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A Remark on the Uniformization in Metric Spaces

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A theorem of Kaniewski states that given a partition of a coanalytic set in a Polish space there is, under some assumptions, a coanalytic selector for this partition. We prove a similar theorem in the non-separable case. As a corollary we obtain a simpler proof of the metric case of a uniformization theorem of Rogers and Willmott and, using a theorem on measurable extensions of mappings, we also obtain a theorem on the uniformization of mappings, that improves a classical theorem of Kondô.

1. Introduction

The uniformization is an important topic of descriptive set theory. We concern ourselves about the co-Souslin uniformization of co-Souslin sets, although other problems (the Borel uniformization of Borel sets) are also reasonable. The most important result on the uniformization in Polish spaces is a theorem of Kondô saying that a coanalytic set in the product of two Polish spaces can be uniformized by a coanalytic set (see [Ku, §39 V]).

The following theorem of Kaniewski generalizes the previous one (see [Ka]):

Let C be a coanalytic subset of a Polish space Z. Let a partition Q of C be given by an equivalence relation $\sim$. Assume that $\mathcal{G}(\sim) = (C \times C) \cap A$ for some analytic $A \subset Z \times Z$. Then there is a coanalytic set S in Z which is a selector for Q.

In the case of non-separable metric spaces, the main known result is due to Rogers and Willmott. Theorem 18 of [RW2] includes even more general topological spaces:

Let X be a space in which open sets are Souslin. Let Y be a Hausdorff space that is a continuous one-to-one image of some closed subset of $\mathbb{N}^{\mathbb{N}}$. Let C be a co-Souslin subset of $X \times Y$. Then C can be uniformized by a co-Souslin set.

We will do some observations on the uniformization in non-separable metric spaces. In Section 3 we prove that the theorem of Kaniewski holds, under certain additional assumption, also in non-separable compete metric spaces.

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In Section 4 we give a simpler proof of the theorem of Rogers and Willmott for metric spaces using our generalization of [Ka].

Another theorem, due essentially to Kondô (see [Ku, §39 V]), says:

Let f be a continuous function defined on a coanalytic subset C of a Polish space. Then there exists a coanalytic set S such that $f(S) = f(C)$ and the partial function $f|_S$ is injective.

In Section 5 we give a non-separable analogue of it. For this purpose we need a theorem on extension of extended Borel-measurable mapping to an extended Borel set. Similar theorems on Borel mappings are in [Ha2], for the case of separable spaces see [Ku, §35].

2. Definitions

A set S in a topological space is called *Souslin* if it is the result of the Souslin operation performed on a system of closed sets, i.e. $S = \bigcup_{i \in \mathbb{N}^{\mathbb{N}}} \bigcap_n S_{i_1 \ldots i_n}$, where $S_{i_1 \ldots i_n}$ is a closed set defined for each $n \in \mathbb{N}$ and $(i_1 \ldots i_n) \in \mathbb{N}^n$.

A set whose complement is a Souslin set is called *co-Souslin*.

In Polish (i.e. separable completely metrizable) spaces the Souslin sets coincide with the analytic sets. Those are defined as continuous images of $\mathbb{N}^{\mathbb{N}}$ (see [Ku §39 II]), and also the empty set is analytic. The complements of analytic sets are called coanalytic sets.

If A a co-Souslin set in the product of topological spaces X and Y, a co-Souslin set $B \subset A$, for which $\pi_X(A) = \pi_X(B)$ and such that for all $x \in \pi_X(A)$ the set $(\{x\} \times Y) \cap B$ is a singleton, is called a *uniformization* of A. (Here π_X denotes the projection of $X \times Y$ to X.)

If f is a mapping defined on a co-Souslin subset A of a space Y into a space X, a uniformization of f is its restriction to a co-Souslin set $B \subset A$ such that $f(A) = f(B)$ and $f|_B$ is injective.

The uniformization of a set $C \subset X \times Y$ is, in fact, the same as the uniformization of the projection $\pi_X : C \to X$.

By a completely metrizable space we mean a space which admits a complete metric compatible with its topology:

3. Uniformization of equivalence relations

The following definitions are taken from [Ka]:

A *partition* Q of a set C is a disjoint system of non-empty sets closed in C whose union is C.

