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Selectivity of Almost Disjoint Families

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Selective properties of almost disjoint families of subsets of a countable set are studied here. In particular, sufficient conditions for the existence of a $+$-Ramsey MAD family are presented. As an application it is shown that the existence of a $+$-Ramsey MAD family implies that two similar versions of a topological game on Fréchet spaces, due to G. Gruenhage, are not equivalent in terms of existence of winning strategies.

I. Introduction

In the current note we investigate selective properties of MAD (maximal almost disjoint) families of subsets of ω. Recall that an infinite family $\mathscr{A} \subseteq [\omega]^{\omega}$ is *almost disjoint (AD)* if every two distinct elements of $\mathscr{A}$ have only finite intersection. A family $\mathscr{A}$ is *MAD* if it is almost disjoint and maximal with this property. Given an almost disjoint family $\mathscr{A}$, $\mathscr{I}(\mathscr{A})$ denotes the ideal of those subsets of ω which can be almost covered by finitely many elements of $\mathscr{A}$, $\mathscr{I}^*(\mathscr{A})$ denotes the dual filter and $\mathscr{I}^+(\mathscr{A}) = \mathscr{P}(\omega) \setminus \mathscr{I}(\mathscr{A})$ the coideal of large sets. We denote by $\mathscr{I}^{++}(\mathscr{A}) = \{A \subseteq \omega : |\{B \in \mathscr{A} : |B \cap A| = \aleph_0\}| \geq \aleph_0\}$ the family of "really" large sets. Note that for a MAD family $\mathscr{I}^+(\mathscr{A}) = \mathscr{I}^{++}(\mathscr{A})$.

The notion of selectivity (Ramseyness) of filters, ideals and coideals has been stud ed extensively in recent decades. The notation connected with this concept is, however, quite far from being unified. Some authors talk about selective or Ramsey filters, ideals or coideals, some about Happy families, some about ideals having weak

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or strong tree properties. We choose to refer to selective coideals as Happy families as it allows for the following pun: *If we rid* $\mathscr{P}(\omega)$ *of a MAD family and its relatives* ($\mathscr{I}(\mathscr{A})$) *the rest* ($\mathscr{I}^+(\mathscr{A})$) *is Happy.* This fact has been known for quite some time (see [BDS] or [Ma]). We will be studying the following strengthening of the notion of selectivity (see [Gr] or [La]), the name $+$*-Ramsey* is probably due to C. Laflamme.

Definition I.1. *A filter* $\mathscr{F}$ *(an ideal* $\mathscr{I}$*) is* $+$*-Ramsey if for every* $\mathscr{F}^+$*-branching tree (for every* $\mathscr{I}^+$*-branching tree)* $T \subseteq \omega^{<\omega}$ *there is a branch* $b \in [T]$ *such that* $rng(b) \in \mathscr{F}^+$ $(rng(b) \in \mathscr{I}^+)$.

In particular, an almost disjoint family $\mathscr{A}$ *will be called* $+$*-Ramsey if the ideal* $\mathscr{I}(\mathscr{A})$ *is* $+$*-Ramsey.*

Recall that a $T \subseteq \omega^{<\omega}$ is a *tree* if for every $s \in T$ and every $t \subseteq s$, $t \in T$. If $\mathscr{S}$ is a family of subsets of ω, a tree T is $\mathscr{S}$*-branching* if $succ_T(t) = \{n \in \omega : t^\frown n \in T\} \in \mathscr{S}$ for every $t \in T$. Finally, $[T] = \{f \in \omega^\omega : \forall n \in \omega\ f \restriction n \in T\}$.

In the second section we introduce related cardinal invariants of the continuum and show that (at least consistently) $+$-Ramsey MAD families exist. It should be mentioned here that not all MAD families are $+$-Ramsey. In the third section we present an application to the theory of Fréchet spaces. In particular, it will be shown there that two similar versions of a game due to G. Gruenhage (see [G]) are not equivalent in terms of the existence of winning strategies.

