Wolfgang Lusky
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On Approximation by Toeplitz Operators

WOLFGANG LUSKY

Paderborn*)

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We show that the set of compact Toeplitz operators is dense in the space of all compact operators for many generalized Bergman-Hardy spaces. Moreover the set of p-Schatten class Toeplitz operators is dense in the p-Schatten class with respect to the p-Schatten class norm for $p \geq 1$.

1. Introduction

We study the richness of classes of compact Toeplitz operators on generalized Bergman-Hardy spaces.

Let $\mathbb{T}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_k| = 1, k = 1, \ldots, n\}$ and let $d\varphi$ be the normalized Haar measure on $\mathbb{T}^n$. We fix a bounded positive Borel measure μ on $\mathbb{R}^n_+$ with $\operatorname{supp} \mu \cap (\text{interior of } \mathbb{R}^n_+) \neq \emptyset$ and define, for $f, g : \mathbb{C}^n \to \mathbb{C}$,

$$\langle f, g \rangle = \int_{\mathbb{R}^n_+} \int_{\mathbb{T}^n} f(r \cdot \exp(i\varphi)) \overline{g(r \cdot \exp(i\varphi))} \, d\varphi \, d\mu(r), \quad \|f\|_2 = \sqrt{\langle f, f \rangle} .$$

(Here, $r \cdot \exp(i\varphi) = (r_1 e^{i\varphi_1}, \ldots, r_n e^{i\varphi_n}) \in \mathbb{C}^n$.) Let $L_2 = L_2(d\varphi \otimes d\mu)$ be the corresponding Hilbert space (of the classes of measurable functions f with $\|f\| < \infty$). We want to consider only such μ where all polynomials on $\mathbb{C}^n$ are elements of L_2 (which is the case, for example, if μ has compact support.) Then put

$$H_2(\mu) = \text{closure of } \{p : \mathbb{C}^n \to \mathbb{C} : p \text{ a polynomial}\} \subset L_2$$

and let $P : L_2 \to H_2(\mu)$ be the orthogonal projection. Now, for $f \in L_\infty = L_\infty(d\varphi \otimes d\mu)$, we define the Toeplitz operator

$$T_f : \begin{cases} H_2(\mu) & \to \quad H_2(\mu) \\ h & \mapsto \quad P(f \cdot h) \end{cases} \quad \text{with symbol } f,$$

*) Fachbereich 17, Universität-Gesamthochschule, Warburger Strasse 100, D-33098 Paderborn, Germany

which, of course, is an element of

$$\mathscr{L} := \{T\colon H_2(\mu) \to H_2(\mu) : T \text{ linear and bounded}\}.$$

Let $\mathscr{K} = \{T \in \mathscr{L} : T \text{ compact}\}$.

It was shown in [2] that, for the measures

$$d\mu(r) = 1_{[0,1]^n}(r)\, r_1 \dots r_n\, dr_1 \dots dr_n \quad \text{(the Bergman space)}$$

and

$$d\mu = r_1\, e^{-r_1^2/2} \dots r_n\, e^{-r_n^2/2}\, dr_1 \dots dr_n \quad \text{(the Fock space)}$$

the compact Toeplitz operators are dense in $\mathscr{K}$ with respect to the operator norm. (In [2] even more general domains $\Omega \subset \mathbb{C}^n$ than polydiscs were treated.) We shall give conditions on μ which show that this result remains true in our setting for a large class of measures. Actually we show that there are more specific density theorems for certain subclasses of Toeplitz operators. In particular, $\{T_f : f \in V, T_f \text{ compact}\}$ is dense in $\mathscr{K}$ where V consists of $L_\infty(d\mu)$-valued trigonometric polynomials. Here f is called $L_\infty(d\mu)$-valued trigonometric polynomial if f has the form $f = \sum_{|k|<j} F_k \xi_k$ for some $j \in \mathbb{Z}_+$ where $F_k(z_1, \dots, z_n) = F_k(|z_1|, \dots, |z_n|)$ and $F_k \in L_\infty$, i.e. where F_k depends only on the radii (called a radial function), and

$$\xi_k(z_1, \dots, z_n) = \left(\frac{z_1}{|z_1|}\right)^{k_1} \dots \left(\frac{z_n}{|z_n|}\right)^{k_n}$$

if $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, $z_1, \dots, z_n \in \mathbb{C} \setminus \{0\}$. $|k|$ means $|k_1| + \dots + |k_n|$ (Corollary 2.7.). Put $\xi_k(z_1, \dots, z_n) = 0$ if $z_j = 0$ for some j.

On the other hand, one needs an additional condition on μ to have sufficiently many compact Toeplitz operators. If μ is the Dirac measure at $(1, \dots, 1) \in \mathbb{R}^n_+$ then $H_2(\mu)$ is the classical Hardy space on $\mathbb{T}^n$. Here it is known that $\{T_f : f \in L_\infty, T_f \text{ compact}\} = \{0\}$, so no such density theorem can hold. However, it is always possible to go over to an equivalent L_2-norm on $H_2(\mu)$ defined by a different measure μ_0 where $\{T_f : f \in L_\infty, T_f \text{ compact}\}$ is dense in $\mathscr{K}$ (Corollary 2.8.).