A partition Q of C can be given by an equivalence relation $\sim$ between elements of C:

$$x \sim y \Leftrightarrow x \text{ and } y \text{ lie in the same element of } Q.$$

A set $S \subset C$ is called a *selector* for the partition Q of the set C, if $S \cap R$ is a singleton whenever $R \in Q$.

Looking for a selector is a problem more general than uniformization. In fact, a uniformization is a selector for the partition of $C \subset X \times Y$ into the sections $(\{x\} \times Y) \cap C$.

3.1. Theorem

Let C be a co-Souslin subset of a completely metrizable space Z. Let a partition Q of C be given by an equivalence relation $\sim$. Let the graph of the relation satisfies $\mathcal{G}(\sim) = (C \times C) \cap A$ with some Souslin $A \subset Z \times Z$. Let the projection p from $Z \times Z$ to Z, defined by $p(x, y) = y$, maps all Souslin subsets of A to Souslin sets. Then there is a co-Souslin set S in Z which is a selector for Q.

This theorem is a generalization of the theorem of [Ka] to non-separable spaces; only the assumption on projections of Souslin sets of A is added. In the separable case every continuous mapping preserves Souslin sets, so this assumption is automatically fulfilled.

The proof also follows that of Kaniewski. It begins with the following lemma.

3.2. Lemma

Let C be a co-Souslin subset of a completely metrizable space Z. Then there exists a relation $\prec$ in Z such that

(i) *its graph $\mathcal{G}(\prec)$ is Souslin in $Z \times Z$,*

(ii) *$\prec$ restricted to C is a linear ordering of C (i.e. it is transitive and satisfies the trichotomy law),*

(iii) *if $x \prec y$ and $y \in C$, then $x \in C$,*

(iv) *in each non-empty set $F \subset C$, closed in C, there is the first element, i.e. an $a \in F$ such that $a \prec x$ for each $x \in F$, $x \neq a$.*

Lemma 2 of [Ka] states the existence of a relation $\prec$ with the same properties as here under the assumption that Z is Polish. But its proof works also in the non-separable case, so we omit it.

Proof of the theorem. Let $\prec$ be as in Lemma 3.2. Let S be the set of the first elements (with respect to $\prec$) of the equivalence classes of $\sim$. According to (iv) of the lemma, S is a selector for Q. It suffices to prove that S is a co-Souslin set. The following characterization holds:

$$y \in C \setminus S \Leftrightarrow y \in C \wedge \exists x \in Z \, (x \sim y \wedge x \prec y).$$

Since $x \sim y$ means that $x, y \in C$ and $(x, y) \in A$, we can write, using (iii),

$$y \in C \setminus S \Leftrightarrow y \in C \wedge \exists x \in Z \, ((x, y) \in A \wedge (x, y) \in \mathcal{G}(\prec)).$$

In other words, $C \setminus S = C \cap p(A \cap \mathcal{G}(\prec))$, hence $S = C \setminus p(A \cap \mathcal{G}(\prec))$.

By (i), $A \cap \mathcal{G}(\prec)$ is Souslin in A, and by the assumption on p, $p(A \cap \mathcal{G}(\prec))$ is Souslin, hence S is a co-Souslin set. □

4.1. Theorem *(Theorem 18 of* [RW2] *in the case of metric spaces)*
Let M be a metrizable space and P a Polish space. Let C be a co-Souslin set in $M \times P$. Then there exists a co-Souslin set $S \subset M \times P$ which uniformizes C, i.e. $\pi_M(S) = \pi_M(C)$ and for each $m \in \pi_M(S)$ the set $(\{m\} \times P) \cap S$ is a singleton.

Proof. 1. For M a complete metric space:
Let $Z = M \times P$. A relation $\sim$ on C let be defined as follows: if $x = (x_M, x_P) \in C$, $y = (y_M, y_P) \in C$, then $x \sim y \Leftrightarrow x_M = y_M$. Set $A = \{(x, y) \in Z \times Z;\ x_M = y_M\}$. It is clear that $\mathscr{G}(\sim) = (C \times C) \cap A$ and A is closed in $Z \times Z$. The map h: $A \to P \times M \times P$, defined by $h((y_M, x_P), (y_M, y_P)) = (x_P, y_M, y_P)$, is a homeomorphism. We denote by p the projection of $Z \times Z$ to Z, $p(x, y) = y$. Then $p|_A = q \circ h$, where q is the projection of $P \times M \times P$ to $M \times P$ defined by $q(x_P, y_M, y_P) = (y_M, y_P)$. Such a q maps Souslin sets to Souslin sets (see [RW1]), hence p maps all Souslin subsets of A to Souslin subsets of Z. Now, Z, C, $\sim$ and A satisfy the requirements of Theorem 3.1 and therefore there exists a co-Souslin set $S \subset Z$ which is a selector for the partition given by $\sim$. Hence S contains exactly one point from each equivalence class $(\{x\} \times P) \cap C$, so it uniformizes C.
2. For M metrizable, let N be the completion of any of its metrization. We can find a uniformization in $N \times P$ and restrict it back to $M \times P$. □