II. Combinatorics and cardinal invariants

Define the following cardinal invariant

$\mathfrak{ra} = \min\{|\mathscr{A}| : \mathscr{A}$ is an AD family which is not $+$-Ramsey$\}$

and recall the definitions of the following standard cardinal invariants of the continuum:

$cov(\mathscr{M}) = \min\{|\mathscr{B}| : \mathscr{B}$ is family of closed nowhere dense subsets of ω^ω such that $\omega^\omega = \bigcup \mathscr{B}\}$,

$\mathfrak{d} = \min\{|\mathscr{D}| : \mathscr{D}$ is a dominating subset of $\omega^\omega\}$,

$\mathfrak{t} = \min\{|\mathscr{T}| : \mathscr{T}$ is a maximal decreasing chain (tower) of infinite subsets of $\omega\}$,

$\mathfrak{a} = \min\{|\mathscr{A}| : \mathscr{A}$ is a maximal AD family$\}$,

$\mathfrak{a}_T = \min\{|\mathscr{C}| : \mathscr{C}$ is a maximal AD family of finitely branching subtrees of $\omega^{<\omega}\}$.

It is well-known an not hard to prove that $\mathfrak{t} \leq cov(\mathscr{M}) \leq \mathfrak{d} \leq \mathfrak{a}_T$. To see this note that $\mathfrak{a}_T$ is the minimal cardinality of a partition of the irrationals ω^ω into compact sets and $\mathfrak{d}$ is the minimal size of a family of compact sets covering ω^ω.

Proposition II.1. $cov(\mathscr{M})$ *is equal to the minimal character of a filter on* ω *which is not* $+$*-Ramsey.*

Proof. Let $\mathscr{F}$ be a filter on ω and $\mathscr{B}$ be its base of size less than $cov(\mathscr{M})$. Let T be an $\mathscr{F}^+$-branching tree. For $B \in \mathscr{B}$ put $A_B = \{b \in [T] : rng(b) \cap B =^* \emptyset\}$.

Each A_B is a meager subset of $[T]$ As $|\mathscr{B}| < cov(\mathscr{M})$ there is a $b \in [T] \setminus \bigcup \{A_B : B \in \mathscr{B}\}$. Hence $rng(b) \in \mathscr{F}^+$, so $\mathscr{F}$ is $+$-Ramsey.

For the other direction let $\mathscr{C}$ be a family of closed nowhere dense subsets of ω^ω covering the whole of ω^ω. Our aim is to define a filter on ω which is not $+$-Ramsey. The working copy of ω will be $\omega^{<\omega}$. For $C \in \mathscr{C}$ let $F_C = \{\sigma \in \omega^{<\omega} : \forall f \in C \ \sigma \not\subseteq f\}$. The F_C's obviously form a base for a filter $\mathscr{F}$ on $\omega^{<\omega}$. To see that it is not $+$-Ramsey define a tree $T \subseteq (\omega^{<\omega})^{<\omega}$ as follows

(1) $\emptyset \in T$,

(2) $\forall s \in T \ \forall \sigma \in \omega^{<\omega} \ s^\frown \sigma \in T_{n+1}$ if and only if $s(n-1) \subset \sigma$,

Now, T is an $\mathscr{F}^+$-branching tree and $\forall b \in [T] \ \exists f \in \omega^\omega$ such that $rng(b) \subseteq P(f)$. In particular, there is a $C \in \mathscr{C}$ such that $rng(b) \cap F_C =^* \emptyset$ so $rng(b)$ is not in $\mathscr{F}^+$. □

It should be noted here that $+$-Ramsey filters of uncountable character exist in ZFC. Recall that a pair of sequences $\{A_\alpha : \alpha < \omega_1\}$, $\{B_\alpha : \alpha < \omega_1\}$ of subsets of ω forms a *Hausdorff gap* provided that

(1) $B_\alpha \subseteq^* B_\beta \subseteq^* A_\beta \subseteq^* A_\alpha$ for all $\alpha < \beta < \omega_1$ and

(2) there is no $C \subseteq \omega$ such that $B_\alpha \subseteq^* C \subseteq^* A_\alpha$ for every α.

A Hausdorff gap is *tight* if for every $C \in [\omega]^\omega$ such that $C \subseteq^* A_\alpha$ for every $\alpha < \omega_1$, there is a $\beta < \omega_1$ such that $C \cap B_\beta \neq^* \emptyset$.