Moreover, we deal with $\mathscr{S}_p = \{T \in \mathscr{K} : T \text{ is of } p\text{-Schatten class}\}$ for $p \geq 1$, i.e. $T \in \mathscr{S}_p$ if there are orthonormal systems $\{g_m\}, \{h_m\}$ in $H_2(\mu)$, and $\lambda_m \in \mathbb{C}$ such that

$$Th = \sum_{m \in \mathbb{Z}_+} \lambda_m \langle h, g_m \rangle h_m, \ h \in H_2(\mu), \quad \text{and} \quad \gamma_p(T) = \left(\sum |\lambda_m|^p\right)^{1/p} < \infty.$$

We show that $\{T_f : f \in V, T_f \in \mathscr{S}_p\}$ is dense in $\mathscr{S}_p$ with respect to γ_p.

While $\{T_f : f \in L_\infty, T_f \text{ compact}\}$ is very often large the set $\{T_f : f \in L_\infty\}$ is small in comparison with $\mathscr{L}$. This is discussed in section 3.

Our considerations concentrate on such operators $T \in \mathscr{L}$ which can be approximated (with respect to the operator norm) by finite combinations of shifts and diagonal operators. In the final section 4 we give characterizations of such operators.

2. Density results

For $m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$ and $r \cdot \exp(i\varphi) = (r_1 e^{i\varphi_1}, \dots, r_n e^{i\varphi_n}) \in \mathbb{C}^n$ put

$$e_m(r \cdot \exp(i\varphi)) = \frac{r^m \xi_m(\exp(i\varphi))}{\sqrt{\int_{\mathbb{R}_+^n} r^{2m} \, d\mu}} .$$

Here $r^m = r_1^{m_1} \dots r_n^{m_n}$. Then $\{e_m\}_{m \in \mathbb{Z}_+^n}$ is an orthonormal basis of $H_2(\mu)$. For $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $h = \sum_{m \in \mathbb{Z}_+^n} \beta_m e_m \in H_2(\mu)$ put

$$S_k h = \sum_{m \geq \max(k,0)} \beta_{m-k} e_m$$

$(m \geq \max(k, 0)$ means $m_1 \geq \max(k_1, 0), \dots, m_n \geq \max(k_n, 0))$.

Now we introduce the main objects of study. At first define

$$T_{\{\alpha_k\}}\left(\sum_{m \in \mathbb{Z}_+^n} \beta_m e_m\right) = \sum_{m \in \mathbb{Z}_+^n} \alpha_m \beta_m e_m .$$

For $p \geq 1$ put

$$\mathcal{M}_p = \{T_{\{\alpha_m\}} : \{\alpha_m\} \in l_p\} \quad \text{and} \quad \mathcal{M}_0 = \{T_{\{a_m\}} : \{\alpha_m\} \in c_0\} .$$

Let $\mathcal{M}_p S_k = \{TS_k : T \in \mathcal{M}_p\}$.

2.1. Lemma. *We have*

(i) closure of span $\left(\bigcup_{k \in \mathbb{Z}^n} \mathcal{M}_0 S_k\right) = \mathcal{K}$ *(closure with respect to the operator norm), and*

(ii) γ_p*-closure of span* $\left(\bigcup_{k \in \mathbb{Z}^n} \mathcal{M}_0 S_k\right) = \mathcal{S}_p$.

Proof. (i) follows from the fact that the finite rank operators are dense in $\mathcal{K}$. To prove (ii) note that $\mathcal{M}_p S_k \subset \mathcal{S}_p$ for each k. Moreover $\mathcal{S}_p^* = \mathcal{S}_q$, if $p^{-1} + q^{-1} = 1$ and $p > 1$, and $\mathcal{S}_1^* = \mathcal{L}$ under the duality

$$\langle S, T \rangle = \sum_{m \in \mathbb{Z}_+^n} \langle TSe_m, e_m \rangle \quad ([5]) .$$

So, let $T \in \mathcal{S}_q$ if $p > 1$ or $T \in \mathcal{L}$ if $p = 1$ such that $\langle S, T \rangle = 0$ for every $S \in \mathcal{M}_p S_k$, $k \in \mathbb{Z}^n$. Fix $l, m \in \mathbb{Z}_+^n$ and put $k = l - m$, $\alpha_{m'} = \begin{cases} 1 & m' = l \\ 0 & \text{otherwise} \end{cases}$. We obtain, with $S = T_{\{a_{m'}\}} S_k$,

$$0 = \langle S, T \rangle = \langle Te_{m+k}, e_m \rangle = \langle Te_l, e_m \rangle .$$

Hence $T = 0$. The Hahn-Banach separation theorem completes the proof. □

As a direct consequence of the definitions using the orthogonality of the ξ_l we obtain (see [4])

