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Steinhaus Chessboard Theorem

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The following statement is due to Hugo Steinhaus [3]; Consider a chessboard with some "mined" squares on. Assume that the king cannot go across the chessboard from the left edge to the right one without meeting a mined square. Then the rook can go across the chessboard from upper edge to the lower one moving exclusively on mined squares.

According to Surowka several proofs of the Steinhaus Theorem seem to be incomplete or use induction on the size of the chessboard [4]. In this note we shall generalize the Steinhaus Theorem assuming that the chessboard (= square) is divided into arbitrary polygons (not necessarily squares) and we shall show an algorithm allowing to find rook's or king's route between chosen opposite edges of the chessboard.

Consider the plane R^2 with Cartesian coordinates and right hand (counterclock) orientation. Let $S := [0, 1]$ be a square and $\mathscr{T}$ a tiling of S into polygons i.e., a covering of S with interior pairwise-disjoint polygons and with a coloring $f : \mathscr{T} \to \{w, b\}$ into two colors; white and black (see Figure 1). The polygons such that $f(P) = w$, $f(P) = b$ will be called white and black tiles respectively.

A segment of an edge which is contained in the intersection of a white and a black tile with the ends between two neighbouring vertices of the tiling $\mathscr{T}$, is said to be a white-black segment. If we establish the beginning and the end then the segment is said to be oriented.

A sequence $P_0, \ldots, P_n$ of white [black] tiles is said to be rook's white [king's black] route if for each $i < n$ the intersection $P_i \cap P_{i+1}$ is a segment [a nonempty-set].

In Gale's paper [1] a hexagonal tiling is considered. In this case there are no differences between rook's and king's route.

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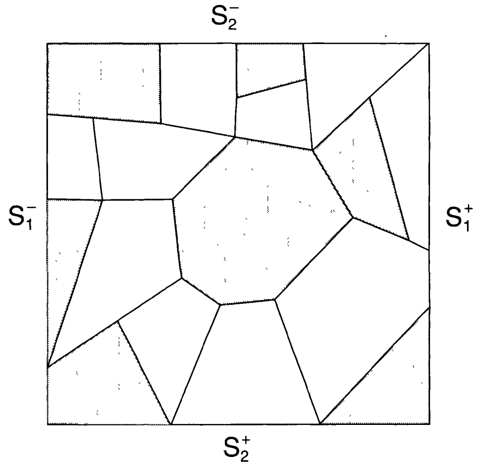

Fig. 1

The Chess Rook-King Moving Theorem. *If T is a tiling of the square S onto white and black tiles then there is a rook's white route connecting two opposite sides of S; the left side $S_1^- := \{0\}\times[0,1]$ and the right side $S_1^+ := \{1\}\times[0,1]$ or there is a king's black route connecting the opposite sides; the upper side $S_2^- := [0,1]\times\{1\}$ and the lower side $S_2^+ := [0,1]\times\{0\}$.*

Proof. We shall give some algorithm which shows how to find a proper route connecting two opposite sides of S.

Fix $a > 0$ and let $K := [-a, 1+a]\times[-a, 1+a]$ be a square which contains S. Extend tiling of S onto tiling of K by adding to $\mathscr{T}$ two white rectangles; $[-a,0]\times[0,1+a]$, $[-a,1]\times[-a,0]$ and $[0,1+a]\times[1,1+a]$, $[1,1+a]\times[-a,1]$ as black rectangles (see Figure 2).

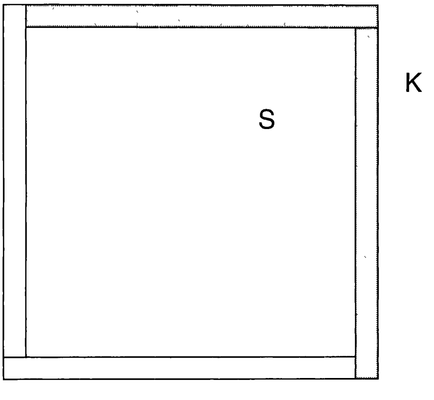

Fig. 2

We shall describe a path L in K from the point $X = (0, 1 + a)$ to $Y = (1, -a)$ consisting of oriented white-black segments of the tiling K such that:

(α) walking along the path from X to Y we have adjacent white tiles on the right hand and the blacks on the left,

(β) if $w \in I$ is the end of a path oriented segment I then accordance with counterclockwise orientation we choose, starting from I, the first white-black segment with the vertex w as the beginning (see Figures 3 and 4).

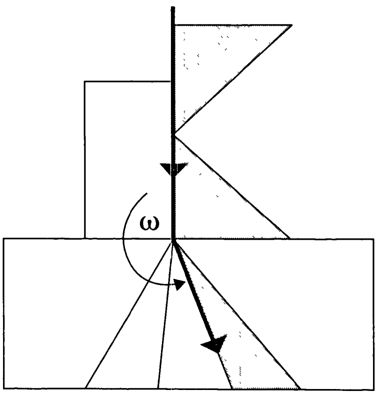

Fig. 3

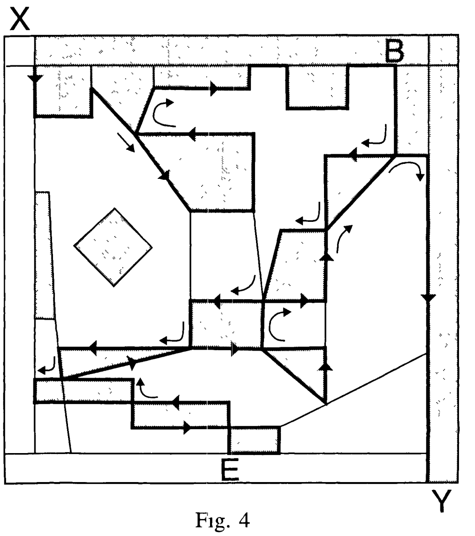

Fig. 4

The conditions (α) and (β) uniquely describe a path L starting from the point X. From our description of the path L it follows that L consists of distinct white-black segments and that the terminal point of this path cannot be an interior point of K. Since X cannot be the terminal point, the only point different from X which is not an interior point of K and which is the end point of an oriented white-black segment, is the point Y.

Let us notice that from definition of the path L it follows that on the right side of L there is in K a rook's white route (the route is obtained by attaching the all tiles lying on the right side of L and containing in L a segment of their edge), and on the left – a king's black route (this route is obtained by attaching the all black tiles lying on the left side of L and meeting L) (see Figure 4).

Starting from X let $E \in L$ (= end) be the first point in the path L which meets the sides $S_1^+ = \{1\} \times [0, 1]$ or $S_2^+ = [0, 1] \times \{0\}$. Now, define a point $B \in L$ (= beginning) such that

(γ) if $E \in S_i^+$, $i = 1, 2$, then B is the last point in the path XE which meets S_i^- (see Figure 4).

Observe that:

if $i = 1$, then the rook's white route along the path BE is contained in S,

if $i = 2$, then the king's black route along the left side of the path BE is contained in S.

This completes the proof.

Acknowledgement

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