It is a remarkable result of Hausdorff that the existence of a Hausdorff gap can be proved in ZFC alone. The existence of a tight Hausdorff gap is known to be equivalent to $\mathfrak{t} = \omega_1$. Consider the following filter associated with a gap:

$$\mathscr{F} = \langle \{A_\alpha : \alpha < \omega_1\} \cup \{\omega \setminus C : \forall \alpha < \omega_1 \ C \cap A_\alpha \neq^* \emptyset \ \& \ C \cap B_\alpha =^* \emptyset\} \rangle.$$

The following is essentially due to P. Nyikos (see [Ny]).

Proposition II.2. *$\mathscr{F}$ is a $+$-Ramsey filter.*

Proof. It is very easy to see that $\mathscr{F}$ is really a filter and that it is uncountably generated. Note that

$$\mathscr{F}^+ = \{A \subseteq \omega : \exists \alpha < \omega_1 \ |A \cap B_\alpha| = \aleph_0\}$$

(if $A \cap B_\alpha =^* \emptyset$ for every α then $\omega \setminus A$ is in the filter, so A is not in $\mathscr{F}^+$). In order to prove that $\mathscr{F}$ is $+$-Ramsey let be an $\mathscr{F}^+$-tree. Fix for every $t \in T$ a $\beta_t < \omega_1$ such hat $|succ_T(t) \cap B_{\beta_t}| = \aleph_0$. Let $\beta = sup\{\beta_t : t \in T\}$. Then B_β intersects $succ_T(t)$ in an infinite set for every $t \in T$ and constructing a branch $b \in [T]$ with $rng(b) \in \mathscr{F}^+$ is now easy. □

Corollary II.3. *There is a $+$-Ramsey filter of character $\aleph_1$.*

Proof. If $cov(\mathscr{M}) > \omega_1$ then by Proposition II.1 any filter of character $\aleph_1$ would do. If $cov(\mathscr{M}) = \omega_1$ then $\mathfrak{t} = \omega_1$ and by the aforementioned result there is a tight gap and the filter $\mathscr{F}$ constructed from the gap has character $\aleph_1$. □

Proposition II.4. $cov(\mathcal{M}) \leq \mathfrak{ra} \leq \mathfrak{a}_T$.

Proof. $cov(\mathcal{M}) \leq \mathfrak{ra}$ follows immediately from the definition and Proposition II.1.

Let $\mathcal{A} = \{T_\alpha : \alpha < \mathfrak{a}_T\}$ be a maximal almost disjoint family of finitely branching subtrees of $\omega^{<\omega}$. Define an infinitely branching tree $T \subseteq (\omega^{<\omega})^{<\omega}$ by $\emptyset \in T$ and $succ_T(t) = \{s \in \omega^{<\omega} : t \subseteq s \text{ and } |s| = |t| + 1\}$. Then T is an $\mathcal{I}^+(\mathcal{A})$-tree as $T_\alpha \cap succ_T(t)$ is finite for every $t \in T$ and $\alpha < \mathfrak{a}_T$. However, every branch of T is a subset of T_α for some α by maximality of $\mathcal{A}$ so $\mathcal{A}$ is not $+$-Ramsey. □

Corollary II.5. *There is a MAD family $\mathcal{A}$ which is not $+$-Ramsey.*

Proof. All we have to do is extend the almost disjoint family given in the construction to a maximal one preserving the fact that the branching sets of T will be in $\mathcal{I}^+(\mathcal{A})$, which is very easy to do. □

More interesting problem, of course, is to construct a $+$-Ramsey MAD family. Unfortunately, we do not know whether such a family can be constructed in ZFC alone, but the following propositions shows that in many models there is one.

Proposition II.6. *($\mathfrak{a} < \mathfrak{ra}$ or $\mathfrak{ra} = \mathfrak{c}$) There is a $+$-Ramsey MAD family.*

Proof. If $\mathfrak{a} < \mathfrak{ra}$ than any MAD family of size $\mathfrak{a}$ is $+$-Ramsey by the definition of $\mathfrak{ra}$.