2.2. Lemma. *Consider* $k \in \mathbb{Z}^n$, $l, m \in \mathbb{Z}_+^n$ *and a radial function* $F \in L_\infty$. *Then*

$$\langle T_{F\xi_k} e_l, e_m \rangle = \begin{cases} \dfrac{\int F r^{2m-k}\, \mathrm{d}\mu}{\sqrt{\int r^{2m}\, \mathrm{d}\mu \int r^{2m+2k}\, \mathrm{d}\mu}} & l = m - k \\ 0 & \text{otherwise} \end{cases}$$

This means $T_{F\xi_k} = T_{\{\langle T_{F\xi_k} e_{m-k}, e_m \rangle\}_{m \geq \max(k,0)}} S_k$ and hence

$$T_{F\xi_k} \in \mathcal{M}_p S_k \text{ if and only if } \sum_{m \geq \max(k,0)} \left| \frac{\int F r^{2m-k}\, \mathrm{d}\mu}{\sqrt{\int r^{2m}\, \mathrm{d}\mu \int r^{2m-2k}\, \mathrm{d}\mu}} \right|^p < \infty,$$

$$T_{F\xi_k} \in \mathcal{M}_0 S_k \text{ if and only if } \lim_{m \to \infty} \frac{\int F r^{2m-k}\, \mathrm{d}\mu}{\sqrt{\int r^{2m}\, \mathrm{d}\mu \int r^{2m-2k}\, \mathrm{d}\mu}} = 0.$$

2.3. Corollary. *Let $b \in \operatorname{supp} \mu$, $\lambda \in]0, 1[$, $k \in \mathbb{Z}^n$ and $F \in L_\infty(\mathrm{d}\mu)$. Then, for any $p \geq 1$, $T_{F1_{[0,\lambda b]}\xi_k} \in \mathcal{M}_p S_k$.*
(Here, with $b = (b_1, \ldots, b_n)$,

$$[0, \lambda b] = \{(c_1, \ldots, c_n) : 0 \leq c_j \leq \lambda b_j,\ j = 1, \ldots, n\}).$$

Proof. Put

$$B = \left\{(t_1, \ldots, t_n) : \left(\frac{1}{2} + \frac{\lambda}{2}\right) b_j < t_j,\ j = 1, \ldots, n\right\}.$$

Since $b \in B \cap \operatorname{supp} \mu$ we have $\mu(B) > 0$. Hence

$$\left| \frac{\int F 1_{[0,\lambda b]} r^{2m-k}\, \mathrm{d}\mu}{\sqrt{\int r^{2m}\, \mathrm{d}\mu \int r^{2m-2k}\, \mathrm{d}\mu}} \right| \leq \frac{\|F\|_\infty}{\mu(B)} \left(\frac{\lambda}{\frac{1}{2} + \frac{\lambda}{2}} \right)^{|2m-k|}.$$

Since $0 < \lambda(\frac{1}{2} + \frac{\lambda}{2})^{-1} < 1$ we obtain

$$\sum_{m \geq \max(k,0)} \left| \frac{\int F 1_{[0,\lambda b]} r^{2m-k}\, \mathrm{d}\mu}{\sqrt{\int r^{2m}\, \mathrm{d}\mu \int r^{2m-2k}\, \mathrm{d}\mu}} \right|^p < \infty. \qquad \square$$

2.4. Definition. μ satisfies condition ($\#$) if there are non-empty sets $I_1, \ldots, I_n \subset \mathbb{R}_+ \setminus \{0\}$ such that $I_2, \ldots, I_n$ are bounded and infinite and satisfy the following:

(i) $I_1 \times \ldots \times I_n \subset \operatorname{supp} \mu$,
(ii) for each $b \in I_1 \times \ldots \times I_n$ there is $\varepsilon \in]0, 1[$ such that cardinality of $([0, 1 - \varepsilon] \cdot b \cap \operatorname{supp} \mu) = \infty$.

To produce examples we note the straightforward

2.5. Lemma. *If $\operatorname{supp} \mu$ contains an interior point with respect to $\mathbb{R}^n$ then μ satisfies condition ($\#$).*

In particular, the measure of the Bergman space on polydiscs and the measure of the Fock space (see introduction) satisfy ($\#$). However we also find easily atomic measures satisfying ($\#$).

Now we come to the main result of this section. For $b = (b_1, \ldots, b_n) \in (\mathbb{C} \setminus \{0\})^n$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ put $b^m = b_1^{m_1} \ldots b_n^{m_n}$.

2.6. Theorem. *Assume that μ satisfies condition (#). Let $k \in \mathbb{Z}^n$.*
(i) Then,

$$\{T_{F\xi_k} : F \in L_\infty(\mathrm{d}\mu),\ \lim_{m \to \infty} \langle T_{F\xi_k} e_{m-k}, e_m \rangle = 0\}$$

is dense in $\mathscr{M}_0 S_k$ with respect to the operator norm.
(ii) For any $p \geq 1$,

$$\{T_{F\xi_k} : F \in L_\infty(\mathrm{d}\mu),\ \sum_{m \geq \max(k,0)} |\langle T_{F\xi_k} e_{m-k}, e_m \rangle|^p < \infty\}$$

is dense in $\mathscr{S}_p$ with respect to γ_p.

