3.1]) and $\mathbf{S}(\mathscr{S})$ is stable to countable unions and intersections (see [JR, Corollary 2.3.3]), it follows that $\mathbf{S}(\mathbf{F}(X \times Y)) \subset \mathbf{S}(\mathbf{F}(X) \times \mathbf{F}(Y))$ and $\mathbf{S}(\mathbf{B}(X \times Y)) \subset \mathbf{S}(\mathbf{B}(X) \times \mathbf{B}(Y))$, whence the other inclusions are obvious. □

Theorem 5. *Let X be a Hausdorff space. Let Y be an analytic space, $S \subset X \times Y$ be in $\mathbf{S}(\mathbf{B}(X \times Y))$. Let $\mathscr{C}$ be a hereditary co-Souslin family of closed subsets of Y.*

(a) Then the sets

$$C_1 = \{x \in X; S_x \in \mathscr{C}^*\} \quad \textit{and} \quad C_2 = \{x \in X; S_x \in \mathscr{C}^*_\sigma\}$$

are complements of sets in $\mathbf{S}(\mathbf{B}(X))$.

(b) If moreover the complement of S is also in $\mathbf{S}(\mathbf{B}(X \times Y))$, then also

$$C_3 = \{x \in X; S_x \in \mathscr{C}\} \quad \textit{and} \quad C_4 = \{x \in X; S_x \in \mathscr{C}_\sigma\}$$

are complements of sets in $\mathbf{S}(\mathbf{B}(X))$.

(c) If moreover Y is a Polish space, $S \subset X \times Y$ is such that both S and S^{C} are in $\mathbf{S}(\mathbf{B}(X \times Y))$, and each element of $\mathscr{C}$ is σ-compact, then the sets

$$C_5 = \{x \in X; \emptyset \neq S_x \in \mathscr{C}\} \quad \textit{and} \quad C_6 = \{x \in X; \emptyset \neq S_x \in \mathscr{C}_\sigma\}$$

are complements of sets in $\mathbf{S}(\mathbf{B}(X))$.

Proof. According to Lemma 4, $\mathbf{S}(\mathbf{B}(X \times Y)) = \mathbf{S}(\mathbf{B}(X) \times \mathbf{B}(Y))$. Moreover, $\mathbf{S}(\mathbf{B}(X) \times \mathbf{B}(Y)) = \mathbf{S}(\mathbf{B}(X) \times \mathbf{F}(Y))$ because of the metrizability of Y. So we can use Theorem 3 with $\mathscr{M} = \mathbf{B}(X)$ to get all assertions of Theorem 5. □

Remark. Of course, if X is a Hausdorff space, where open sets are in $\mathbf{S}(\mathbf{F}(X))$, then C_i's from Theorem 5 are complements of sets from $\mathbf{S}(\mathbf{F}(X))$, since $\mathbf{S}(\mathbf{B}(X)) = \mathbf{S}(\mathbf{F}(X))$ in this case. One statement of Theorem 1 of [L] says that, if X is a topological space with $\mathbf{G}(X) \subset \mathbf{S}(\mathbf{F}(X))$ and Y is a Borel subset of a Polish space, then $\{x \in X;\ \emptyset \neq S_x \in \mathbf{K}_\sigma(Y)\}$ is a complement of a set from $\mathbf{S}(\mathbf{F}(X))$. Obviously, this result follows from Theorem 5 (c). Whence Larman gives in [L] a proof of the classical theorem in the same time, correcting a gap in the original Kunugui's proof, we just deduce our a bit stronger statement from the classical one.

3. Bimeasurable mappings

We use here results of Section 2 to deduce generalizations of two characterizations of Borel bimeasurable projections. The first concerns the combination of theorems of Luzin (see e.g. [K, Theorem 15.1]) and Purves [P] giving a characterization of all Borel measurable mappings between Polish spaces that map Borel sets to Borel sets. The other concerns the combination of the classical theorem of Arsenin and Kunugui and its counterpart proved in [HZ] characterizing those Borel measurable mappings that map closed sets to Borel sets in the classical setting.

In the following assertion we need a perfect set theorem. We might consider complete metric space X and use that a Souslin subset of such an X contains a homeomorphic copy of the Cantor set if it is not σ-discrete. We prefer a more general formulation using isolated-analytic spaces that were introduced by R. W. Hansell under the name "descriptive spaces" in [H]. This class of spaces contains all Banach spaces, that have an equivalent norm with the Kadets property, endowed with their weak topology [H, Theorem 1.5].