So, assume that $\mathfrak{a} = \mathfrak{ra} = \mathfrak{c}$. Enumerate all subtrees of $\omega^{<\omega}$ as $\{T_\alpha : 0 < \alpha < \mathfrak{c}\}$ and let $[\omega]^\omega = \{X_\alpha : 0 < \alpha < \mathfrak{c}\}$. By induction on $\alpha < \mathfrak{c}$ construct an increasing sequence of almost disjoint families $\{\mathcal{A}_\alpha : \alpha < \mathfrak{c}\}$ so that

(1) $\mathcal{A}_0$ is an infinite partition of ω into infinite sets,

(2) $\mathcal{A}_\alpha \setminus \bigcup\{\mathcal{A}_\beta : \beta < \alpha\}$ is countable,

(3) $|X_\alpha \cap A| = \aleph_0$ for some $A \in \mathcal{A}_\alpha$ and

(4) if T_α is an $\mathcal{I}^+(\bigcup\{\mathcal{A}_\beta : \beta < \alpha\})$-tree then there is a $b \in [T_\alpha]$ such that $rng(b) \in I^{++}(\mathcal{A}_\alpha)$.

If we can fulfill the promises (1)$-$(4) it is obvious that $\mathcal{A} = \bigcup\{\mathcal{A}_\alpha : \alpha < \mathfrak{c}\}$ is a MAD family. To see that it is $+$-Ramsey note that if $A \in I^{++}(\mathcal{A}_\alpha)$ for some $\alpha < \mathfrak{c}$ then $A \in I^{++}(\mathcal{A})$.

So assume that the $\mathcal{A}_\beta$ has been defined for every $\beta < \alpha$. If T_α is not an $\mathcal{I}^+(\bigcup\{\mathcal{A}_\beta : \beta < \alpha\})$-tree, or if T_α is an $\mathcal{I}^+(\bigcup\mathcal{A}_\beta : \beta < \alpha\})$-tree and there is a $b \in [T_\alpha]$ such that $rng(b) \in I^{++}(\bigcup\{\mathcal{A}_\beta : \beta < \alpha\})$, extend $\bigcup\{\mathcal{A}_\beta : \beta < \alpha\}$ to $\mathcal{A}_\alpha$ so that (3) is satisfied.

If T_α is an $\mathcal{I}^+(\bigcup\{\mathcal{A}_\beta : \beta < \alpha\})$-tree and no branch of T_α is in $I^{++}(\bigcup\{\mathcal{A}_\beta : \beta < \alpha\})$, let $b \in [T_\alpha]$ be such that $rng(b)$ contains an infinite subset A of ω almost disjoint from every element of $\bigcup\{\mathcal{A}_\beta : \beta < \alpha\}$. Split A into infinitely many infinite sets $\{A_i : i \in \omega\}$ and if X_α is almost disjoint from every element of $\bigcup\{\mathcal{A}_\beta : \beta < \alpha\} \cup \{A_i : i \in \omega\}$ let $\mathcal{A}_\alpha = \bigcup\{\mathcal{A}_\beta : \beta < \alpha\} \cup \{A_i : i \in \omega\} \cup \{X_\alpha\}$ otherwise let $\mathcal{A}_\alpha = \bigcup\{\mathcal{A}_\beta : \beta < \alpha\} \cup \{A_i : i \in \omega\}$. It is obvious that this works. □

Let X be a regular topological space, $x \in X$. We will consider the following two variations on a game introduced by G. Gruenhabe in [G].

Two players, *the hero* and *the villain* take turns, at the n-th inning the hero playing U_n a neighborhood of x and the villain responding with $x_n \in U_n \setminus \{x\}$. After ω-many steps we declare a winner. The hero wins a round of the game if the set $\{x_n : n \in \omega\}$ of points chosen by the villain contains x in the closure. Otherwise the villain wins. This game will be denoted by $G(x, X)$.

A slight modification of the above game is the game $\bar{G}(x, X)$ in which the hero and the villain play as before but the hero wins if the sequence $\langle x_n \rangle$ converges to x, the villain winning otherwise.

As usual a strategy for the hero is a map $\rho : (X \setminus \{x\})^{<\omega} \to \mathfrak{U}_x$ (where $\mathfrak{U}_x$ denotes the set of open neighborhoods of x) and a strategy for the villain is a map $\sigma : \mathfrak{U}_x^{<\omega} \setminus \{\emptyset\} \to X \setminus \{x\}$ such that $\forall n \in \omega\ \forall s \in (U_x)^n\ \sigma(s) \in s(n - 1)$. A strategy ρ for the hero is a winning strategy if for every $f \in X^\omega$ such that $f(n) \in \rho(f \restriction n)$ for every $n \in \omega$ $x \in \overline{rng(f)}$ (in case of $G(x, X)$) or $f(n) \to x$ (in case of $\bar{G}(x, X)$). Similarly, σ is a winning strategy for the villain if for every $f \in (\mathfrak{U}_x)^\omega$ $x \notin \overline{\{\sigma(f \restriction n) : n \in \omega\}}$ (resp. $\sigma(f \restriction n) \not\to x$).

