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On Almost Invariant Subsets of the Real Line

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In this paper we show a new construction of almost invariant subsets of the real line. We first define almost invariant sets ın some linear space and then we transport them by some special linear (over the field of rational numbers) isomorphism between this space and the real line treated as a linear space over rationals. We show a construction of a nonmeasurable almost invariant subsets of the real line and then we discuss the existence of Lebesque measurable almost invariant sets.

1. Introduction

In 1932 W. Sierpiński proved two basic results about the existence of almost invariant subsets of the real line. Namely (see [**5**]) he showed that

- there are nontrivial almost invariant subsets of the real line,
- Continuum Hypothesis implies that there are nontrivial almost invariant, Lebesgue measure subsets of the real line.

We improve the above results and show that some assumptions, like Continuum Hypothesis, are necessary in the second result.

We shall use standard set theoretical notations. We denote by $|X|$ the cardinality of set X. If κ is a cardinal numbe then by $cf(\kappa)$ we denote its cofinality. We identify the first infinite cardinal number ω with the set of natural numbers. By $\mathbf{c}$ we denote the cardinality of continuum, ı.e. $\mathbf{c} = 2^{\omega}$. For any set X and a cardinal number κ by $[X]^{<\kappa}$ we denote the family of all subsets of the set X of cardinality

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less that κ. Similarly, $[X]^{\kappa}$ denotes the family of all subsets of X of cardinality κ. A^c denotes the complement of the set A.

By $\mathbb{R}$ we denote the real line and by $\mathbb{Q}$ the set of rational numbers. The σ-ideal of Lebesgue measurable subsets of real line is denoted by $\mathbb{L}$ and the σ-ideal of first category subsets of real line is denoted by $\mathbb{K}$. We recal that $cov(\mathbb{K}) = \min\{|S| : S \subseteq \mathbb{K} \wedge \bigcup S = \mathbb{R}\}$. Obviously $cov(\mathbb{K})$ is an uncountable cardinal number and $cov(\mathbb{K}) \leq \mathbf{c}$. The Continuum Hypothesis and the Martin's Axiom imply that $cov(\mathbb{K}) = \mathbf{c}$. The theory $ZFC \cup \{cov(\mathbb{K}) < \mathbf{c}\}$ is relatively consistent, too (see e.g. [**1**]). At the end of this article we use forcing terminology from [**3**].

The notion of almost invariant sets were discussed in many papers, see e.g. [**5**], [**6**], [**2**]. Let us recall that if $\mathscr{G} = \langle G, + \rangle$ is an infinite group then the set $A \subseteq G$ is an *almost invariant* subset of $\mathscr{G}$ if

$$(\forall g \in G)\left(|A \triangle (A + g)| < |G|\right),$$

where $\triangle$ denotes the symmetric difference of sets. Notice that if $\mathscr{G} = \langle G, + \rangle$ is an infinite group, $A \subseteq G$ and $|A| < |G|$ or $|G \setminus A| < |G|$, then A is an almost invariant subset of $\mathscr{G}$. We call such sets *trivial almost invariant sets*. Other almost invariant sets are here called *nontrivial almost invariant sets*. In other words, an almost invariant set A is nontrivial if $|A| = |G \setminus A| = |G|$. The family of all almost invariant subsets of an infinite group $\langle G, + \rangle$ is a $cf(|G|)$-additive proper subfield of the field of all subsets of the basic set G (see [**2**]).

2. Basic construction

Let $\mathbb{Q}_{\mathbf{c}} = \{x \in \mathbb{Q}^{\mathbf{c}} : |\mathrm{supp}(x)| < \omega\}$, there $\mathrm{supp}(x) = \{\alpha < \mathbf{c} : x(\alpha) \neq 0\}$. We treat this set as a linear space over the field of rational numbers with operations defined as follows: $(x + y)(\alpha) = x(\alpha) + y(\alpha)$, $(q \cdot x)(\alpha) = q \cdot x(\alpha)$, where $q \in \mathbb{Q}$. The space $\mathbb{Q}_{\mathbf{c}}$ is isomorphic with the direct union of $\mathbf{c}$ many copies of the field $\mathbb{Q}$. By $\bar{0}$ we denote the neutral element of this space. For each $\alpha < \mathbf{c}$ we put $e_\alpha = ((\mathbf{c} \setminus \{\alpha\}) \times \{0\}) \cup \{(\alpha, 1)\}$. Then $\{e_\alpha : \alpha < \mathbf{c}\}$ is a linear base of the space $\mathbb{Q}_{\mathbf{c}}$.

