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Antiproximinal Sets in Banach Spaces

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The aim of the present paper is to survey the results known by the author on so called antiproximinal sets in Banach spaces.

Let X be a real normed space and Z a nonvoid subset of X. For $x \in X$ put

$$\begin{aligned}
d(x, Z) &= \inf \{\|x - z\| : z \in Z\} \\
P(x) = P_Z(x) &= \{z \in Z : \|x - z\| = d(x, Z)\} \\
\text{near}\,(Z) &= \{z \in Z : z \in P_Z(x) \text{ for some } x \in X\} \\
E(Z) &= \{x \in X : P(x) \neq \emptyset\} \\
U(Z) &= \{x \in x : \text{card}\, P(x) \leq 1\} \\
EU(Z) &= \{x \in X : \text{card}\, P(x) = 1\}
\end{aligned}$$

The set Z is called *proximinal* if $E(Z) = X$, *antiproximinal* if $E(Z) = Z$, *a unicity set* if $U(Z) = X$, and *Chebyshev* if $EU(Z) = X$. The points in $P_Z(x)$ are called *nearest points* to x in Z, or *elements of best approximation.* Since nearest points are boundary points of the set Z it is natural to consider the problem of existence of nearest points for closed sets so that we shall suppose in the following the set Z closed.

Remark. The term antiproximinal (antiproximal sometimes [25]) was proposed by R. Holmes [33]. I. Singer [47] proposes the term very non-proximinal which we have used in [11].

1 Support and approximation properties of convex sets

A functional $x^* \in X^*$ (X^*-the conjugate space of X) is called a *support functional* of the set $Z \subset X$ provided there exists $z \in Z$ such that

$$x^*(z) = \sup x^*(Z) \qquad \text{or} \qquad x^*(z) = \inf x^*(Z).$$

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The set of all support functionals of the set Z will be denoted by $\mathscr{S}(Z)$. If $B = B_X$ is the closed unit ball of the space X then

$$x^* \in \mathscr{S}(B) \Leftrightarrow \exists x \in B \quad \text{such that} \quad x^*(x) = \sup x^*(B) = \|x^*\|$$
$$\Leftrightarrow x^*(B) \quad \text{is closed}$$

If $x^* \neq 0$ then any element x in B verifying $x^*(x) = \|x^*\|$ must be of norm one.

Support functionals are very important tools in the study of the geometry of Banach spaces as well as in problems of best approximation and optimization.

Recal two classical results concerning the support functionals of the unit ball of a Banach space.

Theorem 1.1. (R. C. James [35, 37]) *A Banach space X is reflexive if and only $\mathscr{S}(B_X) = X^*$.*

Theorem 1.2. (E. Bishop and R. R. Phelps [7, 8]) *If Z is a closed bounded convex subset of a Banach space X then the set $\mathscr{S}(Z)$ of all support functionals of the set Z is dense in X^*.*

In particular the set $\mathscr{S}(B)$ of norm attaining functionals is dense in X^*.

The denseness property of $\mathscr{S}(B)$ in Theorem 1.1 is called *subreflexivity.*

James' theorem, Theorem 1.1, can be extended to obtain characterizations of weakly compact subsets of Banach spaces.

Theorem 1.3. (R. C. James [36], see also [28]) *A weakly closed subset Z of a Banach space X is weakly compact if and only if every continuous linear functional attains its supremum on Z.*

As the common support functionals to the unit ball and to a set Z will play a key role in the characterization of antiproximinal sets, we mention the following results:

Theorem 1.4. *Let X be a Hausdorff locally convex space.*

1. (V. Klee [38], see also [40]) *Suppose X is quasi-complete. Then X is semi-reflexive if and only if $C_1 + C_2$ is closed for any pair C_1, C_2 of closed convex sets in X.*
2. ([27]) *A convex subset with nonvoid interior C of X is open if and only if $x^*(C)$ is open, for every non-null $x^* \in X^*$.*
3. ([26, 27]) *If $B \subset X$ is absolutely convex with nonvoid interior and $C \subset X$ is convex then $B + C$ is open if and only if either $x^*(B)$ or $x^*(C)$ is open, for every non-null $x^* \in X^*$.*
4. ([26)] *Let X be a normed space, B its unit ball and $C \subset X$ convex. Then $B + C$ is open if and only if*

$$\mathscr{S}(B) \cap \mathscr{S}(C) = \{0\}.$$

The approximation properties of closed hyperplanes can be characterized in terms of the support properties of the corresponding functionals.