It is an open question whether one can find a uniformization in more general cases. The answer is negative in the case of the product $P \times M$ of two metric spaces, P being separable and M non-separable. (Here we mean the uniformization with respect to the projection to P.) Otherwise the existence of a uniformization would imply the existence of a reduction for every (uncountable) system of coanalytic sets in P:

Let $\{U_\alpha\}_{\alpha \in A}$ be a system of coanalytic sets in a separable space P. Consider the product $P \times M$ with M containing a discrete subspace $\{m_\alpha\}_{\alpha \in A}$. Then $\bigcup_{\alpha \in A}(U_\alpha \times \{m_\alpha\})$ would be a co-Souslin set in $P \times M$ and its uniformization would give us a disjoint family of co-Souslin sets $\{V_\alpha\}_{\alpha \in A}$ with $V_\alpha \subset U_\alpha$ and $\bigcup_{\alpha \in A} V_\alpha = \bigcup_{\alpha \in A} U_\alpha$.
But this is impossible because of the following example by G. Hjorth:

4.2. Example. Consider a coanalytic non-Borel set C in $\mathbb{R}$ and denote by C_0 the set $C \times \mathbb{R}$. Let $\{v_\alpha\}_{1 \leq \alpha < c}$ be an enumeration of $\mathbb{R}$, and let $C_\alpha = \mathbb{R} \times \{v_\alpha\}$. (Thus $C_\alpha \cap C_0$ is non-Borel.) Let $\{D_\alpha;\ 1 \leq \alpha < c\}$ be the system of all the coanalytic sets in $\mathbb{R}^2$.

We define a system $\{B_\alpha\}_{0 \leq \alpha < c}$ of sets in $\mathbb{R}^2$ as follows: let $B_0 = C_0$ and for $\alpha \geq 1$ let

$$B_\alpha = \begin{cases} C_\alpha & \text{if } C_\alpha \cap C_0 \cap D_\alpha \text{ is non-Borel} \\ \emptyset & \text{otherwise.} \end{cases}$$

Suppose that for each α there exists a coanalytic set $B_\alpha^* \subset B_\alpha$ such that $\bigcup_{\alpha \in A} B_\alpha^* = \bigcup_{\alpha \in A} B_\alpha$ and $\{B_\alpha^*\}_{0 \leq \alpha < c}$ are disjoint.

Thus for the set B_0^* there exists $\alpha \geq 1$ such that $B_0^* = D_\alpha$. Also $B_\alpha^* \subset B_\alpha$, and B_α equals either to C_α or to $\emptyset$.

If $B_\alpha = \emptyset$, using $\bigcup_{\alpha \in A} B_\alpha^* = \bigcup_{\alpha \in A} B_\alpha$ we infer that $(C_\alpha \cap B_0^*) \cup B_\alpha^* = (C_\alpha \cap B_0) \cup B_\alpha$, thus $B_0^* \supset C_\alpha \cap B_0 = C_\alpha \cap C_0$. Since $B_0^* = D_\alpha$, we have $C_\alpha \cap C_0 \cap D_\alpha = C_\alpha \cap C_0$, which is not Borel, as was mentioned above, and so $B_\alpha = C_\alpha \neq \emptyset$, a contradiction.

If $B_\alpha = C_\alpha$, using $B_\alpha^* \cap B_0^* = \emptyset$ we obtain $B_\alpha^* = C_\alpha \setminus B_0^*$. Using $B_0^* \subset C_0$ we obtain $B_\alpha^* = C_\alpha \setminus (B_0^* \cap C_\alpha \cap C_0) = C_\alpha \setminus (D_\alpha \cap C_\alpha \cap C_0)$. But this is analytic non-Borel, hence B_α^* cannot be coanalytic.

