Ryszard Grząślewicz

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R. GRZASLEWICZ

Wroclaw*)

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In the paper are discussed the geometric properties of the unit ball of the space of operators acting on a Hilbert space. In particular we consider vertices of the unit ball of the space of operators acting on a Hilbert space. The compact and nuclear operators are considered, too.

1 Introduction

Let H be a (real or complex) Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$. By $\mathsf{B}(H)$ and $\mathsf{S}(H)$ we denote the unit ball and sphere of H, respectively. By $\mathscr{K}(H)$, $\mathscr{N}(H)$, $\mathscr{L}(H)$ we denote the space of compact, nuclear and bounded operators acting on H, respectively. The spaces $\mathscr{K}(H)$ and $\mathscr{L}(H)$ are equipped with the standard operator norm and $\mathscr{N}(H)$ is equipped with the nuclear norm $\|\cdot\|_1$. Obviously, $\mathscr{K}(H) = \mathscr{L}(H)$ if and only if $\dim H < \infty$. Note that $\mathscr{K}(H)^* = \mathscr{N}(H)$ and $\mathscr{N}(H)^* = \mathscr{L}(H)$.

For $\mathbf{y}, \mathbf{z} \in H$ we denote by $\mathbf{y} \otimes \mathbf{z}$ the one dimensional operator defined by $(\mathbf{y} \otimes \mathbf{z})(\mathbf{x}) = \mathbf{y}\langle\mathbf{x},\mathbf{z}\rangle$, $\mathbf{x} \in H$. Note that each $T \in \mathscr{K}(H)$ has a representation $T = \sum_i \alpha_i \mathbf{e}_i \otimes \mathbf{f}_i$, with $\|T\| = \sup_i \alpha_i$, where $\{\alpha_i\} \in \mathbf{c}_o$, $\alpha_i > 0$ and $\{\mathbf{e}_i\}$, $\{\mathbf{f}_i\}$ are othonormal systems (not necessarily basis) in H. Moreover, each $T \in \mathscr{N}(H)$ has a representation $T = \sum_i \alpha_i \mathbf{e}_i \otimes \mathbf{f}_i$, with $\|T\|_1 = \sum_i \alpha_i$, where $(\alpha_i) \in l_1$, $\alpha_i > 0$ and $\{\mathbf{e}_i\}$, $\{\mathbf{f}_i\}$ are othonormal systems in H (see for example [7], [8]).

The set of extreme points of the unit ball $\mathsf{B}(\mathscr{L}(H))$ of the space of operators on H coincides with the set of all isometries and coisometries ([5], [1]). Recall that T is coisometry if and only if T^* is an isometry. Note that if H is separable then each extreme point of $\mathsf{B}(\mathscr{L}(H))$ is exposed ([2]). If H is infinite dimensional then the unit ball $\mathsf{B}(\mathscr{K}(H))$ of $\mathscr{K}(H)$ has no extreme points. Point out that the smooth

*) Institute of Mathematics, Politechnika, Vyb. Wyspiańskigo 27, PL-50-370 Wroclaw, Poland

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points of $\mathsf{B}(\mathscr{K}(H))$ and $\mathsf{B}(\mathscr{L}(H))$ are the operators which attain their norm at exactly one linearly independent vector ([4], [6]).

The extreme points of the unit ball $\mathsf{B}(\mathscr{N}(H))$ of $\mathscr{N}(H)$ are of the form $\mathbf{x} \otimes \mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in H$ have norm equal to 1. The extreme points coincide with strongly exposed points. Moreover, the smooth points of $\mathsf{B}(\mathscr{N}(H))$ form the set $\{T \in \mathscr{N}(H) : \|T\|_1 = 1 \text{ and } T \text{ or } T^* \text{ is one-to-one}\}$ ([4]).

A self adjoint operator $T \in \mathscr{L}(H)$ is called *positive semidefinite* if $\langle T\mathbf{x}, \mathbf{x}\rangle \geq 0$ for all $\mathbf{x} \in H$. The set of all positive semidefinite operators in $\mathscr{L}(H)$ will be denoted by $\mathscr{L}_+(H)$ and we put $\mathsf{B}_+(\mathscr{L}(H)) = \mathscr{L}_+(H)) \cap \mathsf{B}(\mathscr{L}(H))$.