Theorem 1.5. *Let X be a normed space, x^* a non-null continuous linear functional on X, $a \in \mathbb{R}$, and*

$$H = \{x \in X : x^*(x) = a\}$$

a closed hyperplane determined by x^*.

(i) If $x^* \in \mathcal{S}(B_X)$ then H is proximinal.

(ii) If $x^* \in X^* \setminus \mathcal{S}(B_X)$ then H is antiproximinal.

From Theorems 1.1, 1.3 and 1.5 we obtain

Corollary 1.6. *For a Banach space X the following assertions are equivalent:*

(i) The Banach space X is reflexive.

(ii) Every weakly closed subset of X is proximinal.

(iii) Every closed hyperplane in X is proximinal.

(iv) Every closed convex subset of X is proximinal.

Nearest points in a closed convex set can be characterized also in terms of support properties.

Theorem 1.7. (see [47, p. 360]) *Let X be normed space, Z a closed convex subset of X, $x \in X \setminus Z$ and $r = d(x, Z) > 0$.*

Then $z_0 \in Z$ is a nearest point to x in Z if and only if there is a non-null functional $x^* \in X^*$ supporting at z_0 both the set Z and the closed ball $B(x, r)$, i.e. there exists $x^* \in X^*$ such that

(i) $\quad \|x^*\| = 1,$

(ii) $\quad x^*(z_0) = \sup x^*(Z),$

(iii) $\quad x^*(x - z_0) = \|x - z_0\|.$

It follows that the functional x^* from Theorem 1.7 supports the closed unit ball B_X at $(x - z_0)/\|x - z_0\|$ and the set Z at z_0.

2 Characterizations of antiproximinal sets

From Theorem 1.7 one obtains the following characterizations of antiproximinal sets:

Theorem 2.1. ([26, 27, 28]) *For a closed convex set Z of a normed space X the following assertions are equivalent:*

(i) Z is antiproximinal

(ii) $\quad \mathcal{S}(B_X) \cap \mathcal{S}(Z) = \{0\}$

(iii) $\quad Z + B_X$ *is open.*

Another useful remark is the following:

Proposition 2.2. ([27]) *Let X be a Banach space and let B denote its closed unit ball. If the set $\mathcal{S}(B)$ of all norm attaining functionals has an interior point then X contains no antiproximinal bounded closed convex sets.*

A.-M. Precupanu and T. Precupanu [44] considered a slightly more general problem. Let X be a topological vector space, $f : X \to \bar{\mathbb{R}}$, $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, an extended real valued function, and let

$$a = \inf f(X). \tag{2.1}$$

Considering the family of optimization problems

$$(P_x^f) \quad \inf_{z \in Z} f(x - z), \quad (x \in X)$$

where Z is fixed nonvoid subset of X, the set Z is called *f-antiproximinal* if the problem P_x^f has no solution, for every $x \in X$ such that $h(x) > a$, where

$$h(x) = \inf_{z \in Z} f(x - z).$$

Let also

$$\mathrm{epi}_a f = \{(x, \lambda) \in X \times \mathbb{R} : \lambda > a \quad \text{and} \quad f(x) \le \lambda\}.$$

The characterization of f-antiproximinal sets is given by:

Theorem 2.3. ([44, 45] *Let X, Z be as above and let $f : X \to \mathbb{R}$ be upper semicontinuous. Then the following assertions are equivalent:*

(i) The set Z is f-antiproximinal;

(ii) The set

$$K^f = epi_a f + Z \times \{0\} \tag{2.2}$$

is open in $X \times \mathbb{R}$.

(iii) The set

$$K_x^f = \bigcup_{z \in Z} [f(x - z), \infty[\tag{2.3}$$

is open in $\mathbb{R}$ whenever $h(x) \in\,]a, \infty[$.

If, in addition, the set K_x^f defined by (2.3) is convex then the above conditions are also equivalent to

(iv) The set

$$R_\lambda^f = Z + f^{-1}(]-\infty, \lambda]) \tag{2.4}$$

is open in X for every $\lambda > a$.

For normed spaces one obtains:

Theorem 2.4. ([45]) *Let X be a Banach, Z a nonvoid bounded closed convex subset of X and $f : X \to \mathbb{R}$ an upper semicontinuous function such that the sets K^f, R_λ^f, defined by (2.2), (2.4), respectively, are convex for every $\lambda > a$.*

If the set of all support functionals of the set $f^{-1}(]-\infty, \lambda])$ has nonvoid interior in X^*, for every $\lambda > a$ then Z is not an f-antiproximinal set.