5. Uniformization of mappings

5.1. Definitions. A family $\{D_\alpha\}_{\alpha \in A}$ of subsets of a topological space X is said to be *discrete* if each $x \in X$ has a neighborhood U_x such that U_x meets at most one of the sets $\{D_\alpha\}_{\alpha \in A}$.

Countable unions of discrete families are called *σ-discrete* families.

A family $\{S_\alpha\}_{\alpha \in A}$ is called *σ-discretely decomposable* (σ-dd for short) if for every α we can write $S_\alpha = \bigcup_n S_\alpha^n$ so that the family $\{S_\alpha^n\}_{\alpha \in A}$ is discrete for each n.

A mapping f: $A \subset X \to Y$ which maps discrete (in the induced topology of A) families of subsets of A to σ-dd families in Y is called *σ-dd-preserving.* (Notice that if X is metrizable and A is its subspace, then a family $\{B_\lambda\}$ of subsets of A is σ-dd in A iff it is σ-dd in X ([Ha1, §1.3.]). So it makes no difference whether we consider families that are discrete in A or in X the definition of σ-dd-preserving mapping.)

A mapping f: $A \subset X \to Y$ such that $f^{-1}(\mathscr{S})$ is σ-dd whenever $\mathscr{S}$ is discrete is called *σ-discrete*. (It is easy to see that continuous mappings are σ-discrete.)

A mapping f: $A \subset X \to Y$ which is both σ-dd-preserving and σ-discrete is called *bi-σ-discrete* here.

The members of the smallest σ-algebra containing the open sets and closed with respect to unions of discrete subfamilies are called the *extended Borel* sets.

Extended Borel sets in a completely metrizable space coincide with the sets that are both Souslin and co-Souslin (see [FH1, Corollary 1.4.]).

A mapping f is called extended Borel-measurable if $f^{-1}(U)$ is extended Borel whenever U is open.

Every extended Borel-measurable σ-dd-preserving map f: $A \subset X \to Y$, where X, Y are completely metrizable and A is extended Borel, maps Souslin sets to Souslin sets ([Ha3, Theorem 7.3.]). Also preimages of Souslin or co-Souslin sets by extended Borel-measurable maps are Souslin or co-Souslin, respectively.

The problems of uniformization of sets and of continuous mappings are equivalent, as we mentioned in Section 2. But in non-separable spaces we can uniformize some sets only. Thus we will uniformize some mappings only — those bi-σ-discrete. (In separable metric spaces any map is bi-σ-discrete, since every discrete family is countable there.) We will not uniformize continuous mappings only, but also extended Borel-measurable ones.

We need the following theorems on extensions of mappings.

5.2. Theorem

Let C be an arbitrary subset of a metrizable space X and f a continuous σ-dd-preserving map of C into a completely metrizable space Y. Then f can be extended to a continuous σ-dd-preserving F defined on a G_δ set $B \supset C$.

Proof. Consider a fixed metric on X. Let $\tilde{A}$ be a G_δ set, $C \subset \tilde{A} \subset \bar{C}$, such that we can extend f onto $\tilde{A}$ to a continuous map $\tilde{f}$ (see [Ku §35 I]).

Let $\tilde{\mathscr{B}}$ be a basis for the topology of $\tilde{A}$, $\tilde{\mathscr{B}} = \bigcup_n \tilde{\mathscr{B}}_n$ with $\tilde{\mathscr{B}}_n$ discrete (in $\tilde{A}$) for all n (see [Ku §21 XVI]). We can suppose that for each n all the elements of $\tilde{\mathscr{B}}_n$ have the diameter at most 1. For each $n, k \in \mathbb{N}$ set $\tilde{\mathscr{B}}^k = \{B \in \tilde{\mathscr{B}};\ \operatorname{diam} B < \frac{1}{k}\}$ and $\tilde{\mathscr{B}}^k_n = \{B \in \tilde{\mathscr{B}}_n;\ \operatorname{diam} B < \frac{1}{k}\}$.