The main purpose of this paper is to discuss geometric properties of the unit ball of operators acting on a Hilbert space H. We consider also positive part of the unit ball.

2 Remarks on the geometry of $\mathscr{L}(\mathbb{R}^2)$

Let $\mathbb{R}^2$ be equipped with the Euclidean norm. The space of operators $\mathscr{L}(\mathbb{R}^2)$ we can identified with the set of all 2×2 matrices. The space $\mathscr{L}(\mathbb{R}^2)$ is a 4 dimensional Banach space with the algebraic basis $\{I, J, K, L\}$, where

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \; K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \; L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We have

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha I + \beta J + \gamma K + \delta L = \begin{bmatrix} \alpha + \gamma & \delta - \beta \\ \delta + \beta & \alpha - \gamma \end{bmatrix}.$$

where $\alpha = \frac{a+d}{2}$, $\beta = \frac{c-b}{2}$, $\gamma = \frac{a-d}{2}$, $\delta = \frac{b+c}{2}$. The norm of any operator T in $\mathscr{L}(\mathbb{R}^2)$ can be in particular calculated as a square root of the maximal eigenvalue of T^*T. After calculation we get

$$\|T\| = \sqrt{\alpha^2 + \beta^2} + \sqrt{\gamma^2 + \delta^2} = \frac{1}{2}\left(\sqrt{(a+d)^2 + (b-c)^2} + \sqrt{(a-d)^2 + (b+c)^2}\right)$$

Therefore $\mathscr{L}(\mathbb{R}^2)$ is norm isomorphic ($=$ isometricaly isomorphic) to $l_2^2 \oplus_1 l_2^2$. The set $\operatorname{ext} \mathsf{B}(\mathscr{L}(\mathbb{R}^2))$ coincides with the set of all isometries ($=$ unitary operators in this case) and consists of two disjoint circles:

$$\mathscr{R} = \left\{\cos\varphi I + \sin\varphi J = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} : \varphi \in [0, 2\pi)\right\} \quad (\textit{rotations})$$

and

$$\mathscr{S} = \left\{\cos\varphi K + \sin\varphi L = \begin{bmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{bmatrix} : \varphi \in [0, 2\pi)\right\}$$

(*rotations composed with symmetry*).

This isometries have determinant equal to 1 or -1, respectively. Each element of the unit sphere of $\mathscr{L}(\mathbb{R}^2)$ is a convex combination of rotation and rotation composed with symmetry.

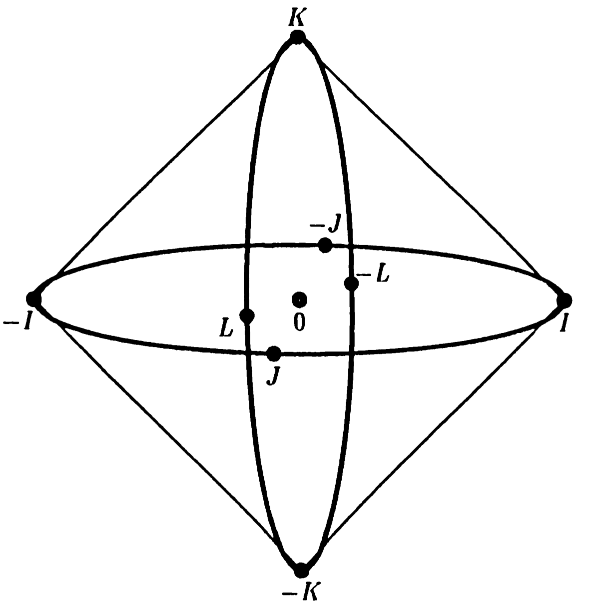

Fig. 1

Therefore the unit ball of $\mathscr{L}(\mathbb{R}^2)$ we can imagine as a convex hull of two disjoint circles in $\mathbb{R}^2$ with center at $\mathbf{0}$ situated in such a way that its border consists of intervals (one dimensional faces) with ends points in both circles. Moreover, there are no two and three dimensional faces in it (see Fig. 1). Note that all points from the border, except the elements of the both circles are smooth points. The elements of the circles are exposed. It is worth here to notice that in complex case extreme points (= unitary operators) form connected set. If we fix $R \in \mathscr{R}$ and $S \in \mathscr{S}$ then the interval $\{\kappa R + (1-\kappa) S : \kappa \in [0,1]\}$ lies on the unit sphere and $\det(\kappa R + (1-\kappa) = 2\kappa - 1$. This means that the middle point of this interval has determinant equal to 0 and it is a one dimensional operator. Clearly each one dimensional operator of norm one is a middle point of some interval from the sphere.