By $\mathbb{CQ}$ we denote the set $\bigcup\{\mathbb{Q}^T : T \in [\mathbf{c}]^{\mathbf{c}}\}$ and for each $f \in \mathbb{CQ}$ we put

$$\langle f \rangle = \{x \in \mathbb{Q}_{\mathbf{c}} \setminus \{\bar{0}\} : ms(x) \in dom(f) \wedge x(ms(x)) = f(ms(x))\}$$

where $ms(x) = \sup(\mathrm{supp}(x))$. Notice that if $f \in \mathbb{Q}^{\mathbf{c}}$ then the definition of the set $\langle f \rangle$ is slightly simpler, namely we have $\langle f \rangle = \{x \in \mathbb{Q}_{\mathbf{c}} \setminus \{\bar{0}\} : x(ms(x)) = f(ms(x))\}$.

Theorem 1. *For each $f \in \mathbb{CQ}$ the set $\langle f \rangle$ is a nontrivial almost invariant subset of $(\mathbb{Q}_{\mathbf{c}}, +)$.*

Proof. Let $t \in \mathbb{Q}_{\mathbf{c}} \setminus \{\bar{0}\}$ and $\alpha = ms(t)$. Notice that if $x \in \mathbb{Q}_{\mathbf{c}} \setminus \{\bar{0}\}$ and $\alpha < ms(x)$ then $ms(x + t) = ms(x)$ and $x(ms(x)) = (x + t)(ms(x))$. Therefore

$$(\langle f \rangle + t) \triangle \langle f \rangle \subseteq \{x \in \mathbb{Q}_{\mathbf{c}} : \mathrm{supp}(x) \leq \alpha\},$$

so $|(\langle f \rangle + t) \triangle \langle f \rangle| \leq |[\alpha]^{<\omega}| \leq |\alpha| \cdot \omega < \mathbf{c}$.

Obviously $|\langle f\rangle| = \mathbf{c}$. Moreover $\langle f\rangle \cap \langle g\rangle = \emptyset$, where $g = \{(\alpha, f(\alpha)+1) : \alpha \in dom(f)\}$, so $\langle f\rangle$ is a nontrivial almost invariant set. $\square$

The next corollary is a generalization of Sierpiński's first result mentioned above about the existence of almost invariant sets.

Corollary 1. *There exists a partition $\{B_q\}_{q\in\mathbb{Q}\setminus\{0\}}$ of $\mathbb{R}\setminus\{0\}$ into Lebesgue non-measurable, nontrivial almost invariant sets such that $(\forall q \in \mathbb{Q}\setminus\{0\})(B_q = q\cdot B_1)$.*

Proof. For each $q\in\mathbb{Q}\setminus\{0\}$ we put $A_q = \langle \bar{q}\rangle$, where $\bar{q} = \{(\alpha, q) : \alpha < \mathbf{c}\}$. Notice that $A_q = q\cdot A_1$ and that the family $\{A_q : q\in\mathbb{Q}\setminus\{0\}\}$ is a partition of the space $\mathbb{Q}_{\mathbf{c}}\setminus\{\bar{0}\}$. Theorem 1 implies that A_q is a nontrivial almost invariant set for each $q\in\mathbb{Q}\setminus\{0\}$.

Let $\varphi : \mathbb{Q}_{\mathbf{c}} \to \mathbb{R}$ be a linear isomorphism over $\mathbb{Q}$ and let $B_q = \varphi(A_q)$. Then the family $\mathscr{F} = \{B_q : q\in\mathbb{Q}\setminus\{0\}\}$ is a partition of $\mathbb{R}\setminus\{0\}$ into nontrivial almost invariant sets and $(\forall q\in\mathbb{Q}\setminus\{0\})(B_q = q\cdot B_1)$.

If one set from the family $\mathscr{F}$ has the Lebesgue measure zero then each set from this family has the Lebesgue measure zero, too, but $\bigcup\{B_q : q\in\mathbb{Q}\setminus\{0\}\} = \mathbb{R}\setminus\{0\}$, which is impossible. Suppose hence that the Lebesgue measure of one set from this family, say B_q is positive. But $B_q + B_q \subseteq B_q \cup B_{2q} \subseteq \mathbb{R}\setminus\{0\}$ and Steinhaus' theorem (see **[4]** implies that there exists a non-empty interval I such that $I \subseteq B_q + B_q$. Let $p\in\mathbb{Q}\cap(1,2)$ be such that $I\cap p\cdot I \neq \emptyset$. But then $\emptyset \neq I\cap p\cdot I \subseteq (B_q\cup B_{2q})\cap(B_{pq}\cup B_{2pq}) = \emptyset$. This contradiction shows that no set from the family $\mathscr{F}$ is Lebesgue measurable. $\square$

Sierpiński in **[6]** showed that there exists an almost invariant set $A\subseteq\mathbb{R}$ such that A and A^c contains a nonempty perfect set. The next corollary is a generalization of this result.

