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Convex Functions with Respect to Logarithmic Mean and Sandwich Theorem

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We will show that, contrary to many classes of functions, some sandwich's theorems are not valid for the classes of convex and concave functions with respect to logarithmic mean. Some properties of convex functions with respect to the logarithmic mean will be also presented.

Let $J \subset \mathbb{R}$ be a fixed interval and let $M : J \times J \to J$ be a mean i.e.

$$\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}, \qquad x, y \in I.$$

A function f defined on an interval $I \subset J$ with the values in J is called

(i) M-convex iff $f(M(x, y)) \leq M(f(x), f(y))$, $x, y \in I$,

(ii) M-concave iff $f(M(x, y)) \geq M(f(x), f(y))$, $x, y \in I$, and

(iii) M-affine iff $f(M(x, y)) = M(f(x), f(y))$, $x, y \in I$,

In the case where $M(x, y) = \frac{x+y}{2}$ we get a notion of J-convex, J-concave and affine functions, respectively, which were extensively studied in a book of Kuczma [5]. An interesting structure has the so-called logarithmic mean, i.e. the mean defined by the formula

$$L(x, y) = \frac{x - y}{\log x - \log y}, \ x \neq y; \ \ L(x, y) = x, \ x = y, \ x, y \in (0, \infty).$$

In his paper [6] J. Matkowski has proved that every continuous L-affine function $f : (0, \infty) \to (0, \infty)$ is either constant or of the form $f(x) = kx$, $x > 0$ where $k > 0$ is an arbitrary constant. We shall use this result to give a negative answer to a "sandwich's" question. This will be done in the following

Theorem 1. *There exist continuous functions $f, g : (0, \infty) \to (0, \infty)$, first of them L-convex and the second one L-concave, $g \leq f$ in $(0, \infty)$, for which there is no continuous L-affine function $h : (0, \infty) \to (0, \infty)$ fulfilling the following inequality*

$$g(x) \leq h(x) \leq f(x), \quad x \in (0, \infty). \tag{1}$$

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Proof. Let us put

$$g(x) = \sqrt{x}, \quad f(x) = \exp(x), \quad x > 0.$$

It is known [7] that f is L-convex and g is L-concave. In view of result of Matkowski mentioned above, it is easily seen that there is no L-affine function h which separates g and f (i.e. which fulfils the inequality (1)).

Recall that if we take the arithmetic mean then a such type result holds true [8]. Also, if g is a superadditive function $(g(x + y) \geq g(x) + g(y),\ x, y \in S)$, f is a subadditive function $(f(x + y) \leq f(x) + f(y), x, y \in S)$, and $g \leq f$ *in* S, where S is an abelian semigroup, then there exists an additive function h $(h(x + y) = h(x) + h(y), x, y \in S)$ which separates g and f ([3], [4]). Theorem 1 shows that convex (concave) functions with respect to the logarithmic mean have essentially different geometrical behaviour than the convex (concave) functions with the usual sense.

Let g and f be defined and with the values in $(0, \infty)$ and we assume that

$$g(L(x, y)) \leq L(f(x), f(y)), \quad x, y \in (0, \infty). \tag{2}$$

Putting $x = y$ we obtain

$$g(x) \leq f(x) \quad x \in (0, \infty). \tag{3}$$

However, if f is L-convex and (3) holds, then (2) holds as well. It follows from our Theorem 1 that we cannot expect to find a continuous L-affine function h which separates g and f. However K. Baron, J. Matkowski and K. Nikodem [1] have proved that functions $g, f : (0, \infty) \to \mathbb{R}$ satisfy the inequality

$$g(tx + (1 - t)\,y) \leq tf(x) + (1 - t)\,f(y), \quad x, y \in (0, \infty),\ t \in [0, 1],$$

iff there exists a convex function $h : (0, \infty) \to \mathbb{R}$ fulfilling (1). We shall show that our condition (2) does not imply the existence of L-convex function $h : (0, \infty) \to (0, \infty)$ separating g and f. For this purpose we shall prove the following

Theorem 2. *Let M be a mean defined on a cartesian product of an interval I by itself satisfying the following conditions:*
(i) for every $x, y \in I$ there is a $z \in I$ such that $M(y, z) = x$
and
(ii) there exists a $\lambda \in (0, 1)$ such that for all $x, y \in I$ the condition $M(x, y) \leq \lambda \max\{x, y\} + (1 - \lambda) \min\{x, y\}$.
Then every upper bounded (on I) M-convex function $h : I \to \mathbb{R}$ is constant.