Let $\mathscr{B}$, $\mathscr{B}_n$, $\mathscr{B}^k$, $\mathscr{B}^k_n$ be the families of sets of $\tilde{\mathscr{B}}$, $\tilde{\mathscr{B}}_n$, $\tilde{\mathscr{B}}^k$, $\tilde{\mathscr{B}}^k_n$, respectively, intersected with C. Now for every n, k the families $\mathscr{B}_n$ and $\mathscr{B}^k_n$ are discrete in C and $\mathscr{B}$, $\mathscr{B}^k$ are bases for the topology of C.

Let $\{\tilde{B}^k_{n,\lambda}\}_{\lambda \in \Lambda_{k,n}}$ be an enumeration of $\tilde{\mathscr{B}}^k_n$, thus $\{B^k_{n,\lambda}\}_{\lambda \in \Lambda_{k,n}}$ is an enumeration of $\mathscr{B}^k_n$. The mapping f maps each $\mathscr{B}^k_n$ to a σ-dd family in Y. In other words, for $\lambda \in \Lambda_{k,n}$ we have $f(B^k_{n,\lambda}) = \bigcup_{m \in \mathbb{N}} T^k_{n,\lambda,m}$, where $\{T^k_{n,\lambda,m}\}_{\lambda \in \Lambda_{k,n}}$ is discrete for each m, n, k. Set $B^k_{n,\lambda,m} = f^{-1}(T^k_{n,\lambda,m}) \cap B^k_{n,\lambda}$.

For fixed m, n, k, the family $\mathscr{S}^k_{n,m} = \{B^k_{n,\lambda,m}\}_{\lambda \in \Lambda_{k,n}}$ is discrete in C. Its image by f, the family $\mathscr{T}^k_{n,m} = \{T^k_{n,\lambda,m}\}_{\lambda \in \Lambda_{k,n}}$, is discrete in Y. We replace each set $T^k_{n,\lambda,m}$ with an open set $U^k_{n,\lambda,m} \supset T^k_{n,\lambda,m}$ in such a way that the family $\mathscr{U}^k_{n,m} = \{U^k_{n,\lambda,m};\ T^k_{n,\lambda,m} \in \mathscr{T}^k_{n,m}\}$ remains discrete. (This is possible since every metric space is collectionwise normal.) Define for each $B^k_{n,\lambda,m} \in \mathscr{S}^k_{n,m}$ a set

$$C^k_{n,\lambda,m} = \bigcup\{B \in \tilde{\mathscr{B}}^k;\ B \cap B^k_{n,\lambda,m} \neq \emptyset,\ \tilde{f}(B) \subset U^k_{n,\lambda,m}\}$$

It is an open subset of $\tilde{A}$. Put $D^k_{n,\lambda,m} = \tilde{B}^k_{n,\lambda} \cap C^k_{n,\lambda,m}$. This $D^k_{n,\lambda,m}$ is also open in $\tilde{A}$ and the family $\mathscr{D}^k_{n,m} = \{D^k_{n,\lambda,m};\ B^k_{n,\lambda,m} \in \mathscr{S}^k_{n,m}\}$ is discrete in $\tilde{A}$, because $\{\tilde{B}^k_{n,\lambda}\}_{\lambda \in \Lambda_{k,n}}$ is discrete. Set

$$G^k = \bigcup_{n,m} \bigcup \mathscr{D}^k_{n,m}.$$

Each G^k is open in $\tilde{A}$; $A = \tilde{A} \cap \bigcap_{k \in \mathbb{N}} G^k$ is of type G_δ and $C \subset A \subset \bar{C}$. Now we extend f to $F = \tilde{f}|_A$.

For each m, n, k, λ set $E^k_{n,\lambda,m} = D^k_{n,\lambda,m} \cap A$ and let $\mathscr{E} = \{E^k_{n,\lambda,m}\}_{\lambda \in \Lambda_{k,n}, m,n,k \in \mathbb{N}}$. This $\mathscr{E}$ is σ-discrete in A and it is a basis of the topology of A. Indeed, for fixed k, for each point x of A there are some n_x, λ_x, m_x such that $x \in D^k_{n_x,\lambda_x,m_x}$. This set is open in $\tilde{A}$, thus $E^k_{n_x,\lambda_x,m_x}$ is open in A, and the diameter of $D^k_{n_x,\lambda_x,m_x}$ is at most $\frac{3}{k}$ (because of the way we defined $D^k_{n,\lambda,m}$). Hence $E^k_{n_x,\lambda_x,m_x}$, $k = 1, 2, \ldots$ form a basis of neighborhoods for x.

