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Marczewski Sets in the Hashimoto Topologies for Measure and Category

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We characterize the classes (s) and (s^0) of Marczewski sets for the Hashimoto topologies associated with the ideals of meager sets and of Lebesgue null sets in $\mathbb{R}$.

1. Introduction

A perfect set in a topological space will mean a nonempty closed dense-in-ifself set. In [S], Szpirlajn (called later Marczewski) introduced the notion of a set with property (s), and of a set with property (s^0) in a separable complete metric space X. These notion are the following. A set $A \subset X$ is called an (s)-set if for each perfect set $P \subset X$ there exists a perfect set $Q \subset P$ such that either $Q \subset A$ or $Q \cap A = \emptyset$. A set $A \subset X$ is called an (s^0)-set if for each perfect set $P \subset X$ there exists a perfect set $Q \subset P$ such that $Q \cap A = \emptyset$. Obviously, every (s^0)-set is an (s)-set. In turns out [S] that the family of all (s)-sets forms a σ-algebra and the family of all (s^0)-sets forms a σ-ideal.

The above notions make sense in arbitrary topological spaces. For instance, Marczewski (s)-sets and (s^0)-sets were studied in [R] for the density topology in $\mathbb{R}$ and for the Ellentuck topology in $[\omega]^\omega$. We continue these studies and investigate Marczewski sets for the Hashimoto topologies in $\mathbb{R}$ (see [H]; in the monograph [LMZ] called "ideal topologies") associated with the σ-ideals $\mathcal{M}$ and $\mathcal{N}$ of meager sets and of Lebesgue null sets in $\mathbb{R}$. Many analogs between Lebesgue measure and the Baire category are described in [O]. There exist also dissimilarities and our results are of that kind.

If $\langle X, \tau \rangle$ is a topological space, by (s)-τ and (s^0)-τ we denote the families of all (s)-sets and (s^0)-sets with respect to the topology τ in X. If τ is fixed, we write briefly (s) and (s^0).

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Proposition 1.1. *For any topological space $\langle X, \tau \rangle$ the family (s) is an algebra of sets and the family (s^0) is an ideal of sets.*

Proof. It is obvious that (s) is stable under taking complements. Let $A, B \in (s)$ and let $P \subset X$ be a perfect set. We consider two cases.

1^0: There exists a perfect set $Q \subset P$ such that $Q \subset A$ or $Q \subset B$. Then clearly $Q \subset A \cup B$.

2^0: For each perfect set $Q \subset P$ we have $Q \not\subset A$ and $Q \not\subset B$. Since $A \in (s)$, there is a perfect set $D \subset P$ such that $D \cap A = \emptyset$. Since $B \in (s)$, there is a perfect set $E \subset D$ such that $E \cap B = \emptyset$. Thus E is a perfect subset of P and $E \cap (A \cup B) = \emptyset$.

Hence, in both cases 1^0 and 2^0 we conclude that $A \cup B \in (s)$.

It is obvious that the family (s^0) is hereditary which means that is stable under taking subsets. Let $A, B \in (s^0)$. Then we can follow the argument for case 2^0 and show that $A \cup B \in (s^0)$. □

Note that in general the family of (s^0)-sets need not be a σ-ideal. (See [R, Cor. 1.2].) For a separable complete metric space X, the family

$$H(s) = \{E \subset X : (\forall D \subset E)(D \in (s))\}$$

of all hereditary (s)-sets is identical with the family of (s^0)-sets [S, 3.1]. That is not true in general since for the Ellentuck topology on $[\omega]^\omega$ we have $(s^0) \subsetneq H(s)$ [R, Cor. 1.10].