3 Existence of antiproximinal sets

V. Klee [39] proposed the following classification of Banach spaces: a Banach space X is called of N_1-type if it contains a nonvoid proper closed antiproximinal convex set and of N_2-type if it contains a nonvoid bounded closed antiproximinal convex set. It is immediate, from James' theorem (Th. 1.1) and Theorem 1.5, that a Banach space is of N_1-type if and only if it is non-reflexive.

The characterization of Banach spaces of N_2-type is more complicated. The first example of a Banach space of N_2-type was given by M. Edelstein and A. C. Thompson:

Theorem 3.1. ([26]) *The Banach space c_0 contains an antiproximinal bounded symmetric closed convex body.*

We agree to call a bounded symmetric convex body a *convex cell.* (A convex body is a convex set with nonvoid interior). The Minkowski functional of a convex cell in a normed space X is a norm, equivalent with the original norm. The proof of the antiproximinality of a given set Z in a Banach space X is based on the characterization given in Theorem 2.1

$$\mathscr{S}(Z) \cap \mathscr{S}(B) = \{0\}$$

so that we need to know how the support functionals of the unit ball look like. Characterizations of support functionals in some concrete Banach spaces were given by R. R. Phelps [42, 43], and by the author in [15, 17]. The case of the space of continuous Banach valued functions was considered in [49] (see also [10, 48]).

Later [13], we have shown that the space c is also of N_2-type, more exactly, it contains an antiproximinal convex cell and that this property is shared by any Banach space of continuous functions which is isomorphic to c_0 ([14]). If T is a compact Hausdorff topological space such that $C(T)$ is isomorphic to c_0 then T is homeomorphic to a space of the form $[1, \alpha]$, where α is a countable ordinal (see [1] and [6, 31]). The space $c = C([1. \omega])$ is isomorphic to c_0, where ω denotes the first infinite ordinal number. Extensions of these results to more general classes of compact spaces T, including $[0, 1]^n$ and the Hilbert cube, were given by V. P. Fonf [29]. The definitive answer to the problem of the existence of antiproximinal set in Banach spaces of continuous functions on compacta was given by S. V. Balaganskii [3]:

Theorem 3.2. ([3]) *If T is an infinite compact Hausdorff space then the space $C(T)$ contains a nonvoid bounded closed antiproximinal convex body. If T is a non-compact locally compact Hausdorff space then the same is true for the space $C_0(T)$.*

There are some extensions of these results to Banach spaces of vector valued functions:

Theorem 3.3. ([16, 18, 19]) *Let X be a non-trivial Banach space. Then the Banach spaces $c_0(X)$, $c(X)$, and $C([1, \omega^k]; X)$ contain antiproximinal convex cells.*

Remark. As in the scalar case, the Banach space $C([1, \omega^k]; X)$ is isomorphic to $c_0(X)$. Concerning the space L_1 we mention the following result:

Theorem 3.4. ([12]) *Let $(S, \mathscr{A}, \mu)$ be a measure space containing at least one atom and such that $(L_1(S, \mathscr{A}, \mu))^*$ is canonically isometric to $L_\infty(S, \mathscr{A}, \mu)$.*
Then $L_1(S, \mathscr{A}, \mu)$ is not of N_2-type.

Extensions of this result to the case of f-antiproximinal sets, with an appropriate f, were given by A.-M. Precupanu and T. Precupanu [44] and A.-M. Precupanu [45].
There are also some results concerning the existence of antiproximinal closed bounded convex sets in Banach spaces whose norms satisfy some suplementary geometric conditions:

Theorem 3.5. ([11]) *The Banach space c_0 equipped with Day's locally uniformly convex norm contains an antiproximinal convex cell.*
2. ([2]) *There is an equivalent strictly convex Fréchet differentiable norm on c_0 such that the corresponding space contains an antiproximinal convex cell.*

It was shown by Ka-Sing Lau [41] that for every closed bounded subset Z of a reflexive locally uniformly convex Banach space X the set $E(Z)$ is G_δ and dense in X.
A closed nonvoid subset of a Banach space X is called *smooth* provided that for every $x \in X$, if $x^*, y^* \in S^*$ (the unit sphere of X^*) are such that $x^*(x) = \sup x^*(Z)$ and $y^*(x) = \sup y^*(Z)$ then $x^* = y^*$, where $S(X^*)$ denotes the unit sphere of the space X^*. As remarked V. S. Balaganskii [5], a Banach space contains a smooth compact convex set if and only if it is separable. Concerning the existence of smooth antiproximonal sets we mention the following results proved by V. S. Balaganskii [5]:

Theorem 3.6. *If a separable Banach space X contains a bounded closed convex antiproximinal set Y then it contains also a smooth bounded closed convex antiproximinal set Z.*

Remark. The proof given in [5] shows that if Y is symmetric then Z is also symmetric, and if Y is a convex body then Z is a convex body, too.