Corollary 2. *There exists a family $\{N_\alpha : \alpha < \mathbf{c}\}$ of pairwise disjoint almost invariant subsets of $\mathbb{R}$ such that N_α contains a nonempty porfect set for each $\alpha < \mathbf{c}$.*

Proof. Let $P\subseteq\mathbb{R}$ be a nonempty perfect set of algebraically independent elements. Let $\{P_\alpha : \alpha < \mathbf{c}\}$ be a family of nonempty pairwise disjoint perfect subsets of P. Let $\{T_\alpha : \alpha < \mathbf{c}\} \subseteq [\mathbf{c}]^{\mathbf{c}}$ be a family of pairwise disjoint sets such that $|\mathbf{c}\setminus\bigcup\{T_\alpha : \alpha < \mathbf{c}\}| = \mathbf{c}$. Let $\varphi : \mathbb{Q}_{\mathbf{c}} \to \mathbb{R}$ be a linear isomorphism over $\mathbb{Q}$ such that $\varphi(\{e_\xi : \xi\in T_\alpha\}) = P_\alpha$ for each $\alpha < \mathbf{c}$. Then $\{\varphi(\langle T_\alpha \times \{1\}\rangle) : \alpha < \mathbf{c}\}$ is a required family of almost invariant sets. $\square$

3. Almost invariant Lebesgue measure zero sets

In this section we discuss the existence of Lebesgue measurable, almost invariant subsets of the real line. It is easy to check that if L is such a set then $L\in\mathbb{L}$ or $\mathbb{R}\setminus L\in\mathbb{L}$.

Let $Odd = \{\alpha < \mathbf{c} : (\exists\zeta)(\exists n\in\omega)(\zeta \text{ is a limit ordinal} \wedge \alpha = \zeta + 2\cdot n + 1)\}$.

Theorem 2. *Suppose that* $cov(\mathbb{K}) = \mathbf{c}$. *Let* $\mathbb{R} = K \cup L$ *where* $K \in \mathbb{K}$ *and* $L \in \mathbb{L}$. *Then there exists a linear isomorphism* $\varphi : \mathbb{Q}_{\mathbf{c}} \to \mathbb{R}$ *over the field of rational numbers such that* $\varphi(\{x \in \mathbb{Q}_{\mathbf{c}} \setminus \{\bar{0}\} : ms(x) \in Odd\}) \subseteq L$.

Proof. Let $\preceq$ be a fixed well-ordering of $\mathbb{R}$ of the order type $\mathbf{c}$. We shall build by a transfinite recursion of length $\mathbf{c}$ some Hamel base $\{h_\xi\}_{\xi<\mathbf{c}}$. Suppose that $\alpha < \mathbf{c}$ and that the sequence $\{h_\xi\}_{\xi<\alpha}$ is defined. Let H_α be the linear span of the set $\{e_\xi : \xi < \alpha\}$ and let $\varphi_\alpha : H_\alpha \to \mathbb{R}$ be the unique linear function such that $\varphi_\alpha(e_\xi) = h_\xi$ for each $\xi < \alpha$. If $\alpha \in \mathbf{c} \setminus Odd$ then let h_α be the $\preceq$ – first element of the set $\mathbb{R} \setminus \varphi_\alpha(H_\alpha)$ (note that $H_0 = \{0\}$). If $\alpha \in Odd$ then let h_α be any element of the set

$$\bigcap \{q \cdot (L - \varphi_\alpha(y)) : q \in \mathbb{Q} \setminus \{0\} \wedge y \in H_\alpha\} \setminus \varphi_\alpha(H_\alpha).$$

The assumption $cov(\mathbb{K}) = \mathbf{c}$ implies that the above set is non-empty. Finally, let φ be the unique linear over $\mathbb{Q}$ isomorphism between $\mathbb{Q}_{\mathbf{c}}$ and $\mathbb{R}$ such that $(\forall \alpha < \mathbf{c})\,(\varphi(e_\alpha) = h_\alpha)$. This is the required mapping. □

The next result is a generalization of the second, mentioned above, Sierpiński's result about the existence of almost invariant sets.