Put $\mathbf{e}_\alpha = \cos\alpha \mathbf{e}_1 + \sin\alpha \mathbf{e}_2$. Each one dimensional projection is of the form

$$\mathbf{e}_\alpha \otimes \mathbf{e}_\alpha = \begin{bmatrix} \cos^2\alpha & \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha & \sin^2\alpha \end{bmatrix} = \frac{1}{2}\left(I + \cos 2\alpha K + \sin 2\alpha L\right).$$

Thus the positive semidefinite part of the unit ball is a three dimensional convex hull of the operators I, 0 and of the circle $\frac{1}{2}(I + \cos\varphi K + \sin\varphi L)$, $\varphi \in [0, 2\pi)$

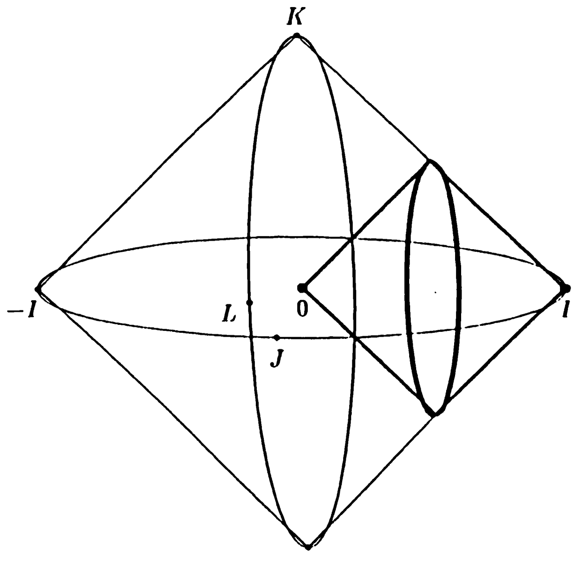

Fig. 2

which consists of the middle points of all intervals joining I with the circle of operators representing rotations composed with symmetry (see Fig. 2 and 3). In fact it is a circle $\mathcal{S}$ reduced two times (with radius equal to $\frac{1}{2}$) shifted in such a way that its center is at $\frac{1}{2}I$. Note the all extreme points are exposed and additionally 0, I are vertices (see Section 3).

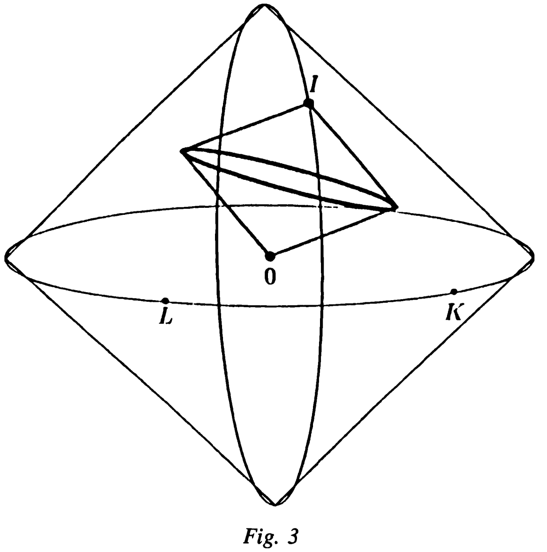

Fig. 3

The set of positive semidefinite operators forms a three dimensional cone with the vertex 0 and with the base formed by the circle $I + \mathscr{S}$. The line crossing O and I is a symmetry axis of this cone. Extreme rays of this cone (see Fig. 4) are half lines induced by positive one dimensional operators (orthogonal projections multiplied by positive reals).