Proof. We proceed in two steps. At first we prove the following.
(a) Let $b \in I_1 \times \ldots \times I_n$ and $0 < \varepsilon < 1$ as in condition (#). Consider $\{\alpha_m\} \in l_\infty$ and put

$$G(r) = \sum_{m \geq \max(-k,0)} \alpha_m \frac{r^{2m+k}}{\sqrt{\int r^{2m}\,\mathrm{d}\mu \int r^{2m+2k}\,\mathrm{d}\mu}}.$$

Then, we claim, for any $\lambda \in]0, 1[$, the series defining G is uniformly convergent on $[0, \lambda b]$. Moreover, if $G 1_{[0,(1-\varepsilon)b]} = 0$ μ-a.e. then $\alpha_m = 0$ for all m.

Indeed, with $B = \{(t_1, \ldots, t_n) : (1/2 + \lambda/2)\, b_j < t_j, j = 1, \ldots, n\}$ we obtain

$$\left\| \frac{r^{2m+k}}{\sqrt{\int r^{2m}\,\mathrm{d}\mu \int r^{2m+2k}\,\mathrm{d}\mu}} 1_{[0,\lambda b]}(r) \right\|_\infty \leq \left(\frac{\lambda}{\frac{1}{2} + \frac{\lambda}{2}} \right)^{|2m+k|} \frac{1}{\mu(B)}.$$

Since $\lambda(1/2 + \lambda/2)^{-1} < 1$ and $\mu(B) > 0$ (in view of $b \in \operatorname{supp} \mu \cap B$) this implies that the series defining G is uniformly convergent. In particular G is continuous on $[0, b[$. Assume that $G 1_{[0,(1-\varepsilon)b]} = 0$ μ-a.e. Use condition (#) to find $b(m) \in [0, (1-\varepsilon)[\cdot b \cap \operatorname{supp} \mu$ with $b(m) \neq b(m')$ if $m \neq m'$. Consider open δ-balls $U_\delta(b(m))$ centered at $b(m)$ and take into account $\mu(U_\delta(b(m))) > 0$. Since $G 1_{[0,(1-\varepsilon)b]} = 0$ μ-a.e. find $b(\delta, m) \in U_\delta(b(m))$ with $G(b(\delta, m)) = 0$. We have $\lim_{\delta \to 0} b(\delta, m) = b(m)$. Hence continuity yields $G(b(m)) = 0$ for all m.

Fix ϱ_m with $0 < \varrho_m \leq 1 - \varepsilon$, $b(m) = \varrho_m b$ and $\varrho_m \neq \varrho_{m'}$ if $m \neq m'$. Put

$$g(t) = \sum_{m \geq \max(-k,0)} \alpha_m \frac{b^{2m+k}}{\sqrt{\int r^{2m}\,\mathrm{d}\mu \int r^{2m+2k}\,\mathrm{d}\mu}} t^{|2m+k|}.$$

Then g is a uniformly converging power series for $t \in [0, 1 + \varepsilon]$. Since $g(\varrho_m) = 0$ for all m we obtain

$$\sum_{|2m+k|=j} \alpha_m \frac{b^{2m+k}}{\sqrt{\int r^{2m}\,\mathrm{d}\mu \int r^{2m+2k}\,\mathrm{d}\mu}} = 0 \quad \text{for all } j \in \mathbb{Z}_+.$$

For fixed j this is true for infinitely many $b_2 \in I_2, \ldots, b_n \in I_n$, where $b = (b_1, b_2, \ldots, b_n)$. Using the identity theorem successively in each component for $b_n, b_{n-1}, \ldots, b_2$ we obttain eventually $\alpha_m = 0$ for all m with $|2m + k| = j$.

(b) Now we prove the theorem. Recall that $\mathscr{S}_p^*$ can be identified with $\mathscr{S}_q$, if $p > 1$ and $p^{-1} + q^{-1} = 1$, and $\mathscr{S}_1^* \cong \mathscr{S}_1$ (see [5]). Fix $p \geq 1$ or $p = 0$ and consider $\psi \in (\mathscr{M}_p S_k)^*$ such that $\psi(T_{F\xi_k}) = 0$ for every $F \in L_\infty(\mathrm{d}\mu)$ with $T_{F\xi_k} \in \mathscr{M}_p S_k$. By Hahn-Banach we find $T \in \mathscr{S}_p^*$ if $p \geq 1$ and $T \in \mathscr{K}^*$ if $p = 0$ with $T|_{\mathscr{M}_p S_k} = \psi$. Using the duality $\mathscr{S}_p^* \cong \mathscr{S}_q$, $\mathscr{S}_1^* \cong \mathscr{L}$ and $\mathscr{K}^* \cong \mathscr{S}_1$, we obtain with Lemma 2.2.