A Hausdorff space X is *isolated-analytic* if there is a continuous mapping $f : M \to X$ of some complete metric space onto X such that, for every discrete family of subsets D_a, $a \in A$, of M, we have $f(D_a) = E_a = \bigcup_{n \in \mathbb{N}} E_n^a$, where the indexed families $\{E_n^a;\ a \in A\}$ are *relatively discrete* (or, equivalently, *isolated*) in X. It means that every point of $\bigcup_{a \in A} E_n^a$ has a neighbourhood intersecting E_n^a for at most one index $a \in A$. It is not difficult to notice that an isolated subset of a topological space, i.e. a set whose one-point subsets form an isolated family, is the intersection of an open set and a closed set and so any σ-isolated set is Borel (cf. [H, Lemma 3.4]).

We need the facts that a Souslin subset of an isolated-analytic space is isolated-analytic [H, Theorem 5.3] and that a regular isolated-analytic space that is not σ-isolated contains a homeomorphic copy of the Cantor set. The last statement can be proved similarly as [FH, Lemma 5.3] replacing the uniform discreteness by the relative discreteness. We have to take into account that every isolated-analytic space has a σ-relatively discrete network [H, Theorem 5.1] and consequently that locally σ-isolated sets are σ-isolated.

We use Π_X to denote the project on to X in the following theorem.

Theorem 6. *Let X be an isolated-analytic regular space and Y be a Polish space. Let $B \subset X \times Y$ be in bi-*$\mathbf{S}(\mathbf{B}(X \times Y))$.

(a) Then $\Pi_X(E)$ *is in bi-*$\mathbf{S}(\mathbf{B}(X))$ *for every* $E \in$ *bi-*$\mathbf{S}(\mathbf{B}(B))$ *if and only if the set* $\{x \in X;\ B_x$ *is uncountable*$\}$ *is σ-isolated.*

(b) Then $\Pi_X(F)$ *is in bi-*$\mathbf{S}(\mathbf{B}(X))$ *for every* $F \in \mathbf{F}(B)$ *if and only if the set* $\{x \in X;$ B_x *is not* $\mathbf{K}_\sigma(Y)\}$ *is σ-isolated.*

Proof. (a) If set $\{x \in X;\ B_x$ is uncountable$\}$ is σ-isolated, then obviously $N_E = \{x \in X;\ E_x$ is uncountable$\}$ is σ-isolated for each bi-$\mathbf{S}(\mathbf{B}(B))$ subset E of B. Each N_E, being σ-isolated, is $(\mathbf{F} \wedge \mathbf{G})_\sigma$, so Borel. The set $E' = E \setminus (N_E \times Y)$ is bi-$\mathbf{S}(\mathbf{B}(X \times Y))$ and such that each section E'_x is countable. So the complement of $\Pi_X E'$ is in $\mathbf{S}(\mathbf{B}(X))$ by Theorem 5(c) applied to $\mathscr{C}$ consisting of the sets that contain at most one point. The set $\Pi_X E'$ itself is also in $\mathbf{S}(\mathbf{B}(X))$ by Theorem 5(a) used to the family $\mathscr{C} = \{\emptyset\}$ this time. Since $\Pi_X(E) = \Pi_X(E') \cup N_E$, so $\Pi_X(E)$ is also in bi-$\mathbf{S}(\mathbf{B}(X))$.

Conversely, let $N_B = \{x \in X;\ B_x$ is uncountable$\}$ be not σ-isolated. It is also in $\mathbf{S}(\mathbf{B}(X))$ by Theorem 5(a) applied to $\mathscr{C}$ consisting of at most one-point sets again. So N_B is isolated-analytic and it contains a homeomorphic copy of the Cantor set C as mentioned before Theorem 6.

Due to [P, Theorem] there exists a Borel set $E \subset B \cap (C \times Y)$ with the projection to C being non-Borel in C. Since $C \cap \Pi_X(E)$ is not Borel, $\Pi_X(E)$ can not be in bi-$\mathbf{s}(\mathbf{B}(X))$.

(b) If the set $\{x \in X;\ B_x \notin \mathbf{K}_\sigma\}$ is σ-isolated, then also $N_F = \{x \in X;\ F_x \notin \mathbf{K}_\sigma\}$ is σ-isolated for each closed $F \subset B$. Thus it is also in $(\mathbf{F} \wedge \mathbf{G})_\sigma$. The set $F' = F \setminus (N_F \times Y)$ is thus in bi-$\mathbf{S}(\mathbf{B}(X \times Y))$, every section F'_x is in $\mathbf{K}_\sigma(Y)$. So the complement of $\Pi_X(F')$ is in $\mathbf{S}(\mathbf{B}(X))$ by Theorem 5(c) with $\mathscr{C} = \mathbf{K}(Y)$ and the set itself is in $\mathbf{S}(\mathbf{B}(X))$ by Theorem 5(a) applied to $\mathscr{C} = \{\emptyset\}$. Since $\Pi_X(F) = \Pi_X(F') \cup N_F$, $\Pi_X F$ is also bi-$\mathbf{S}(\mathbf{B}(X))$.