Proof. Let us put

$$c := \sup\{h(t); t \in I\}.$$

Thus $h(x) \leq c$, $x \in I$. Fix an arbitrary $\varepsilon > 0$ and choose a $y \in I$ such that

$$h(y) > c - (1 - \lambda)\,\varepsilon. \tag{1}$$

It is enough to show that

$$(5) \qquad h(x) \geq c - \varepsilon, \quad x \in I.$$

For indirect proof we assume that $h(x_0) < c - \varepsilon$ for some $x_0 \in I$. By virtue of (i) there exists a $z \in I$ such that $y = M(x_0, z)$ and according to the M-convexity of h, (4) and (5) we get

$$c - (1 - \lambda)\,\varepsilon < h(y) = h(M(x_0, z)) \leq \lambda \max\{h(x_0), h(z)\} + (1 - \lambda)\min\{h(x_0), h(z)\} <$$
$$< \lambda c + (1 - \lambda)(c - \varepsilon) = c - (1 - \lambda)\,\varepsilon.$$

This contradiction completes the proof of Theorem 2.

Since $L(x, y) \leq \frac{x+y}{2}$ ([2]), as a consequence of Theorem 2 we obtain

Corollary 1. *Every bounded on $(0, \infty)$ L-convex function $h : (0, \infty) \to (0, \infty)$ is constant.*

An another consequence of Theorem 2 is following known result.

Corollary 2. *Every upper bounded J-convex function $h : \mathbb{R} \to \mathbb{R}$ is constant.*

Now, we shall construct functions $g, f : (0, \infty) \to (0, \infty)$ having required properties.

Theorem 3. *There exist functions $g, f : (0, \infty) \to (0, \infty)$ satisfying condition* (2) *for which there is no L-convex function $h : (0, \infty) \to (0, \infty)$ fulfilling the inequality* (1).

Proof. Let $x_0 > 0$ be chosen such that the inequalities

$$1 < \frac{\dfrac{1}{\log 2} + 1}{2} < \frac{\mathrm{e}^{-x}}{\log(1 + \mathrm{e}^{-x})}, \quad x \in (0, x_0)$$

are fulfilled. Let us put

$$z_0 := \min\left\{x_0, \frac{1}{2}\left(\frac{1}{\log 2} - 1\right)\right\},$$
$$f(x) := 1 + \mathrm{e}^{-x}, \quad x > 0,$$

and

$$g(x) := \begin{cases} -x + 1 + z_0, & x \in (0, \infty) \\ 1, & x \in [z_0, \infty). \end{cases}$$

We shall show that g and f satisfy the inequality (2) and there is no L-convex function $h : (0, \infty) \to (0, \infty)$ fulfilling condition (1). At first we note that for all $x, y \in [z_0, \infty)$ we have

$$L(x, y) \in [z_0, \infty) \quad \text{and} \quad g(L(x, y)) = 1.$$

Moreover, $f(x), f(y) \in (1, 2)$ and therefore $L(f(x), f(y)) \in (1, 2)$. Consequently

$$(7) \qquad g(L(x, y)) \leq L(f(x), f(y)), \quad x, y \in (z_0, \infty).$$

Fix an $x \in (0, z_0]$ and take an arbitrary $y > 0$. By the definition of g we get

$$g(L(x, y)) \leq 1 + z_0 \leq 1 + \frac{\frac{1}{\log 2} - 1}{2} = \frac{1 + \frac{1}{\log 2}}{2}. \tag{8}$$

Since f is decreasing and $1 < f(y) < 2$, by virtue of the monotonicity of $L(f(x), .)$ and (6) we obtain

$$L(f(x), f(y)) > L(f(x), 1) = \frac{e^{-x}}{\log(1 + e^{-x})} > \frac{1 + \frac{1}{\log 2}}{2}. \tag{9}$$

It follows from (7), (8) and (9) that (2) holds true.

Suppose now that $h : (0, \infty) \to (0, \infty)$ is a L-convex function fulfilling condition (1). Similarly to f, function h is bounded on $(0, \infty)$. According to Corollary 1 there exists a constant $c > 0$ such that

$$h(x) = c, \quad x \in (0, \infty). \tag{10}$$

In view of definitions g, f and (11) we infer that $c = 1$ which is impossible, because $g(x) > 1$ for each $x \in (0, z_0)$. This contradiction proves that there is no L-convex function h separating g and f.

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