Taking into account the above mentioned results on the existence of antiproximinal closed bounded convex sets in concrete Banach spaces, one obtains:

Corollary 3.7. 1. *The Banach spaces $C(Q)$, for Q a metrizable compact, and $C_0(T)$, for T a metrizable locally compact space, contain nonvoid smooth bounded closed antiproximinal convex bodies.*

2. *The spaces $C([0, 1])$, c, c_0 contain smooth bounded closed symmetric antiproximinal convex bodies.*

There are also some general results concerning classes of Banach spaces which are not of N_2-type:

Theorem 3.8. *Let X be a Banach space with closed unit ball B.*

1. ([23]) *If X is a separable conjugate Banach space then X is not of N_2-type.*

2. ([9]) *If X is a space with Radon-Nikodym property then $Z =$ cl-co (near Z) for any bounded closed convex subset Z of X. If Z is weakly compact then the above equality is valid in an arbitrary Banach space X.*

3. ([27]) *If X has a reflexive l^1-factor, i.e. $X = X_1 \otimes X_2$, $\|x_1 + x_2\| = \|x_1\| + \|x_2\|$, with $X_2 \neq \{0\}$ reflexive, then* int $\mathscr{S}(B) \neq \emptyset$, *so that, by* Proposition 1.1, *the space X contains no antiproximinal bounded convex sets.*

Some interesting connections between antiproximinal sets and sets without farthest points were established by M. Edelstein [24] and V. S. Balaganskii [4]. For a nonvoid bounded subset Z of a normed space X put

$$q(x, Z) = \sup\{\|x - z\| : z \in Z\}$$
$$Q(x) = Q_Z(x) = \{z \in Z : \|x - z\| = q(x, Z)\}$$
$$F(Z) = \{x \in X : Q(x) \neq \emptyset\}.$$

The points in $Q(x)$ are called *farthest points from* x in Z, and the set Z is called *without farhest points* if $F(Z) = \emptyset$.

M. Edelstein [24] showed that the space l^1 contains a bounded closed symmetric (with respect to 0) convex set without farthest points, and that l^1 admits an equivalent renorming such that the corresponding space contain a closed antiproximinal set whose complement is bounded convex and symmetric. Extensions of these results to more general classes of Banach spaces were given by V. S. Balaganskii [4]. To present these results we need to recall some geometric properties of Banach spaces.

Let X be a Banach space. We denote by $X \in (RN)$, $X \in (WCG)$ the fact the X has the Radon-Nikodym property or that X is weakly compactly generated, respectively. Also $X \in (CLUR)$ if $x_n, x \in S$ (the unit sphere of X) and $\|x_n + x\| \to 2$ imply that (x_n) contains a convergent subsequence, and $X \in (CLD)$ provided $x_n, x \in S$, $x^* \in S(X^*)$ and $x^*(x_n) \to x^*(x)$ imply that (x_n) contains a convergent subsequence.

Concerning the relations between sets without farthest points and antiproximinal sets we mention the following fact: if $X \in (CLUR)$ and Z is a nonvoid bounded closed subset of X without farthest points then the set $\{x \in X : q(x, Z) \geq \alpha\}$ is antiproximinal for any $\alpha > 0$ ([4]). In the same paper V. S. Balaganskii has shown that

$$A = \left\{x \in L^1[0,1] : \int_0^1 (1+t)\,|x(t)|\,\mathrm{d}t \leq 1\right\}$$

is a set without farthest points in $L^1[0,1]$ and that $B = \mathrm{cl}(L^1[0,1] \setminus A)$ is antiproximinal in the same space.

We mention also some other interesting results from this paper.

Theorem 3.9. ([4]) *If $X \notin (RN)$ then it contains a bounded closed symmetric set (a convex body if, furthermore, $X \in (CLD)$) without farthest points.*

In particular:

Corollary 3.10. ([4]) 1. *If $X \in (CLD) \setminus (RN)$ then X admits a renorming such that the corresponding space contains an antiproximinal closed symmetric set whose complement is a bounded convex body.*

2. *If $X \in (WCG) \setminus (RN)$ then it admits a renorming such that the corresponding space contains a closed bounded symmetric convex body without farthest points.*

Also

Theorem 3.11. ([4]) *If $X \in (CLUR) \setminus (RN)$ then it contains a closed antiproximinal set whose complement is a closed bounded symmetric convex body.*

If $X \in (WCG) \setminus (RN)$ then the above assertion holds for an equivalent renorming of X.

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