$$0 = \sum_{m \in \mathbb{Z}_+^n} \langle T T_{F\xi_k} e_m, e_m \rangle = \sum_{m \geq \max(-k, 0)} \frac{\int F r^{2m+k} \, \mathrm{d}\mu}{\sqrt{\int r^{2m} \, \mathrm{d}\mu \int r^{2m+2k} \, \mathrm{d}\mu}} \langle T e_{m+k}, e_m \rangle .$$

Put $\alpha_m = \langle T e_{m+k}, e_m \rangle$. Then $\{\alpha_m\} \in l_\infty$. Define

$$G(r) = \sum_{m \geq \max(-k,0)} \alpha_m \frac{r^{2m+k}}{\sqrt{\int r^{2m} \, \mathrm{d}\mu \int r^{2m+2k} \, \mathrm{d}\mu}} .$$

which, according to (a), is well-defined on $[0, b[$ for all $b \in I_1 \times \ldots \times I_n$. Take $\tilde{F} \in L_\infty(\mathrm{d}\mu)$ arbitrarily and put, for some $\lambda \in]0, 1[$, $F = \tilde{F} 1_{[0, \lambda b]}$. Then, by Corollary 2.3., $T_{F\xi_k} \in \mathscr{M}_p S_k$. We obtain

$$0 = \sum_{m \in \mathbb{Z}_+^n} \langle T T_{F\xi_k} e_m, e_m \rangle = \int_{\mathbb{R}_+^n} G 1_{[0, \lambda b]} \tilde{F} \, \mathrm{d}\mu .$$

Since $\tilde{F} \in L_\infty(\mathrm{d}\mu)$ was arbitrary we have $G 1_{0, \lambda b]} = 0$ μ-a.e. Then, in view of (a), $\langle T e_{m+k}, e_m \rangle = \alpha_\mathrm{m} = 0$ for all m. This implies $\psi = 0$ and the Hahn-Banach separation theorem proves Theorem 2.6. □

Lemma 2.1. implies

2.7. Corollary. *Let μ satisfy (#). Put*

$$V = \{f \in L_\infty : f \text{ an } L_\infty(\mathrm{d}\mu)\text{-valued trigonometric polynomial}\} .$$

(i) Then $\{T_f : f \in V, T_f \in \mathscr{K}\}$ is dense in $\mathscr{K}$ with respect to the operator norm.
(ii) For any $p \geq 1$ the set $\{T_f : f \in V, T_f \in \mathscr{S}_p\}$ is dense in $\mathscr{S}_p$ with respect to γ_p.

Recall that, for $\mu =$ the Dirac measure at $(1, \ldots, 1)$, $\{T_f : T_f \in \mathscr{K}\} = \{0\}$.However, the "richness" of $\{T_f : T_f \in \mathscr{K}\}$ does not depend on the topology of $H_2(\mu)$.

2.8. Corollary. *Let μ be any positive bounded Borel measure on $\mathbb{R}_+^n$ with* $\operatorname{supp} \mu \cap$ *interior of $\mathbb{R}_+^n \neq \emptyset$. Then there is a positive bounded Borel measure μ_0 on $\mathbb{R}_+^n$ satisfying condition (#) such that $H_2(\mu) = H_2(\mu_0)$ algebraically and topologically.*

Remark. For μ_0 the density results 2.6. and 2.7. hold.

Proof of Corollary 2.8. Fix $b = (b_1, \dots, b_n) \in \operatorname{supp} \mu \cap$ interior of $\mathbb{R}^n_+$. Let $d\mu_1 = 1_{[0,b/2]}\, dr_1 \dots dr_n$ and put $\mu_0 = \mu + \mu_1$. In view of Lemma 2.5., μ_0 satisfies (#). Let

$$B = \left\{(t_1, \dots, t_n) : \frac{1}{2} b_j < t_j,\ j = 1, \dots, n\right\}.$$

Then $\mu(B) > 0$ since $b \in B \cap \operatorname{supp} \mu$. For any polynomial h we obtain, using the maximum principle,

$$\begin{aligned}\int_{\mathbb{R}^n_+}\int_{\mathbb{T}^n} |h|^2\, d\varphi\, d\mu &\leq \int_{\mathbb{R}^n_+}\int_{\mathbb{T}^n} |h|^2\, d\varphi\, d\mu_0 \\ &\leq \int_{\mathbb{R}^n_+}\int_{\mathbb{T}^n} |h|^2\, d\varphi\, d\mu + \mu_1([0, \tfrac{1}{2}b])\left(\int_{\mathbb{T}^n} |h(\tfrac{1}{2}b \cdot \exp(i\varphi))|^2\, d\varphi\right) \\ &\leq \int_{\mathbb{R}^n_+}\int_{\mathbb{T}^n} |h|^2\, d\varphi\, d\mu + \frac{\mu_1([0, \tfrac{1}{2}b])}{\mu(B)} \int_{\mathbb{R}^n_+}\int_{\mathbb{T}^n} |h|^2\, df\, d\mu\,.\end{aligned}$$

Hence the L_2-norms with respect to $d\varphi \otimes d\mu$ and $d\varphi \otimes d\mu_0$ are equivalent. □

3. A non-density result

While $\{T_f : f \in L_\infty,\ T_f \in \mathscr{K}\}$ is often quite large the set $\{T_f : f \in L_\infty\}$ is small in comparison with $\mathscr{L}$.

