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On Club-Like Principles on Regular Cardinals above $\beth_\omega$

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We prove that for regular λ above a strong limit singular μ certain guessing principles follow just from cardinal arithmetic assumptions. The main result is that for such λ and μ there are coboundedly many regular $\kappa < \mu$ such that $\clubsuit^-(S^\lambda_\kappa)$ holds whenever $\lambda = \lambda^{<\kappa}$.[1]

0 Introduction

The main result of this note is that for any regular cardinal λ above $\beth_\omega$ there are unboundedly many regular cardinals $\kappa < \beth_\omega$ such that provided $\lambda = \lambda^{<\kappa}$, the combinatorial principle $\clubsuit^-(S^\lambda_\kappa)$ holds. That principle is defined in the following Definition 0.1, and the notation S^λ_κ is recalled in 0.2.

Definition 0.1 Suppose that S is a stationary subset of a regular cardinal λ. Then $\clubsuit^-(S)$ is the statement claiming the existence of a sequence $\langle \mathscr{P}_\delta : \delta \in S\rangle$ such that

(i) each $\mathscr{P}_\delta$ is a family of $< \lambda$ many subsets of δ and
(ii) for every unbounded subset A of λ there are stationarily many δ such that for some $X \in \mathscr{P}_\delta$ with $\sup(X) = \delta$ we have $X \subseteq A$.

We also prove that some other similar combinatorial principles on such λ follow just from the assumptions on cardinal arithmetic. In fact, the same theorems hold more generally in a situation in which $\beth_\omega$ is replaced by any strong limit singular cardinal. Our proofs are an application of a (consequence of) a powerful theorem

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of Shelah in [Sh 3], Theorem 0.3 below. The methods are similar to the ones used in [Sh 3] to prove e.g. that for λ as above the assumption $2^\lambda = \lambda^+$ implies that $\diamondsuit$ holds at λ.

Throughout the note we use the notation given below. Note that cov as used here is a special case of a more general notation used in pcf theory, but to increase readability we only quote the instance of it that we actually use.

Notation 0.2 Suppose that κ is a regular cardinal and $\alpha > \kappa$ an ordinal. Then

$$S^\alpha_\kappa \stackrel{\text{def}}{=} \{\beta < \alpha : \mathrm{cf}(\beta) = \kappa\}, \tag{1}$$

$$S^\alpha_{<\kappa} \stackrel{\text{def}}{=} \{\beta < \alpha : \mathrm{cf}(\beta) < \kappa\}, \tag{2}$$

$$\begin{aligned}\mathrm{cov}(\alpha, \kappa^+, \kappa^+, \kappa) \stackrel{\text{def}}{=} \min\{\theta : (\exists \mathscr{P} \subseteq [\alpha]^{\leq\kappa})\\ |\mathscr{P}| = \theta \,\&\, (\forall A \in [\alpha]^{\leq\kappa})\,(\exists X \in [\mathscr{P}]^{<\kappa})\, A \subseteq \bigcup X\}.\end{aligned} \tag{3}$$

(4) For a subset A of κ we let $\lim(A) \stackrel{\text{def}}{=} \{\beta < \kappa : \beta = \sup(A \cap \beta)\}$ and $\mathrm{cl}(A) = A \cup \lim(A)$.

The theorem we need for our application is given below as Theorem 0.3. Its statement is modulo the notation an easy consequence of Theorem 1.1. of [Sh 3] combined with another deep theorem of cardinal arithmetic, the 'cov versus pp' theorem of Shelah. As this may not be immediate from reading [Sh 3], for the benefit of an interested reader we briefly comment on how the connection can be seen.

Theorem 0.3. *(Shelah) Suppose that μ is a strong limit singular cardinal. Then for $\lambda > \mu$, for every regular large enough $\kappa < \mu$, we have that for all $\alpha < \lambda$,*

$$\mathrm{cov}(\alpha, \kappa^+, \kappa^+, \kappa) < \lambda.$$

Sketch of the proof. The statement of Theorem 1.1 of [Sh 3] is that in the situation as described by the assumptions of Theorem 0.3, there are only boundedly many $\kappa < \mu$ such that for some $\lambda^* \in (\mu, \lambda)$ we have $\mathrm{pp}_{\Gamma(\mu^+,\kappa)}(\lambda^*) \geq \lambda$. The notation to the extent needed here will be described below.