In the sequel T denotes the natural topology in $\mathbb{R}$. By $\text{int}(E)$, $\text{cl}(E)$ and $\text{bd}(E)$ we denote (respectively) the interior, the closure and the boundary of a set $E \subset \mathbb{R}$ in the topology T. Assume that $\mathscr{I}$ is a σ-ideal of subsets of $\mathbb{R}$, containing all singletons. The topology $T^*_{\mathscr{I}}$ given by $T^*_{\mathscr{I}} = \{U \setminus A : U \in T \,\&\, A \in \mathscr{I}\}$ will be called the Hashimoto topology associated with $\mathscr{I}$. (See [H], [JH], [LMZ, Chap. 1C].) Another description of that topology can be obtained by the use of the operation $A \mapsto A^*_{\mathscr{I}}$ where $A^*_{\mathscr{I}}$ is the set of all $x \in X$ such that $U \cap A \notin \mathscr{I}$ for each neighbourhood $U \in T$ of x. Then $A \mapsto A^*_{\mathscr{I}}$ coincides with the derived set operator in $T^*_{\mathscr{I}}$. Thus a nonempty set $F \subset \mathbb{R}$ is $T^*_{\mathscr{I}}$-perfect if and only if $F = F^*_{\mathscr{I}}$. It is known and easy to check that $F^*_{\mathscr{I}}$ is T-closed. Since $\mathscr{I}$ contains all countable sets, each point of $F^*_{\mathscr{I}}$ is a T-accumulation point of F. Thus we easily obtain:

Proposition 1.2. *Let $\mathscr{I}$ be a σ-ideal of subsets of $\mathbb{R}$ containing all singletons. A set $F \subset \mathbb{R}$ is $T^*_{\mathscr{I}}$-perfect if and only if F is T-perfect and $U \cap F \notin \mathscr{I}$ for each T-open set U such that $U \cap F \neq \emptyset$.*

Remark 1.1. *It is easy to verify that one gets an equivalent definition of $A^*_{\mathscr{I}}$ and an equivalent version of Proposition 1.2 if, in their formulations, T-open sets are replaced by T-open sets from a fixed countable base of T.*

In the paper, we will characterize the families (s)-$T^*_{\mathscr{M}}$, (s^0)-$T^*_{\mathscr{M}}$, (s)-$T^*_{\mathscr{N}}$, (s^0)-$T^*_{\mathscr{N}}$. Also it will be shown that $H((s)\text{-}T^*_{\mathscr{M}}) = (s^0)\text{-}T^*_{\mathscr{M}}$ and $H((s)\text{-}T^*_{\mathscr{N}}) = (s^0)\text{-}T^*_{\mathscr{N}}$.

From Proposition 1.2 we immediately get the following:

Lemma 2.1. *A nonempty set* $F \subset \mathbb{R}$ *is* $T^*_{\mathcal{M}}$*-perfect if and only if* $F = \mathrm{cl}(\mathrm{int}(F))$.

Theorem 2.1. *For a set* $A \subset \mathbb{R}$ *the following conditions are equivalent:*
(1): $A \in (s)\text{-}T^*_{\mathcal{M}}$;
(2): *for every closed nondegenerate interval* I *there exists a closed nondegenerate interval* $J \subset I$ *such that either* $J \subset A$ *or* $J \cap A = \emptyset$;
(3): $\mathrm{int}(\mathrm{bd}(A)) = \emptyset$;
(4): *there are a* T*-open set* U *and a* T*-nowhere dense set* E *such that* $A = U \cup E$.

Proof. (1) $\Rightarrow$ (2) Let I be a closed nondegenerate interval. From Lemma 2.1 it follows that I is a $T^*_{\mathcal{M}}$-perfect set, hence there exists a $T^*_{\mathcal{M}}$-perfect set $Q \subset I$ such that either $Q \subset A$ or $Q \cap A = \emptyset$. Then it suffices to pick a closed nondegenerate interval $J \subset Q$, which is possible by Lemma 2.1.

(2) $\Rightarrow$ (3) Suppose that (2) holds true and that $\mathrm{int}(\mathrm{bd}(A)) \neq \emptyset$. Thus we pick a closed nondegenerate interval $I \subset \mathrm{bd}(A)$. Consequently, each closed nondegenerate interval $J \subset I$ intersects both sets A and $\mathbb{R} \setminus A$ which contradicts (2).

(3) $\Rightarrow$ (4) Put $U = \mathrm{int}(A)$ and $E = A \setminus \mathrm{int}(A)$. Thus $E \subset \mathrm{bd}(A)$ and therefore, by (3), the set E is T-nowhere dense.