For a function h on $\mathbb{C}^n$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{T}^n$ put $h_\lambda(z_1, \dots, z_n) = h(\lambda_1 z_1, \dots, \lambda_n z_n)$. For $T \in \mathscr{L}$ let T_λ be the operator with $T_\lambda h = (T h_{\bar\lambda})_\lambda$, $h \in H_2(\mu)$. Using the fact that $d\varphi$ is a Haar measure we conclude, for $f \in L_\infty$,

$$(T_f)_\lambda = T_{(f_\lambda)}\,.$$

If $k \in \mathbb{Z}^n$ define $\int_{\mathbb{T}^n} T_{\exp(i\varphi)} \xi_{-k}(\exp(i\varphi))\, d\varphi$ by

$$\left(\int_{\mathbb{T}^n} T_{\exp(i\varphi)} \xi_{-k}(\exp(i\varphi))\, d\varphi\right) h = \int_{\mathbb{T}^n} (T_{\exp(i\varphi)} h)\, \xi_{-k}(\exp(i\varphi))\, d\varphi,\ h \in H_2(\mu)\,.$$

Clearly, $T_\lambda \in \mathscr{L}$ and $\int_{\mathbb{T}^n} T_{\exp(i\varphi)} \xi_{-k}(\exp(i\varphi))\, d\varphi \in \mathscr{L}$. If $k = 0$ (i.e. $\xi_0 = 1$) and $T = T_{\{\alpha_m\}}$ for some $\{\alpha_m\} \in l_\infty$ then $\int_{\mathbb{T}^n} T_{\exp(i\varphi)}\, d\gamma = T$.

3.1. Lemma. *Let $f \in L_\infty$ and $F(r) = \int_{\mathbb{T}^n} f(r \cdot \exp(i\varphi))\, \xi_{-k}(\exp(i\varphi))\, d\varphi$. Then*

$$\int_{\mathbb{T}^n} (T_f)_{\exp(i\varphi)}\, \xi_{-k}(\exp(i\varphi))\, d\varphi = T_{F\xi_k}\,.$$

Proof. Let $h, \tilde{h} \in H_2(\mu)$. With Fubini's theorem we obtain

$$\left\langle\left(\int_{\mathbb{T}^n} (T_f)_{\exp(i\varphi)}\, \xi_{-k}(\exp(i\varphi))\, \mathrm{d}\varphi\right) h, \tilde{h}\right\rangle$$

$$= \int_{\mathbb{T}^n} \langle f h_{\exp(i\varphi)}, \tilde{h}_{\exp(-i\varphi)}\rangle\, \xi_{-k}(\exp(i\varphi))\, \mathrm{d}\varphi$$

$$= \int_{\mathbb{R}^n_+}\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} f_{\exp(i\varphi)}\xi_{-k}(\exp(i\varphi))\, h\bar{\tilde{h}}\, \mathrm{d}\varphi\, \mathrm{d}\psi\, \mathrm{d}\mu$$

$$= \int_{\mathbb{R}^n_+}\int_{\mathbb{T}^n}\left(\int_{\mathbb{T}^n} f(r\cdot\exp(i(\varphi+\psi))\, \xi_{-k}(\exp(i\varphi))\, \mathrm{d}\varphi\right) h(r\cdot\exp(i\psi))\, \overline{\tilde{h}(r\cdot\exp(i\psi))}\, \mathrm{d}\psi\, \mathrm{d}\mu$$

$$= \int_{\mathbb{R}^n_+}\int_{\mathbb{T}^n} F(r)\, \xi_k(\exp(i\psi))\, h(r\cdot\exp(i\psi))\, \overline{\tilde{h}(r\cdot\exp(i\psi))}\, \mathrm{d}\psi\, \mathrm{d}\mu = \langle T_{F\xi_k} h, \tilde{h}\rangle . \qquad \square$$

3.2. Definition. Let ν be a positive bounded Borel measure on $\mathbb{R}_+$. ν satisfies condition $(*)$ if

$$\lim_{l\to\infty}\int_{\mathbb{R}_+}\left|\frac{\varrho^{2l}}{\int_{\mathbb{R}_+}\varrho^{2l}\,\mathrm{d}\nu} - \frac{\varrho^{2l+2}}{\int_{\mathbb{R}_+}\varrho^{2l+2}\,\mathrm{d}\nu}\right| \mathrm{d}\nu = 0 .$$

Similar conditions were treated in [4]. An elementary calculation shows that $(*)$ holds if supp ν is bounded (provided that supp $\nu \neq \{0\}$). Moreover, $(*)$ holds, for example, if $\mathrm{d}\nu(\varrho) = \varrho\varepsilon^{-\varrho^2/2}\,\mathrm{d}\varrho$.