For the converse implication we use that the set $\{x \in X;\ B_x$ is not $\mathbf{K}_\sigma\}$ is in $\mathbf{S}(\mathbf{B}(X))$ (Theorem 5(b) with $\mathscr{C} = \mathbf{K}(Y)$). If it is not σ-isolated, it contains a copy of the Cantor set C and using [HZ, Main Theorem] we find a closed set $F \subset B \cap (C \times Y)$ with the projection to X being non-Borel, thus also not in bi-$\mathbf{S}(\mathbf{B}(X))$. $\square$

Remark. Another generalization of the results of Luzin and Purves for mappings between nonseparable complete metric spaces, and thence between "point-Luzin" spaces, was proved in [FH, Theorem 5.3].

4. Uniformization

We shall give here an application of the separable reduction described above to derive an improvement of the classical Kondô uniformization theorem (see e.g. [K, 36.14]). Let us notice that some generalization of Kondô's theorem were proved in [RW]. As concerns the uniformization, our result is stronger than that of [RW, Theorem 17] as we uniformize complements of sets from $\mathbf{S}(\mathbf{B}(X \times Y))$ and not only from $\mathbf{S}(\mathbf{F}(X \times Y))$. Of course, for the case of X with $\mathbf{G}(X) \subset \mathbf{S}(\mathbf{F}(X))$, we get just another proof of [RW, Theorem 17].

Theorem 7. *Let X be a Hausdorff topological space and Y be a Polish space. Let the complement of $C \subset X \times Y$ be in $\mathbf{S}(\mathbf{B}(X \times Y))$.*

Then there exists a set $U \subset C$ whose complement is also in $\mathbf{S}(\mathbf{B}(X \times Y))$ which uniformizes C, i.e. $\Pi_X(U) = \Pi_X(C)$ and U_x is a singleton for every $x \in \Pi_X(U)$.

Proof. Using Lemma 4 and Lemma 1 to the complement S of C, we get the Borel measurable mapping $f : X \to f(X) \subset \{0,1\}^{\mathbb{N}}$ and a set $T \in \mathbf{S}(\mathbf{F}(f(X) \times Y))$ as in Lemma 1. Applying Kondô's theorem we get the co-Souslin uniformization V for the complement of T in $f(X) \times Y$. The set $U = (f \times id)^{-1}(V)$ is a uniformization of C whose complement $(f \times id)^{-1}(V^{\complement})$ is in $\mathbf{S}(\mathbf{B}(X \times Y))$ since the mapping $f \times id$ is Borel measurable. □

Remark. A uniformization theorem for equivalence classes in nonseparable metric spaces is proved in [Ko, Theorem 3.1]. We do not see how to receive this result, or its corresponding improvement, by methods used above.

References

[D1] DELLACHERIE C., *Ensembles analytiques: théorèmes de séparation at applications*, Séminaires de Probabilités IX, Lecture Notes in Mathematics 465, Springer Verlag, Berlin, Heidelberg, New York, 1975, pp. 336 – 372.

[D2] DELLACHERIE C., *Un cours sur les ensembles analytiques*, Analytic sets (C. A. Rogers, ed.), Academic Press, London, 1980, pp 183 – 316.

[FH] FROLIK Z. and HOLICKÝ P., *Applications of Luzinian separation principles (non-separable case)*, Fundamenta Mathematicae **117** (1983), 165 – 185.

[H] HANSELL R. W., *Descriptive sets and the topology of non-separable Banach spaces.*

[HZ] HOLICKÝ P. and ZELENÝ M., *A converse of the Arsenin-Kunugui theorem on Borel sets with σ-compact sections*, Fundamenta Mathematicae **165** (2000), 191 – 202.

[JR] JAYNE J. E. and ROGERS C. A., *K-analytic sets*, Analytic sets (C. A. Rogers, ed.), Academic Press, London, 1980, pp. 1 – 181.

[K] KECHRIS A. S., *Classical Descriptive Set Theory*, Springer-Verlag, New York etc., 1995.

[Ko] KOMINEK V., *A remark on the uniformization in metric spaces*, Acta Universitatis Carolinae Mathematica et Physica **40** (1999), 65 – 74.

[L] LARMAN D. G., *Projecting and uniformi ing Borel sets with $\mathbf{K}_\sigma$ sections I.*, Mathematica **19** 1972), 231 – 244.

[P] PURVES R., *B measurab e functions* Fundamenta Mathematicae **58** (1966), 149 – 157.

[RW] ROGERS C. A. and WILLMOTT R. O. *On the uniformization of sets in topological spaces*, Acta mathematica **120** (968), 1 – 52.