Corollary 3. *Suppose that* $cov(\mathbb{K}) = \mathbf{c}$. *Then there exists a Lebesgue measure zero, nontrivial almost invariant subset of the real line.*

Proof. Let $\varphi : \mathbb{Q}_{\mathbf{c}} \to \mathbb{R}$ be a linear isomorphism over $\mathbb{Q}$ such that $\varphi(\{x \in \mathbb{Q}_{\mathbf{c}} \setminus \{\bar{0}\} : ms(x) \in Odd\}) \in \mathbb{L}$. Let $f = \{(\alpha, 1) : \alpha \in Odd\}$. Then $\langle f \rangle \subseteq \{x \in \mathbb{Q}_{\mathbf{c}} \setminus \{\bar{0}\} : ms(x) \in Odd\}$ and, by Theorem 1, $\langle f \rangle$ is a nontrivial almost invariant subset of the space $\mathbb{Q}_{\mathbf{c}}$. Therefore $\varphi(\langle f \rangle)$ is a required subset of $\mathbb{R}$. □

We show now that the existence of Lebesgue measure zero, nontrivial almost invariant subset of the real line cannot be proved in the theory ZFC without any additional assumption.

Theorem 3. *The theory* $ZFC \cup \{\neg(\exists A \subseteq \mathbb{R})\,(|A| = 2^\omega \wedge A \in \mathbb{L} \wedge A$ *is almost invariant*$)\}$ *is relatively consistent.*

Proof. Let M be a countable standard model of the theory $ZFC \cup \{2^\omega = \aleph_2\}$. In the model M for each infinite subset T of ω_1 we define the family B_T of Borel subsets of the space $\{0,1\}^T$, the σ-ideal L_T of measure zero subsets of the space $\{0,1\}^T$ with respect to the standard product measure and the boolean algebra $R_T = B_T/L_T$. Note that R_ω is the standard measure algebra and that R_{ω_1} is the standard measure algebra which adds simultaneously ω_1 random reals. Note that if $T \subseteq \omega_1$ then we may consider R_T as a complete subalgebra of the algebra R_{ω_1}. Let G be a R_{ω_1}-generic set over model M and let $N = M^{R_{\omega_1}}[G]$. Then $N \models 2^\omega = \aleph_2$.

Suppose that $A \in N$ and $N \models (|A| = \aleph_2 \wedge A \notin \mathbb{L})$. Let $a \in \omega^\omega \cap N$ be a Borel code of a measure zero set such that $N \models (A \subseteq \#a)$, where $\#a$ denotes the Borel set coded by a. It follows from the countable chain condition of the boolean algebra R_{ω_1} that there exists a set $S \in [\omega_1]^\omega$ such that $a \in M[G \cap R_S]$. Let us fix an increasing family $\{T_\alpha\}_{\alpha<\omega_1}$ of countable sets such that $\bigcup T_0 = S$ and $\bigcup\{T_\alpha : \alpha < \omega_1\} = \omega_1$. The sequence $\langle \mathbb{R} \cap M[G \cap B_T] : \alpha < \omega_1 \rangle$ is definable in the model N. Notice that $N \models (A = \bigcup\{A \cap M[G \cap B_{T_\alpha}] : \alpha < \omega_1\})$, so there exists $\zeta < \omega_1$ such that $N \models (|A \cap M[G \cap B_{T_\zeta}]| = \omega_2)$. Let $B = A \cap M[G \cap B_{T_\zeta}]$ and let $U \in [\omega_1]^\omega$ be such that $U \cap T_\zeta = \emptyset$. Let r be the canonical random real generated by $G \cap R_U$. Then r is a random real over the model $M[G \cap B_{T_\zeta}]$. Notice that $B \subseteq \mathbb{R}^{M[G \cap B_{T_\zeta}]}$. Therefore for each $x \in B$ the real $r + x$ is a random real over $M[G \cap B_{T_\zeta}]$. Notice that $a \in M[G \cap B_{T_\zeta}]$. Hence $N \models ((r + B) \cap \#a = \emptyset$. But $A \subseteq \#a$ and $B \subseteq A$. This shows that $|A \triangle (A + r)| \geq |B| = 2^\omega$, so A is not an almost invariant set. □

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