As the topology outside the given point x is completely irrelevant to the outcome of the game we may assume that every point other than x is isolated. The most interesting cases seem to occur when X is a countable space, so we restrict ourselves to spaces of the form $\omega \cup \{\mathcal{F}\}$, where $\mathcal{F}$ is a free filter on ω and is treated both as the distinguished point x and the filter of its neighborhoods. In this case we refer to the games as $G(\mathcal{F})$ and $\bar{G}(\mathcal{F})$.

The following lemma can be found in [La]. We include the proof for the sake of completeness.

Lemma III.1. *(Laflamme) Let $\mathcal{F}$ be a filter on ω. Then the following are equivalent:*

(1) *The hero has a winning strategy in the game $G(\mathcal{F})$*

(2) *The hero has a winning strategy in the game $\bar{G}(\mathcal{F})$*

(3) $\chi(\mathcal{F}) = \omega$.

and the villain has a winning strategy in the game $G(\mathcal{F})$ if and only if the filter $\mathcal{F}$ is not $+$-Ramsey.

Proof. If the character of $\mathcal{F}$ is countable then hero has a obvious winning strategy in $\bar{G}(\mathcal{F})$. He simply plays all sets from a countable local base.

A winning strategy for the hero in $\bar{G}(\mathcal{F})$ is obviously also a winning strategy in $G(\mathcal{F})$ and if $\sigma : \omega^{<\omega} \to \mathcal{F}$ is a winning strategy for the hero in the game $G(\mathcal{F})$ then it is easy to see that $\sigma[\omega^{<\omega}]$ is a base of $\mathcal{F}$.

If there is a tree witnessing that $\mathcal{F}$ ιs not $+$-Ramsey the villain can just play along the tree. That is his winning strategy.

If $\sigma : \mathscr{F}^{<\omega} \to \omega$ is a winning strategy for the villain, construct a tree T by induction by $\emptyset \in T$ and if $s \in T$ then there is a sequence $\bar{s} \in \mathscr{F}^{<\omega}$ such that $dom(s) = dom(\bar{s})$ and for every n in $dom(s)$ $s(n) = \sigma(\bar{s} \restriction n)$ and $\overline{s \restriction n} = \bar{s} \restriction n$. Then $s^\frown y \in T$ if and only if there is a $U \in \mathscr{F}$ such that $\sigma(\bar{s}^\frown U) = y$. Fix this U and put $\overline{s^\frown n} = \bar{s}^\frown U$. Obviously T is an $\mathscr{F}^+$-tree as σ was a strategy and it does not contain a branch in $\mathscr{F}^+$ since σ was a winning strategy. $\square$

We further restrict our attention to Fréchet spaces. A space X is *Fréchet* if whenever $x \in \bar{A}$, there is a sequence $\langle x_n \rangle \subseteq A$ such that $x_n \to x$. Consequently, a filter $\mathscr{F}$ is *Fréchet* if the space $\omega \cup \{\mathscr{F}\}$ is Fréchet, in other words, if for every $A \in \mathscr{F}^+$ there is a $B \subseteq A$ such that for every $F \in \mathscr{F} : B \subseteq^* F$. Let $C_{\mathscr{F}} = \{B : \forall F \in \mathscr{F}\ B \subseteq^* F\}$ denote the set of all convergent sequences in $\omega \cup \{\mathscr{F}\}$. We show that (perhaps not in a very natural way) the notion of $+$-Ramseyness fits into the hierarchy of α_i-spaces introduced by Archangelskii in [Ar].

Definition III.2. *Let X be a regular space and let $x \in X$. The point x is said to be a $+$-Ramsey point if the villain does not have a winning strategy in the game $G(x, X)$.*

Definition III.3. *(Archangelskii) Let X be a Fréchet space, $x \in X$. We say that x is an α_i-point (for $i \in \{1, 2, 3, 4\}$) if for every countable collection of sequences converging to x there is a sequence converging to x intersecting:*
α_1: each of them in a cofinite set
α_2: each of them in an infinite set
α_3: infinitely many of them in an infinite set.
α_4: infinitely many of them.