Suppose $\alpha < \lambda$. As clearly $\mathrm{cov}(\alpha, \kappa^+, \kappa^+, \kappa) = \mathrm{cov}(|\alpha|, \kappa^+, \kappa^+, \kappa)$ for any κ, we can assume that α is a cardinal θ. Let $\kappa < \mu$ be large enough uncountable such that for no $\lambda^* \in (\mu, \lambda)$ do we have $\mathrm{pp}_{\Gamma(\mu^+,\kappa)}(\lambda^*) \geq \lambda$. The notation used here is that for a cardinal σ

$$\begin{aligned}\Gamma(\sigma, \kappa) \stackrel{\text{def}}{=} \{I : \text{for some cardinal } \theta_I < \sigma\\ I \text{ is a proper } \kappa\text{-complete ideal on } \theta_I\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{pp}_{\Gamma(\mu^+,\kappa)}(\lambda^*) \stackrel{\text{def}}{=} \sup\{\mathrm{tcf}(\Pi\mathfrak{a}/J) : \mathfrak{a} \text{ is a set of regular cardinals unbounded in } \lambda^*,\\ J \in \Gamma(\mu^+, \kappa) \text{ and } \mathrm{tcf}(\Pi\mathfrak{a}/J) \text{ is well defined}\}.\end{aligned}$$

For our purposes here it is not important what the notation $\mathrm{tcf}(\Pi\mathfrak{a}/J)$ means exactly, one should simply observe that $\Gamma(\kappa^+,\kappa) \subseteq \Gamma(\mu^+,\kappa)$ and hence $\mathrm{pp}_{\Gamma(\mu^+,\kappa)}(\lambda^*) \geq \mathrm{pp}_{\Gamma(\kappa^+,\kappa)}(\lambda^*)$ for all relevant λ^*. This implies that for no $\lambda^* \in (\mu, \lambda)$ do we have $\mathrm{pp}_{\Gamma(\kappa^+,\kappa)}(\lambda^*) \geq \lambda$.

Now we quote Shelah's 'cov versus pp' theorem, [Sh 1], II 5.4., which says that

$$\mathrm{cov}(\theta, \kappa^+, \kappa^+, \kappa) + \theta = \sup\{\mathrm{pp}_{\Gamma(\kappa^+,\kappa)}(\lambda^*) : \lambda^* \in [\kappa, \theta]\} + \theta,$$

leading us to conclude that $\mathrm{cov}(\theta, \kappa^+, \kappa^+, \kappa) < \lambda$. $\bigstar_{0.3}$

We shall also use another staple of cardinal arithmetic, namely the club guessing principle quoted in the following.

Theorem 0.4. *(Shelah, [Sh 1], III, §2) Suppose that $\aleph_0 < \mathrm{cf}(\kappa) = \kappa$ and $\kappa^+ < \lambda = \mathrm{cf}(\lambda)$. Then there is a sequence $\bar{e} = \langle e_\delta : \delta \in S^\lambda_\kappa\rangle$ of sets such that for each δ we have $\mathrm{otp}(e_\delta) = \kappa$ and e_δ is a club subset of δ consisting of points of cofinality $< \kappa$, and for every club E of λ there are stationarily many δ such that $e_\delta \subseteq E$.*

If $\kappa = \aleph_0$, then there is a sequence $\bar{e}$ of the above form such that each e_δ is a cofinal subset of δ of order type ω, and for every club E of λ there are stationarily many δ such that $e_\delta \subseteq E$.

1 The results

To simplify the notation, which involves dealing with many cardinals at a time, we first formulate and prove the theorem in lesser generality where $\beth_\omega$ is the strong limit singular we work with. The same proof gives the fully general result, as indicated in Theorem 1.2.