Let us return to the given measure μ on $\mathbb{R}^n_+$. We say that μ_j is a boundary measure of μ if $\mu_j(B) = \mu(\mathbb{R}^{j-1}_+ \times B \times \mathbb{R}^{n-1}_+)$ for all Borel sets $B \subset \mathbb{R}_+$.

3.3. Theorem. *Assume that μ has a boundary measure μ_j satisfying $(*)$. Let $\alpha_m = (-1)^{m_j}$, $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n_+$.*

Then $T_{\{\alpha_m\}} \notin$ closure of $\{T_f : f \in L_\infty\}$.

Proof. Assume $\|T_{\{\alpha_m\}} - T_f\| \le 1/2$ for some $f \in L_\infty$. Put $F(r) = \int_{\mathbb{T}^n} f(r\cdot\exp(i\varphi))\,\mathrm{d}\varphi$. Then $\langle T_F e_m, e_m\rangle = \left(\int F r^{2m}\,\mathrm{d}\mu\right)\left(\int r^{2m}\,\mathrm{d}\mu\right)^{-1}$ (Lemma 2.2.) and we obtain

$$\begin{aligned}\sup_m\left|\alpha_m - \frac{\int F r^{2m}\,\mathrm{d}\mu}{\int r^{2m}\,\mathrm{d}\mu}\right| &= \|T_{\{\alpha_m\}} - T_F\| \\ &= \left\|T_{\{\alpha_m\}} - \int_{\mathbb{T}^n}(T_f)_{\exp(i\varphi)}\,\mathrm{d}\varphi\right\| \\ &\le \|T_{\{\alpha_m\}} = T_f\| \le \frac{1}{2}\,.\end{aligned}$$

Put $m(l) = (\underbrace{0, \ldots, 0}_{j-1}, l, 0, \ldots, 0)$. Then we have

$$\left|\frac{\int F r^{2m(l)}\,\mathrm{d}\mu}{\int r^{2m(l)}\,\mathrm{d}\mu} - \frac{\int F r^{2m(l+1)}\,\mathrm{d}\mu}{\int r^{2m(l+1)}\,\mathrm{d}\mu}\right| \le \|F\|_\infty \int\left|\frac{\varrho^{2l}}{\int\varrho^{2l}\,\mathrm{d}\mu_j} - \frac{\varrho^{2l+2}}{\int\varrho^{2l+2}\,\mathrm{d}\mu_j}\right|\mathrm{d}\mu_j\,.$$

In view of $(*)$, it follows that

$$2 = \limsup_{l \to \infty} |\alpha_{m(l)} - \alpha_{m(l+1)}| \leq 1 ,$$

a contradiction. □

In [2] it was shown that, in the case of the Bergman space for $n = 1$ and the Fock space, even the C^*-algebra generated by $\{T_f : f \in L_\infty\}$ is not dense in $\mathscr{L}$.

4. The space $\mathscr{M}_\infty S_k$

Here we deal with $\mathscr{M}_\infty = \{T_{\{\alpha_m\}} : \{\alpha_m\} \in l_\infty\}$. We have seen that $\mathscr{M}_\infty \not\subset$ closure of $\{T_f : \in L_\infty\}$ in general. Note that $T \in \mathrm{span}(\bigcup_{k \in \mathbb{Z}^n} \mathscr{M}_\infty S_k)$ if and only if there is $j \in \mathbb{Z}_+$ such that $\langle Te_l, e_m \rangle = 0$ whenever $|l - m| > j$. For $T \in \mathscr{L}$ let

$$\sigma_j T = \sum_{|k|<j} \frac{j - |k|}{j} \int_{\mathbb{T}^n} T_{\exp(i\varphi)} \xi_{-k}(\exp(i\varphi))\, \mathrm{d}\varphi .$$

Then $\sigma_j T \in \mathrm{span}(\bigcup_{k \in \mathbb{Z}^n} \mathscr{M}_\infty S_k)$. Moreover, $(\sigma_j T_f) = T_{\sigma_j f}$ where

$$\sigma_j f = \sum_{|k|<j} \frac{j - |k|}{j} \int_{\mathbb{T}^n} f_{\exp(i\varphi)} \xi_{-k}(\exp(i\varphi))\, \mathrm{d}\varphi .$$

It is easily seen that $\sigma_j f$ is an $L_\infty(\mathrm{d}\mu)$-valued trigonometric polynomial. (See Lemma 3.1.)

Let $q : \mathscr{L} \to \mathscr{L}/\mathscr{K}$ be the quotient map.

4.1. Theorem. *The following are equivalent*

(a) $T \in$ *closure of* $\mathrm{span}(\bigcup_{k \in \mathbb{Z}^n} \mathscr{M}_\infty S_k)$.

(b) The map $\begin{cases} \mathbb{T}^n \to \mathscr{L} \\ \lambda \mapsto T_\lambda \end{cases}$ *is continuous*

(c) The map $\begin{cases} \mathbb{T}^n \to \mathscr{L}/\mathscr{K} \\ \lambda \mapsto qT_\lambda \end{cases}$ *is continuous*

(d) $\lim_{j \to \infty} \|qT - q\sigma_j T\| = 0$

(e) $\lim_{j \to \infty} \|T - \sigma_j T\| = 0$

Proof. $(a) \Rightarrow (b)$ follows from the fact that the map

$$\lambda \mapsto (T_{\{\alpha_m\}} S_k)_\lambda = (T_{\{\alpha_m\}} S_k)\, \lambda^k$$

is continuous.