The α_i-properties have proved to be very useful in determining when the product of Fréchet spaces is Fréchet. They have been studied by many mathematicians, most notably by Archangelskii, Nogura, Nyikos, Dow and Steprāns, and Simon. It is well known and not hard to see that:

Proposition III.4. *Let X be Fréchet space and let $x \in X$. Then x is an α_2-point if and only if the villain does not have a winning strategy in the game $\bar{G}(x, X)$.*

It follows from the definition that if x is an α_i-point it is also α_j-point for every $j \geq i$. It is not hard to see that the filter $\mathscr{F}$ used in the proposition II.2 is a Fréchet uncountably generated α_2-filter. It is even consistent that $\mathscr{F}$ is α_1. In fact, it has been shown by A. Dow and J. Steprāns that there are no honest (ZFC) examples of countable α_1-spaces which are not first countable, nor there are ZFC examples of α_2-spaces which are not α_1.

Proposition III.5. *Let X be a Fréchet space and let $x \in X$. Then:*
(1) *If x is a $+$-Ramsey point then x is a α_4-point.*
(2) *If x is an α_2-point then x is a $+$-Ramsey point.*

Proof. For (1) suppose that the villain does not have a winning strategy in the game $G(x, X)$. That means that for every $\mathfrak{U}_x^+$-tree there is a branch in $\mathfrak{U}_x^+$. Given a set of sequences $\{\sigma_n : n \in \omega\}$ construct a tree branching everywhere on a level n to $rng(\sigma_n)$. By the assumption there is a branch in $\mathfrak{U}_x^+$ and since the space is Fréchet there is a subsequence of this branch converging to x.

For (2) consider the contrapositive and recall that a winning strategy for the villain in game the $G(x, X)$ is also winning in the game $\bar{G}(x, X)$. $\square$

We conclude by showing that the property of being $+$-Ramsey is incomparable with α_3, assuming the existence of a $+$-Ramsey MAD family. In particular, this shows that under the assumption there is a countable Fréchet space X and a point $x \in X$ such that the villain has a winning strategy in the game $\bar{G}(x, X)$ but not in the game $G(x, X)$.

First recall the standard construction of an AD family of size $\mathfrak{c}$. Consider the Cantor ree $2^{<\omega}$ and let $\mathscr{A} = \{A_f : f \in 2^\omega\}$, where $A_f = \{f \restriction n : n \in \omega\}$. We will show that $\mathscr{I}^*(\mathscr{A})$ is Fréchet, α_3 and not $+$-Ramsey. For $s \in 2^{<\omega}$ let $u(s) = \{t \in 2^{<\omega} : s \subseteq t\}$.

Proposition III.6. *The filter $\mathscr{I}^*(\mathscr{A})$ is a Fréchet α_3-point which is not $+$-Ramsey.*

Proof. To see that $\mathscr{I}^*(\mathscr{A})$ is Fréchet note that every set in $\mathscr{F}^+$ contains an infinite antichain and that every infinite antichain is in $C_{\mathscr{F}}$.

In order to show that $\mathscr{I}^*(\mathscr{A})$ is α_3 let $\{A_n : n \in \omega\}$ be a set of infinite antichains in $2^{<\omega}$. The aim is to find an antichain A in $2^{<\omega}$ having infinite intersection with infinitely many A_n's. To do this find a $b \in 2^\omega$ such that $|u(b \restriction n) \cap A_i| = \aleph_0$ for every $n \in \omega$ and infinitely many $i \in \omega$. Then either

$$\exists I \in [\omega]^\omega \ \forall n \in \omega \ \forall i \in I \ |u(b \restriction n) \cap A_i| = \aleph_0$$

or

$$\forall I \in [\omega]^\omega \ \exists i \in I \ \exists n \in \omega \ |u(b \restriction n) \cap A_i| < \aleph_0.$$

In the first case fix a bijection $\phi : \omega \to \omega \times I$ and by induction following the branch b choose $s_n \in 2^{<\omega}$ so that for every n, $s_n \not\subset b$, $s_n \cap b \subsetneq s_{n+1} \cap b$ and if $\phi(n) = (i, j)$ then $s_n \in A_j$. Then $A = \{s_n : n \in \omega\}$ is as required.