Theorem 1.1. *Suppose that λ is a regular cardinal with $\lambda > \beth_\omega$.*

Then there are coboundedly many regular $\kappa < \beth_\omega$ such that the following statements hold:

(1) If $\lambda^{<\kappa} = \lambda$, then $\clubsuit^-(S^\lambda_\kappa)$ holds. Precisely, there is a sequence $\langle \mathscr{P}_\delta : \delta \in S^\lambda_\kappa\rangle$ such that

(i) each $\mathscr{P}_\delta$ is a family of $< \lambda$ elements of $[\delta]^{\leq\kappa}$ and

(ii) for every $A \in [\lambda]^\lambda$, there are stationary many δ such that for some X in $\mathscr{P}_\delta$ we have $X \subseteq A$ and $\sup(X) = \kappa$.

(2) There is a sequence $\langle \mathscr{P}^0_\delta : \delta \in S^\lambda_\kappa\rangle$ satisfying (1)(i) above and such that for all $A \in [\lambda]^\lambda$ there is a club E of λ such that for every $\delta \in E \cap S^\lambda_\kappa$, for some $a_\delta \in \mathscr{P}^0_\delta$ we have $\sup(A \cap a_\delta) = \delta$.

(3) If $\theta < \lambda \Rightarrow \theta^{<\kappa} < \lambda$, then there is a sequence $\langle \mathscr{R}_\delta : \delta \in S^\lambda_\kappa\rangle$ satisfying (1)(i) above and

(ii)$^+$ for every sequence $\langle a_\delta : \delta \in S^\lambda_\kappa\rangle$ of sets such that each a_δ is a subset of δ of order type $\leq \kappa$, there is a club C of λ such that $\delta \in C \cap S^\lambda_\kappa \Rightarrow a_\delta \in \mathscr{R}_\delta$.

Proof. For $\alpha < \lambda$, let

$$R_\alpha \overset{\text{def}}{=} \{\kappa \text{ regular} < \beth_\omega : \text{cov}(\alpha, \kappa^+, \kappa^+, \kappa) < \lambda\}.$$

By Theorem 0.3, for each such α there is $n_\alpha < \omega$ such that R_α contains all regular cardinals in the interval $[\beth_{n_\alpha}, \beth_\omega)$. Hence there is $n^* < \omega$ such that for unboundedly many $\alpha < \lambda$ we have that $n_\alpha = n^*$. As it is easily seen that

$$\alpha < \beta \Rightarrow \text{cov}(\alpha, \kappa^+, \kappa^+, \kappa) \leq \text{cov}(\beta, \kappa^+, \kappa^+, \kappa),$$

it follows that for all $\alpha < \lambda$, the set R_α contains all regular cardinals in $[\beth_{n^*}, \beth_\omega)$. Let us fix a regular cardinal $\kappa > \aleph_0$ in the interval $[\beth_{n^*}, \beth_\omega)$ and show that all three statements of the Theorem hold for such κ.

For each $\alpha < \lambda$ let $\mathscr{P}^0_\alpha$ be a family exemplifying that $\text{cov}(\alpha, \kappa^+, \kappa^+, \kappa) < \lambda$. The sequence needed for (2) is in fact $\langle \mathscr{P}^0_\delta : \delta \in S^\lambda_\kappa \rangle$, a point to which we shall briefly return later, but for the moment we go on to the main part of the proof, which is the proof of (1).

Proof of (1). As we are assuming $\lambda^{<\kappa} = \lambda$, let us enumerate $[\lambda]^{<\kappa} = \{A_i^* : i < \lambda\}$ so that each set in the enumeration appears λ many times. For $\delta \in S^\lambda_\kappa$ let

$$\mathscr{P}^1_\delta = \{(\bigcup_{i \in B} A_i^*) \cap \delta : B \in \mathscr{P}^0_\delta\},$$

hence each $X \in \mathscr{P}^1_\delta$ is a subset of δ of size $\leq \kappa$ and $|\mathscr{P}^1_\delta| < \lambda$. Fixing $\delta \in S^\lambda_\kappa$ for a moment, we have that for each $X \in \mathscr{P}^1_\delta$ the size of X is $\leq \kappa$, so the síze of $\mathscr{P}(X)$ is $\leq 2^\kappa < \beth_\omega < \lambda$, leading us to conclude that

$$\mathscr{P}_\delta \overset{\text{def}}{=} \{Y : (\exists X \in \mathscr{P}^1_\alpha)\ Y \subseteq X\}$$