(4) $\Rightarrow$ (1) Let $P \subset \mathbb{R}$ be a $T^*_{\mathcal{M}}$-perfect set. If $P \cap U \neq \emptyset$ then by Lemma 2.1 we have $\mathrm{int}(P) \cap U \neq \emptyset$ and so, there is a closed nondegenerate interval $Q \subset P \cap U \subset P \cap A$. If $P \cap U = \emptyset$ then there is a closed nondegenerate interval $Q \subset \mathrm{int}(P) \setminus E$ since E is T-nowhere dense. Hence $Q \subset P \setminus A$. In both cases Q is $T^*_{\mathcal{M}}$-perfect and thus $A \in (s)\text{-}T^*_{\mathcal{M}}$. □

Corollary 2.1. $H((s)\text{-}T^*_{\mathcal{M}}) = (s^0)\text{-}T^*_{\mathcal{M}}$ *and* $(s^0)\text{-}T^*_{\mathcal{M}}$ *is exactly the family of all* T*-nowhere dense sets.*

Proof. By Lemma 2.1 and arguments similar to those for (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1) we easily obtain the second assertion. Since the inclusion $(s^0)\text{-}T^*_{\mathcal{M}} \subset H((s)\text{-}T^*_{\mathcal{M}})$ is clear, it suffices to show that each set from $H((s)\text{-}T^*_{\mathcal{M}})$ is T-nowhere dense. Suppose otherwise that $E \in H((s)\text{-}T^*_{\mathcal{M}})$ and that E is dense in some interval (a, b). We shall find a set $D \subset E$ such that $D \notin (s)\text{-}T^*_{\mathcal{M}}$. This will imply that $E \notin H((s)\text{-}T^*_{\mathcal{M}})$, a contradiction. Choose a set $D \subset E$ countable and dense in (a, b). Then $D \notin (s)\text{-}T^*_{\mathcal{M}}$. Indeed, consider the $T^*_{\mathcal{M}}$-perfect set $P = [a, b]$. Then every $T^*_{\mathcal{M}}$-perfect set $Q \subset P$ is uncountable (by Proposition 1.2), hence it is not contained in D. Also, Q cannot be disjoint from D since $\mathrm{int}(Q) \neq \emptyset$ by Lemma 2.1 and thus $D \cap Q \neq \emptyset$. □

By the above corollary, $(s^0)\text{-}T^*_{\mathcal{M}}$ is not a σ-ideal. From Proposition 1.2 and the equivalence (1) $\Leftrightarrow$ (4) in Theorem 2.1 we infer that $(s)\text{-}T^*_{\mathcal{M}}$ is the smallest algebra containing T-open sets and T-nowhere dense sets. Note that (s)-$T^*_{\mathscr{M}}$ is not σ-additive since it contains all singletons but the set of all rationals does not belong to (s)-$T^*_{\mathscr{M}}$ (e.g. by (3)). In the next section we shall see that the measure case is different.

3. The measure case

The results and proofs in this section are similar to those from [R] concerning the density topology. Let λ denote Lebesgue measure on $\mathbb{R}$.

Lemma 3.1. [Bu], [R, Lemma 3.1] *A set $E \subset \mathbb{R}$ is nonmeasurable if and only if there exists a T-perfect set P of positive measure such that for every T-perfect set $Q \subset P$ of positive measure, Q intersects E and $\mathbb{R} \setminus E$.*

Lemma 3.2. *If $E \subset \mathbb{R}$ be a measurable set of positive measure, then there exists a $T^*_{\mathscr{N}}$-perfect set $Q \subset E$.*

Proof. We choose a T-closed set $P \subset E$ of positive measure. Consider a fixed countable base $\{U_n : n \in \mathbb{N}\}$ of T. Let Q consist of points $x \in P$ such that $\lambda(U_n \cap P) > 0$ for any set U_n containing x. Obviously $Q \subset E$. The set $P \setminus Q$ can be expressed as $\bigcup_{n \in K}(U_n \cap P)$, where $K = \{n \in \mathbb{N} : \lambda(U_n \cap P) = 0\}$, and so $\lambda(P \setminus Q) = 0$. Hence $\lambda(Q) = \lambda(P)$. From Proposition 1.2 and Remark 1.1 it follows that Q is a $T^*_{\mathscr{N}}$-perfect set. □