In the latter case go along the branch and choose whole infinite blocks of A_i's in a similar manner.

The filter $\mathscr{I}^*(\mathscr{A})$ is not $+$-Ramsey as the villain has an obvious winning strategy in $G(\mathscr{F})$ by playing an increasing chain. $\square$

Next it will be shown how to use $+$-Ramsey MAD families to construct Fréchet filters which are $+$-Ramsey and not α_3. The construction depends heavily on ideas of P. Simon. Recall that an AD family $\mathscr{A}$ is *nowhere MAD* if for every $X \in \mathscr{I}^+(\mathscr{A})$ there is a $Y \subset X$ almost disjoint from every $A \in \mathscr{A}$.

Theorem III.7. *(Simon) For every MAD family $\mathscr{A}$ there is an $X \in \mathscr{I}^+(\mathscr{A})$ such that $\mathscr{A} \upharpoonright X = \{A \cap X : A \in \mathscr{A}$ and $|A \cap X| = \aleph_0\}$ can be partitioned into two nowhere MAD subfamilies $\mathscr{A}_1$ and $\mathscr{A}_2$.*

Proposition III.8. *Let $\mathscr{A}$ be a $+$-Ramsey MAD family. Then there is a $+$-Ramsey Fréchet filter $\mathscr{F}$ which is not α_3.*

Proof. Let $\mathscr{A}$ be a $+$-Ramsey MAD family. Find X, and $\mathscr{A}_1$ and $\mathscr{A}_2$ as in the above theorem. Note that $\mathscr{A} \upharpoonright X$ is a $+$-Ramsey MAD family (of subsets of X) and let $\mathscr{F} = \mathscr{I}^*(\mathscr{A}_1) \subseteq \mathscr{P}(X)$.

Then $\mathscr{F}^+ = \{B \subseteq X : \exists A \in \mathscr{I}(\mathscr{A}_2)\, |B \cap A| = \aleph_0\}$ and $C_{\mathscr{F}} = \mathscr{I}(\mathscr{A}_2) \cap [X]^\omega$. So $\mathscr{F}$ is Fréchet and not α_3. To see that $\mathscr{F}$ is $+$-Ramsey let T be an $\mathscr{F}^+$-tree. WLOG $succ_T(t) \cap succ_T(s) = \emptyset$ for every $t \neq s \in T$. Hence for every $n \in \bigcup_{t \in T} succ_T(t)$ there is a unique $s_n \in T$ such that $n \in succ_T(s_n)$. Define a new tree T' by letting $\emptyset \in T'$, $succ_{T'}(\emptyset) = \bigcup_{t \in T} succ_T(t)$, and for $t \in T'$ $succ_{T'}(t) = \bigcup\{succ_T(s) : s_{t(|t|-1)} \subseteq s\}$. Note that $succ_{T'}(t) \subseteq succ_{T'}(s)$ whenever $s \subseteq t$.

If T' is not an $\mathscr{I}^+(\mathscr{A} \upharpoonright X)$-tree then there is a $t \in T$ such that $\bigcup_{t \subseteq s} succ_T(s) \in \mathscr{I}(\mathscr{A} \upharpoonright X)$. Then there exists a $b \in [T]$ such that $t \subseteq b$ and $rng(b) \in \mathscr{I}(\mathscr{A}_2)$ and it is easy to find one with infinite range.

If T' is a $\mathscr{I}^+(\mathscr{A} \upharpoonright X)$-tree then let $b \in [T']$ be such that $rng(b) \in \mathscr{I}^+(\mathscr{A} \upharpoonright X)$ and note that there is a branch $b' \in [T]$ such that $rng(b) \subseteq rng(b')$. This finishes the proof. $\square$

Open problems. *The following is a list of questions the author does not know the answer to:*

(1) *Is there a $+$-Ramsey MAD family in ZFC?*

(2) *Is there (in ZFC) a Fréchet filter on ω which is $+$-Ramsey and not α_2? In other words, is there a Fréchet filter on ω such that the villain has a winning strategy in the game $\bar{G}(\mathscr{F})$ but not in the game $G(\mathscr{F})$?*

(3) *Is $cov(\mathscr{M}) < \mathfrak{ra}$ consistent?*

(4) *Is $\mathfrak{d} < \mathfrak{a}_T$ consistent?*

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