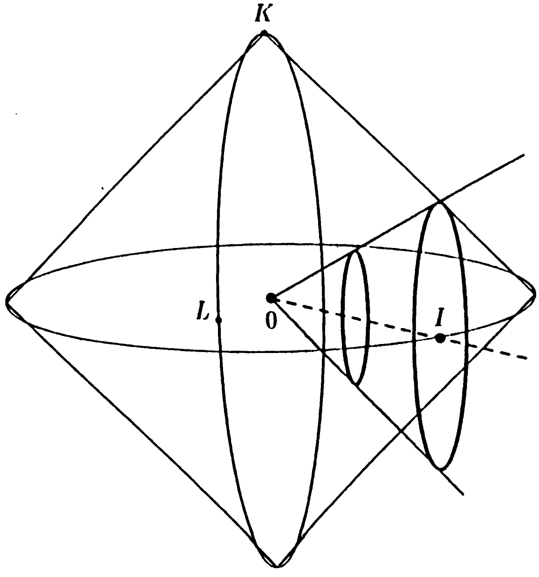

Fig. 4

3 Vertices

Recall that a point $\mathbf{q}_o$ of a convex subset Q of a normed linear space E is called a *vertex* of Q if $\{\xi \in E^* : \xi(\mathbf{q}_o) = \sup Re\xi(Q)\}$ is total over E. In other words supporting Q functionals at $\mathbf{q}_o$ separate vectors from E (or equivalently $\xi(\mathbf{x}) = 0$ for all supporting ξ implies that $\mathbf{x} = 0$). The set of all vertices of Q we denote by $\mathsf{vertex}\, Q$. We have $\mathsf{vertex}\, Q \subseteq \mathsf{ext}\, Q$ because if a functional ξ supports a convex set Q at $\mathbf{q}_o = \frac{1}{2}(\mathbf{x} + \mathbf{y})$, $\mathbf{x}, \mathbf{y} \in Q$ then $\xi(\mathbf{x}) = \xi(\mathbf{y})$ and $\xi(\mathbf{x} - \mathbf{y}) = 0$.

Theorem 1. *If* $\dim H \geq 2$ *then* $\mathsf{vertex}\, \mathsf{B}(\mathscr{L}(H)) = \emptyset$.

Proof. Fix an orthonormal base $\{\mathbf{e}_i\}$ of H. Let T be an isometry. And let a functional ξ supports $\mathsf{B}(\mathscr{L}(H))$ at T. Put $R = T\mathbf{e}_2 \otimes \mathbf{e}_1 - T\mathbf{e}_1 \otimes \mathbf{e}_2$. It is not difficult to check that $\|T + \lambda R\| = \sqrt{1 + \lambda^2}$, $\lambda \in \mathbb{R}$. Thus $\xi(T + \lambda R) \leq \sqrt{1 + \lambda^2}\, \xi(T)$. Hence $\lambda\xi(R) \leq (\sqrt{1 + \lambda^2} - 1)\, \xi(T)$ for all $\lambda \in \mathbb{R}$. This implies that $\xi(R) = 0$. Which means that functionals supporting $\mathsf{B}(\mathscr{L}(H))$ at T are not total over $\mathscr{L}(H)$. For coisometries we use the same arguments. □

In the proof of Theorems 2 and 3, we denote by $\mathbf{x} \otimes \mathbf{y}$ a functional on $\mathscr{L}(H)$ defined by $(\mathbf{x} \otimes \mathbf{y})(T) = \langle T\mathbf{x}, \mathbf{y} \rangle$ for $\mathbf{x}, \mathbf{y} \in H$.

Theorem 2. $\operatorname{vertex} \mathscr{L}_+(H) = \operatorname{vertex} \mathscr{K}_+(H) = \operatorname{vertex} \mathscr{N}_+(H) = \{0\}$ *for any Hilbert space H.*