F maps $\mathscr{E}$ to a σ-discrete family. Indeed, $\tilde{f}(E^k_{n,\lambda,m}) \subset \tilde{f}(C^k_{n,\lambda,m}) \subset U^k_{n,\lambda,m}$ and $\mathscr{U}^k_{n,m}$ is discrete. Thus F maps $\mathscr{E}$ to a σ-dd family. According to Corollary 3.9 of [Ha3], F maps every discrete family to σ-dd. □

5.3. Theorem

Let X be a metrizable space, Y a completely metrizable space, A a subset of X and $f\colon A \to Y$ an extended Borel-measurable σ-discrete mapping. Then f can be extended to an extended Borel-measurable F defined on an extended Borel set A^*. If X is completely metrizable, then F will be σ-discrete.

Remark. This is a non-separable analogue of Theorem 1 of [Ku §35 VI]. In [Ha2] there is Theorem 9 saying that a σ-discrete Borel mapping of nonlimit class α defined on a subset of a paracompact space X into a complete metric space Y can be extended to a Borel mapping of the same class defined on a Borel set of multiplicative class $\alpha + 1$.

Proof. Consider a fixed complete metric ρ on Y. Let $\{B^1_{k,\lambda};\ \lambda \in \Lambda\}, k = 1\ldots$ be discrete families of open sets of the diameter at most $\frac{1}{2}$ that form a σ-discrete covering of Y. For each $k \in \mathbb{N}$, let us do the following: Put $C_{k,\lambda} = f^{-1}(B^1_{k,\lambda})$ for each $\lambda \in \Lambda$. Since f is σ-discrete and extended Borel-measurable, each $C_{k,\lambda}$ is extended Borel in A and $\{C_{k,\lambda};\ \lambda \in \Lambda\}$ is σ-dd and disjoint in A. We need to find sets $\{G_{k,\lambda};\ \lambda \in \Lambda\}$ that are extended Borel, disjoint and σ-dd in X, and such that $G_{k,\lambda} \cap A = C_{k,\lambda}$ for each λ.

We can find extended Borel sets $\{D_{k,\lambda};\ \lambda \in \Lambda\}$ in X such that $D_{k,\lambda} \cap A = C_{k,\lambda}$ for each λ. Since $\{C_{k,\lambda};\ \lambda \in \Lambda\}$ is σ-dd, we can write $C_{k,\lambda} = \bigcup_m C_{k,\lambda,m}$ with $\{C_{k,\lambda,m};\ \lambda \in \Lambda\}$ discrete in X for each m. Let $E_{k,\lambda,m} \supset C_{k,\lambda,m}$ be open in X and such that $\{E_{k,\lambda,m};\ \lambda \in \Lambda\}$ is discrete. Put $E_{k,\lambda} = \bigcup_m E_{k,\lambda,m}$ and $F_{k,\lambda} = E_{k,\lambda} \cap D_{k,\lambda}$. The sets $F_{k,\lambda}$, $\lambda \in \Lambda$, are extended Borel and σ-dd in X and $F_{k,\lambda} \cap A = C_{k,\lambda}$ for each λ. Put $G_{k,\lambda} = F_{k,\lambda} \setminus \bigcup\{F_{k,\alpha};\ \alpha \neq \lambda\}$. Now the family $\{G_{k,\lambda} \in \Lambda\}$ has all the properties we required.

Finally, set $H^1_{1,\lambda} = G_{1,\lambda}$ and $H^1_{k,\lambda} = G_{k,\lambda} \setminus \bigcup_{j<k} \bigcup_{\alpha \in \Lambda} H^1_{j,\alpha}$ for $k > 1$. Now $\{H^1_{k,\Lambda};\ \lambda \in \Lambda, k \in \mathbb{N}\}$ is disjoint. Take $y^1_{k,\lambda} \in B^1_{k,\lambda}$ and put $f_1 = y^1_{k,\lambda}$ on $H^1_{k,\lambda}$. So $\rho(f(x), f_1(x)) \leq \frac{1}{2}$ on A. Set $A_1 = \bigcup\{H^1_{k,\lambda};\ \lambda \in \Lambda, k \in \mathbb{N}\}$. It is clear that f_1 is extended Borel-measurable on A_1.