also has size $< \lambda$. We shall proceed to show that $\langle \mathscr{P}_\delta : \delta \in S^\lambda_\kappa \rangle$ is a sequence as required. Part (i) of our requirement is clearly satisfied, so let us proceed to part (ii). For this we shall first need to fix a club guessing sequence $\langle e_\delta : \delta \in S^\lambda_\kappa \rangle$ as provided by Theorem 0.4. For each $\delta \in S^\lambda_\kappa$, let $e_\delta = \{\zeta^\delta_\gamma : \gamma < \kappa\}$ be the increasing enumeration of e_δ.

Let $A \in [\lambda]^\lambda$ be given. For $\varepsilon \in S^\lambda_{<\kappa}$ define $X_\varepsilon = X^A_\varepsilon$ to be a subset of A of size $< \kappa$ with $\sup(X_\varepsilon) = \varepsilon$, if such a set exists. Now define a function $h_A : S^\lambda_{<\kappa} \to \lambda$ by the following recursive definition

$$h_A(\varepsilon) \overset{\text{def}}{=} \begin{cases} \min\{i > \sup_{\beta \in S^\varepsilon_{<\kappa}} h_A(\beta) : X_\varepsilon = A_i^* & \text{if } X_\varepsilon \text{ is defined,} \\ \sup_{\beta \in S^\varepsilon_{<\kappa}} h_A(\beta) + 1 & \text{otherwise} \end{cases}$$

Let $E \overset{\text{def}}{=} \text{cl}(\{\delta < \lambda : (\forall \varepsilon \in S^\delta_{<\kappa})\ h_A(\varepsilon) \leq \delta\})$, hence a club of λ. Note that if $\delta \in E \cap S^\lambda_\kappa$, then for all $\varepsilon \in S^\delta_{<\kappa}$ we actually have $h_A(\varepsilon) < \delta$. Let us choose $\delta \in E \cap S^\lambda_\kappa$ such that $e_\delta \subseteq \lim(A)$. This in particular means that for every $\gamma < \kappa$ the set $X_{\zeta^\delta_\gamma}$ has been defined. For such γ, let $i_\gamma \overset{\text{def}}{=} h_A(\zeta^\delta_\gamma)$, hence $\langle i_\gamma : \gamma < \kappa \rangle$ is a strictly increasing sequence and for each γ we have $A^*_{i_\gamma} = X_{\zeta^\delta_\gamma}$. As $\{i_\gamma : \gamma < \kappa\} \in [\delta]^\kappa$, there are sets

$\{B_j : j < j^* < \kappa\}$ in $\mathscr{P}^0_\delta$ such that $\{i_\gamma : \gamma < \kappa\} \subseteq \bigcup_{j<j^*} B_j$. By the regularity of κ, there is $j < j^*$ such that $|\{i_\gamma : \gamma < \kappa\} \cap B_j| < \kappa$. Let $B = B_j$ for some such j.

Consider $(\bigcup_{i \in B} A_i^*) \cap \delta$. Clearly, this set is a superset of $\bigcup_{i_\gamma \in B} X_{\zeta_\gamma^\delta}$ (so it has size κ) and is a member of $\mathscr{P}^1_\delta$. For this reason, $\bigcup_{i_\gamma \in B} X_{\zeta_\gamma^\delta} \in \mathscr{P}_\delta$, and this set is clearly an unbounded subset of $A \cap \delta$ of size κ.

Proof of (2). This follows trivially with $\langle \mathscr{P}^0_\delta : \delta \in S^\lambda_\kappa \rangle$ as above, since by taking $\delta \in \lim(A) \cap S^\lambda_\kappa$, we obtain that $A \cap \delta$ is unbounded in δ and covered by $< \kappa$ many elements of $\mathscr{P}^0_\delta$. Hence, by the regularity of κ we obtain that there is an element X of $\mathscr{P}^0_\delta$ with $\sup(A \cap X) = \delta$.