$(b) \Rightarrow (c)$, $(e) \Rightarrow (a)$ and $(e) \Rightarrow (d)$ are clear. $(d) \Rightarrow (a)$ follows from the fact that $\sigma_j T \in \mathrm{span}(\bigcup_{k \in \mathbb{Z}^n} \mathscr{M}_\infty S_k)$ and $\mathscr{K} \subset$ closure of $\mathrm{span}(\bigcup_{k \in \mathbb{Z}^n} \mathscr{M}_\infty S_k)$.

$(c) \Rightarrow (a)$: By assumption the map $\lambda \mapsto qT_\lambda$ is Bochner-integrable with respect to $\mathrm{d}\varphi$. In particular, $\{qT_\lambda : \lambda \in \mathbb{T}^n\}$ is separable. Moreover, $\mathscr{K}$ is separable in view of

Lemma 2.1.(i). We conclude that $\{T_\lambda : \lambda \in \mathbb{T}^n\}$ is separable and, hence, $\lambda \mapsto T_\lambda$ is Bochner-integrable. This implies

$$\sigma_j qT := \sum_{|k|<j} \frac{j-|k|}{j} \int_{\mathbb{T}^n} qT_{\exp(i\varphi)} \xi_{-k}(\exp(i\varphi))\, d\varphi = q(\sigma_j T).$$

For any $\psi \in (\mathscr{L}/\mathscr{K})^*$, $\psi(qT_\lambda)$ is continuous in λ. We obtain

$$\sum_{|k|<j} \frac{j-|k|}{j} \int_{\mathbb{T}^n} \psi(qT_{\exp(i\varphi)})\, \xi_{-k}(\exp(i\varphi))\, d\varphi = \psi(\sigma_j qT) = \psi(q(\sigma_j T)).$$

and $\lim_{j\to\infty}\psi(qT) = \psi(qT)$. ($\psi(\sigma_j qT)$ are the "usual" Cesaro means of $\psi(qT_\lambda)$ at $\lambda = (1, \ldots, 1)$, se [3]).

By Mazur's theorem ([1]), $\lim_{l\to\infty}\|qT_l - qT\| = 0$ for suitable convex combinations T_l of the $\sigma_j T$. Since $T_l \in \operatorname{span}(\bigcup_{k\in\mathbb{Z}^n}\mathscr{M}_\infty S_k)$ this yields $qT \in q$(closure of $\operatorname{span}(\bigcup_{k\in\mathbb{Z}^n}\mathscr{M}_\infty S_k)$). Since $\mathscr{K} \subset$ closure of $\operatorname{span}(\bigcup_{k\in\mathbb{Z}^n}\mathscr{M}_\infty S_k)$ we derive (a).

$(a) \Rightarrow (e)$: Find $T_l \in \operatorname{span}(\bigcup_{k\in\mathbb{Z}^n}\mathscr{M}_\infty S_k)$ with $\lim_{l\to\infty}\|T - T_l\| = 0$. We easily obtain $\|\sigma_j(T - T_l)\| \le \|T - T_l\|$ for each j and l. Moreover, since T_l is a finite sum of operators of the form $T_{\{\alpha_m\}}S_k$, we have $\lim_{j\to\infty}\|T_l - \sigma_j T_l\| = 0$ for each l. Fix $\varepsilon > 0$, l and j_0 with

$$\|T - T_l\| \le \frac{\varepsilon}{3} \text{ and } \|\sigma_j T_l - T_l\| \le \frac{\varepsilon}{3} \text{ for } j \ge j_0.$$

Hence

$$\|T - \sigma_j T\| \le \|T - T_l\| + \|T_l - \sigma_j T_l\| + \|\sigma_j T_l - \sigma_j T\| \le \varepsilon$$

and $\lim_{j\to\infty}\|T - \sigma_j T\| = 0$. □

4.2. Corollary. *Let $f \in L_\infty$. Then the following are equivalent*

(a) $T_f \in$ *closure of* $\operatorname{span}(\bigcup_{k\in\mathbb{Z}^n}\mathscr{M}_\infty S_k)$.

(b) The map $\begin{cases} \mathbb{T}^n \to \mathscr{L} \\ \lambda \mapsto T_{f_\lambda} \end{cases}$ *is continuous*

(c) The map $\begin{cases} \mathbb{T}^n \to \mathscr{L}/\mathscr{K} \\ \lambda \mapsto qT_{f_\lambda} \end{cases}$ *is continuous*

(d) $\lim_{j\to\infty}\|qT_f - qT_{\sigma_j f}\| = 0$

(e) $\lim_{j\to\infty}\|T_f - T_{\sigma_j f}\| = 0$

Toeplitz operators satisfying Corollary 4.2. were studied in [4].

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