Proceeding inductively, using σ-discrete coverings $\{B^n_{k,\lambda};\ \lambda \in \Lambda,\ k \in \mathbb{N}\}$ of Y by open sets of diameter at most 2^{-n}, we obtain σ-dd families $\{H^n_{k,\lambda};\ k \in \mathbb{N},\ \lambda \in \Lambda\}$ of disjoint extended Borel sets. It can be so arranged that $\{H^{n+1}_{k,\lambda};\ k \in \mathbb{N},\ \lambda \in \Lambda\}$ will be a refinement of $\{H^n_{k,\lambda};\ k \in \mathbb{N},\ \lambda \in \Lambda\}$. Set $A_n = \bigcup\{H^n_{k,\lambda};\ k \in \mathbb{N},\ \lambda \in \Lambda\}$ and $f_n = y^n_{k,\lambda}$ on $H^n_{k,\lambda}$, where $y^n_{k,\lambda} \in B^n_{k,\lambda}$.

Thus $\{A_n\}$ is a decreasing sequence of extended Borel sets, $A_n \supset A$, and each f_n is an extended Borel-measurable mapping on A_n such that $\rho(f_n(x), f_{n+1}(x)) \leq 2^{-n+1}$ on A_{n+1}. To see this, consider a point $x \in A_{n+1}$. So $x \in H^{n+1}_{k_1,\lambda_1} \subset H^n_{k_2,\lambda_2}$ for some $k_1, k_2, \lambda_1, \lambda_2$. There is some $z \in A \cap H^{n+1}_{k_1,\lambda_1}$. For this z, $\rho(f(z), f_{n+1}(z)) \leq 2^{-n-1}$

and $\rho(f(z), f_n(z)) \leq 2^{-n}$. Since $f_n(z) = f_n(x)$ and $f_{n+1}(z) = f_{n+1}(x)$, the inequallity follows.

Put $A^* = \bigcap_n A_n$ and $F = \lim f_n$ on A^*. With the obvious modifications, it follows from [Ku, §31 VIII] that the limit of a sequence of extended Borel-measurable mappings is extended Borel-measurable. Also $F|_A = f$.

If X is completely metrizable, let $\{U_\lambda\}_{\lambda \in \Lambda}$ be a discrete family of open sets in Y. Since the union of each its subfamily is open, the union of each subfamily of $\{F^{-1}(U_\lambda)\}_{\lambda \in \Lambda}$ is extended Borel. The family $\{F^{-1}(U_\lambda)\}_{\lambda \in \Lambda}$ is disjoint and therefore, using Theorem 2 of [Ha1], it is σ-dd. Hence F is σ-discrete. □

5.4. Theorem

Let C be an arbitrary subset of a metrizable space X and f an extended Borel-measurable bi-σ-discrete map of C into a completely metrizable space Y. Then f can be extended to an extended Borel-measurable F defined on an extended Borel set $B \supset C$ so that F will be σ-dd-preserving. If X is completely metrizable, then F will be bi-σ-discrete.

Remark. In [Ha2] there is Theorem 10 on extension of bi-σ-discrete Borel isomorphisms between complete metric spaces.

Proof. Let $\tilde{X}$ be the completion of some metrization of X. According to Theorem 5.3., we find an extended Borel set $E \supset C$ in $\tilde{X}$ and an extended Borel-measurable σ-discrete extension $\tilde{f}$ of f defined on E. The graph of $\tilde{f}$ is extended Borel in $\tilde{X} \times Y$ (see Lemma 6.4. of [Ha3]).

Since f is σ-dd-preserving, the projection $\pi_Y\colon (x, f(x)) \mapsto f(x)$ is also σ-dd-preserving (see e.g. [FH2, Lemma 2.5.]). Consider $\mathcal{G}(\tilde{f})$, the graph of $\tilde{f}$, as a metric space. We find a G_δ set G in $\mathcal{G}(\tilde{f})$ with $\mathcal{G}(f) \subset G$ such that π_Y will be σ-dd-preserving on G (Theorem 5.2.). Hence G is extended Borel in the complete space $\tilde{X} \times Y$.

The projection $\pi_{\tilde{X}}$ restricted to G is one-to-one and continuous. It is also σ-dd-preserving. Indeed, $\pi_{\tilde{X}}|_G = \tilde{f}^{-1} \circ \pi_Y|_G$, where $\tilde{f}$ is σ-discrete and $\pi_Y|_G$ is σ-dd-preserving. Thus $\tilde{B} = \pi_{\tilde{X}}(G)$ is extended Borel in $\tilde{X}$ (Theorem 7.3. of [Ha3]), and $B = \tilde{B} \cap X$ is extended Borel in X.