Proof of (3). For each relevant δ, we form the family $\mathscr{P}^0_\delta$ as in the proof of (1). Fixing $\delta \in S^\lambda_\kappa$ for a moment and letting $\theta = |\mathscr{P}^0_\delta|$, se have $\theta^{<\kappa} < \lambda$, so we can let $\mathscr{P}^2_\delta$ consist of the unions of all subfamilies of $\mathscr{P}^1_\delta$ which have size $< \kappa$ and obtain a family of elements of elements of $[\delta]^{\leq \kappa}$ of size $< \lambda$. The proof now follows the proof of (1), but we give the details for the sake of completeness.

As $\theta < \lambda \Rightarrow \theta^{<\kappa} < \lambda$ and λ is regular, we have $\lambda^{<\kappa} = \lambda$. We enumerate $[\lambda]^{<\kappa} = \{A_i^* : i < \lambda\}$. Let $\mathscr{P}^3_\delta \stackrel{\text{def}}{=} \{(\bigcup_{i \in B} A_i^*) \cap \delta : B \in \mathscr{P}^2_\delta\}$, and let $\mathscr{R}_\delta \stackrel{\text{def}}{=} \{Y : (\exists X \in \mathscr{P}^3_\delta)\, X \subseteq Y\}$, for each relevant δ. Let

$$E \stackrel{\text{def}}{=} \mathrm{cl}(\{\beta < \lambda : (\forall X \in [\beta]^{<\kappa})\, (\exists i < \beta)\, X = A_i^*\}).$$

Note that if $\delta \in E \cap S^\lambda_\kappa$ then for all $X \in [\delta]^{<\kappa}$ we have $X = A_i^*$ for some $i < \delta$. We claim that for each such δ the set a_δ is in $\mathscr{P}_\delta$. Let $f_\delta : \kappa \to a_\delta$ be the increasing enumeration of a_δ and for $\gamma < \kappa$ let $X_\gamma = \mathrm{ran}(f_\delta \restriction \gamma)$. For each such γ let $i_\gamma < \delta$ be such that $X_\gamma = A_{i_\gamma}^*$. Hence there are sets $\{B_j^* : j < j^* < \kappa\}$ in $\mathscr{P}^0_\delta$ such that $\{i_\gamma : \gamma < \kappa\} \subseteq \bigcup_{j<j^*} B_j \stackrel{\text{def}}{=} B$. We have that $B \in \mathscr{P}^2_\delta$, hence $(\bigcup_{i \in B} A_i^*) \cap \delta \in \mathscr{P}^3_\delta$ and is a superset of a_δ, so $a_\delta \in \mathscr{R}_\delta$. $\bigstar_{1.1}$

A more general theorem is

Theorem 1.2. *The analogue of Theorem 1.1 holds when $\beth_\omega$ is replaced by any other strong limit singular cardinal μ.*

Proof. Exactly the same as that of Theorem 1.1, using the full generality of Theorem 0.3 and replacing $\beth_\omega$ by μ throughout. $\bigstar_{1.2}$

2 Concluding Remarks

The main result we proved is that when μ is a strong limit singular cardinal and λ is a regular cardinal above μ, there are coboundedly many regular $\kappa < \mu$ such that

$$\lambda = \lambda^{<\kappa} \Rightarrow \clubsuit^-(S^\lambda_\kappa),$$

hence the existence of the guessing sequence follows simply from the cardinal arithmetic assumed. When combined with the result of Shelah in [Sh 3] which under these conditions shows the equivalence of the assumption $\lambda = \lambda^{<\lambda}$ with $\diamondsuit^{-}$, an immediate consequence is that $\clubsuit^{-}(\lambda)$ and $\diamondsuit^{-}(\lambda)$ are different, a fact whose analogue at ω_1 requires a rather serious proof (Shelah, see [Sh 2] e.g.). It would be interesting to know if when we in addition assume that λ as above is a successor cardinal, then $\lambda = \lambda^{<\kappa} \Rightarrow \clubsuit$. The analogue of this for $\diamondsuit$ follows from the above mentioned result of [Sh 3] and Kunen's argument on the equivalence bektween $\diamondsuit$ and $\diamondsuit^{-}$ at successor cardinals (see [Ku] e.g.). We have the impression that the answer to the question is negative, since it is known by [DžSh] that $\clubsuit^{-}$ and $\clubsuit$ differ at $\aleph_1$.

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