Theorem 3.1. *The family (s)-$T^*_{\mathscr{N}}$ equals the σ-algebra $\mathscr{L}$ of all Lebesgue measurable sets in $\mathbb{R}$.*

Proof. Suppose that $E \in (s)$-$T^*_{\mathscr{N}} \setminus \mathscr{L}$. By Lemma 3.1 there is a T-perfect set P of positive measure such that for every T-perfect set $D \subset P$ of positive measure, D intersects E and $\mathbb{R} \setminus E$. Choose a $T^*_{\mathscr{N}}$-perfect set $Q \subset P$, using Lemma 3.2. Since $E \in T^*_{\mathscr{N}}$, there exists a $T^*_{\mathscr{N}}$-perfect set $D \subset Q$ such that $D \subset E$ or $D \cap E = \emptyset$. But, by Proposition 1.2, the set D is T-perfect of positive measure and so we get a contradiction with the above statement derived from Lemma 3.1.

Suppose now that $E \in \mathscr{L}$. Let P be a $T^*_{\mathscr{N}}$-perfect set. If $\lambda(P \cap E) > 0$ then by Lemma 3.2 we find a $T^*_{\mathscr{N}}$-perfect set $Q \subset P \cap E$. If $\lambda(P \cap E) = 0$ then $\lambda(P \setminus E) > 0$ and by Lemma 3.2 we find a $T^*_{\mathscr{N}}$-perfect set $Q \subset P \setminus E$. Thus $E \in (s)$-$T^*_{\mathscr{N}}$. □

Corollary 3.1. $H((s)\text{-}T^*_{\mathscr{N}}) = \mathscr{N} = (s^0)\text{-}T^*_{\mathscr{N}}$.

Proof. It is known that $H(\mathscr{L}) = \mathscr{N}$. Hence the first equality holds, by Theorem 3.1. The inclusion (s^0)-$T^*_{\mathscr{N}} \subset H((s)$-$T^*_{\mathscr{N}})$ is obvious. Conversely, suppose that $E \in H((s)\text{-}T^*_{\mathscr{N}}) = \mathscr{N}$. Thus $\lambda(E) = 0$ and for each $T^*_{\mathscr{N}}$-perfect set $P \subset \mathbb{R}$ we choose (by Lemma 3.2) a $T^*_{\mathscr{N}}$-perfect set $Q \subset P \setminus E$ which witnesses that $E \in (s^0)$-$T^*_{\mathscr{N}}$. □

Finally, let us mention that our results can be stated in a more general fashion. For instance, Theorem 2.1 works for an arbitrary regular Baire space (a closed interval should be replaced by the closure of an open set). The referee has pointed out that our Lemma 3.2 uses a particular case of an abstract theorem concerning unions of open null sets [O, Th. 16.4]. In fact, Lemmas 3.1, 3.2 and Theorem 3.1 are valid for any regular τ-additive measure on a Hausdorff space.

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References

[Bu] BURSTIN, C., *Eigenschaften messbaren und nichtmessbaren Mengen*, Wien Ber. **123** (1914), 1525–1551.

[H] HASHIMOTO, H., *On the *topology and its application*, Fund. Math. **91** (1976), 5–10.

[JH] JANKOVIČ, D. and HAMLETT, T. R., *New topologies from old via ideals*, Amer. Math. Monthly, **97**, No 4, April 1990, 295–310.

[LMZ] LUKEŠ, J., MALÝ, J. and ZAJÍČEK, L., *Fine Topology Methods in Real Analysis and Potential Theory*, Lecture Notes in Math. 1189, Springer, New York, 1986.

[O] OXTOBY, J. C., *Measure and Category*, Springer, New York, 1971.

[R] REARDON, P., *Lebesgue and Marczewski sets and the Baire property*, Fund. Math. **149** (1996), 191–203.

[S] SZPILRAJN, E., *Sur une classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles*, Fund. Math. **24** (1935), 17–34.