Denote $\tilde{f}|_B$ by F. Then F is extended Borel-measurable on B. It is also σ-dd-preserving. Indeed, if $\{B_\lambda\}_{\lambda \in \Lambda}$ is a discrete family in B, then $\{(B_\lambda \times Y) \cap \mathcal{G}(F)\}_{\lambda \in \Lambda}$ is discrete in $B \times Y$, the set $f(B_\lambda)$ coincides with the Y-projection of $(B_\lambda \times Y) \cap \mathcal{G}(F)$ and this projection is σ-dd-preserving on $\mathcal{G}(F)$.

Similarly to the proof of Theorem 5.3. we observe that, if X is a completely metrizable space, then F is σ-discrete, hence bi-σ-discrete. □

5.5. Theorem

Let E be a co-Souslin subset of a completely metrizable space X and f an extended Borel-measurable bi-σ-discrete map of E into a metrizable space Y. Then there is a co-Souslin set $U \subset E$ such that $f(U) = f(E)$ and $f|_U$ is injection.

Remark. This is an analogue of a theorem of Kondô ([Ku §39 V, Remark 5]).

Proof. Let $\tilde{Y}$ be the completion of any metrization of Y. According to Theorem 5.4., we can extend f to an extended Borel-measurable $F\colon B \to \tilde{Y}$, where B is an extended Borel set and F is bi-σ-discrete. Let $Z = X \times \tilde{Y}$. According to Lemma 6.4. of [Ha3], the graph $\mathcal{G}(F)$ is extended Borel in Z. Thus the set $C = \mathcal{G}(f) = \mathcal{G}(F) \cap (E \times \tilde{Y})$ is co-Souslin in Z. A relation $\sim$ on C let be defined as follows: if $a = (a_X, a_Y) \in C$, $b = (b_X, b_Y) \in C$, then $a \sim b \Leftrightarrow a_Y = b_Y \Leftrightarrow f(a_X) = f(b_X)$. Let $A = \{(a,b) \in \mathcal{G}(F) \times \mathcal{G}(F);\, a_Y = b_Y\}$. It is clear that $\mathcal{G}(\sim) = (C \times C) \cap A$ and A is an extended Borel set in $Z \times Z$.

Now we will show that the projection p of $Z \times Z$ onto the second coordinate maps Souslin subsets of A to Souslin sets. Similarly to the proof of Theorem 4.1., $p|_A$ is composed from the projection q of A to $\tilde{Y} \times X \times \tilde{Y}$ defined by $q(a_X, b_Y, b_X, b_Y) = (b_Y, b_X, b_Y)$, and from the homeomorphism between the set $\{(b_Y, b_X, b_Y);\, b_Y \in \tilde{Y}, b_X \in X\}$ and $X \times \tilde{Y}$ defined by $h(b_Y, b_X, b_Y) = (b_X, b_Y)$.

So it suffices to investigate q. The map F is σ-dd-preserving and so is $\pi_Y\colon \mathcal{G}(F) \to \tilde{Y}$ ([FH2, Lemma 2.5.]). Since $q(a_X, b_Y, b_X, b_Y) = (\pi_{\tilde{Y}}(a_X, b_Y), b_X, b_Y)$, it follows that q is σ-dd-preserving. Since it is also continuous, it maps Souslin sets to Souslin sets ([Ha3, Theorem 7.3.]).

Thus the requirements of Theorem 3.1. are satisfied for Z, C, $\sim$, and A. So there is a co-Souslin selector S for C and $\sim$.

Similarly to the proof of Theorem 5.4., the projection π_X of $\mathcal{G}(F)$ is σ-dd-preserving. So it is an extended Borel isomorphism ([Ha3, Theorem 7.4.]). Thus it maps co-Souslin sets to co-Souslin sets, hence $U = \pi_X(S)$ is a co-Souslin set such that $f|_U$ uniformizes f. $\square$

5.6. Remark. We do not know whether it is possible to replace the requirement "f is extended Borel-measurable and bi-σ-discrete" in Theorem 5.5 by "f maps Souslin subsets of E to Souslin subsets of $f(E)$".

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