Proof. Obviously a functional $\mathbf{x} \otimes \mathbf{x}$ supports $\mathscr{L}_+(H)$ at 0. Let $\mathbf{x}, \mathbf{y}$ be orthonormal. We claim that a functional $\eta = -\mathbf{x} \otimes \mathbf{x} - \mathbf{y} \otimes \mathbf{y} + \lambda \mathbf{x} \otimes \mathbf{y}$, $|\lambda| = 1$ supports $\mathscr{L}_+(H)$ at 0. Indeed, suppose that $T \in \mathscr{L}_+(H)$. Put $t_{uv} = \langle T\mathbf{u}, \mathbf{v} \rangle$. We have $|t_{xy}|^2 \le t_{xx} t_{yy}$. Thus $-t_{xx} - t_{yy} + |t_{xy}| = -(\sqrt{t_{xx}} - \sqrt{t_{yy}})^2 - 2\sqrt{t_{xx} t_{yy}} + |t_{xy}| \le 0$. Therefore $\eta(T) \le -t_{xx} - t_{yy} + |t_{xy}| \le 0 = \eta(0)$, which proves our claim. Now suppose that $T \in \mathscr{L}(H)$ be such that $\xi(T) = 0$ for all functionals ξ supporting $\mathscr{L}_+(H)$ at 0. By the above $t_{xx} = 0$ and $t_{xy} = 0$, so $T = 0$. Thus supporting $\mathscr{L}_+(H)$ až 0 functionals forms a total set over $\mathscr{L}_+(H)$ and 0 is a vertex. For compact and nuclear operators the proof is the same. □

Theorem 3. $\operatorname{vertex} \mathsf{B}_+(\mathscr{L}(H)) = \{0, I\}$ *for any Hilbert space H.*

Proof. Let P be a projection different from 0 and I. Choose unit vectors $\mathbf{x} \in P(H)$, $\mathbf{y} \in \operatorname{Ker} P$. Obviously $\mathbf{x} \perp \mathbf{y}$. Suppose that a functional ξ supports $\mathsf{B}_+(\mathscr{L}(H))$ at P. Put $R = \frac{1}{2}(\mathbf{x} \otimes \mathbf{y} + \mathbf{y} \otimes \mathbf{x})$ and $S = \mathbf{x} \otimes \mathbf{x} - \mathbf{y} \otimes \mathbf{y}$. Note that $P - \frac{1}{2}S \in \mathsf{B}_+(\mathscr{L}(H))$ and $\frac{1}{2}\xi[S + (2P - S)] = \xi(P) \ge \xi(P - \frac{1}{2}S)$ which implies that $\xi(S) \ge 0$. It is not difficult to check that $T = P + \frac{1}{2}(\cos\alpha - 1) + \frac{1}{2}\sin\alpha R \in \mathsf{B}_+(\mathscr{L}(H))$ for all $\alpha \in [0, 2\pi)$. We have $\xi(T) \le \xi(P)$. Thus $\sin\alpha\xi(R) \le (1 - \cos\alpha)\,\xi(S)$. This implies that $\xi(R) = 0$. Which means that the set of supporting functionals is not total.

Now consider the identity operator I. Fix an orthonormal base $\{\mathbf{e}_i\}$ of H. Let $i \neq j$. Obvisously a functional $\xi = \mathbf{e}_i \otimes \mathbf{e}_i + \mathbf{e}_j \otimes \mathbf{e}_j$ supports $\mathsf{B}_+(\mathscr{L}(H))$ at I. We claim that a functional $\eta = \xi + \lambda \mathbf{e}_i \otimes \mathbf{e}_j$, $|\lambda| = 1$ supports $\mathsf{B}_+(\mathscr{L}(H))$ at I, too. Indeed, for a positive contraction $T = (t_{ji})$ we have $t_{ii} + t_{jj} + \sqrt{4|t_{ij}|^2 + (t_{ii} - t_{jj})^2} \le 2$. Hence $t_{ii} + t_{jj} + |t_{ij}| \le 2$. We have $\eta(T) \le t_{ii} + t_{jj} + |t_{ij}| \le 2 = \eta(I)$. Suppose that $\xi(T) = \eta(T) = (\mathbf{x} \otimes \mathbf{x})(T) = 0$ for some $T \in \mathscr{L}(H)$. Then $t_{ij} = \eta(T) - \xi(T) = 0$ and $t_{ii} = (\mathbf{e}_i \otimes \mathbf{e}_i)(T) = 0$. This implies that $T = 0$. Hence supporting $\mathsf{B}_+(\mathscr{L}(H))$ functionals at I form a total set over $\mathscr{L}(H)$ and $I \in \operatorname{vertex} \mathsf{B}_+(\mathscr{L}(H))$. By Theorem 1, $0 \in \operatorname{vertex} \mathsf{B}_+(\mathscr{L}(H))$. □

Using the same arguments we can get the next result.

Theorem 4. *If* $\dim H \ge \aleph_o$ *then* $\operatorname{vertex} \mathsf{B}_+(\mathscr{K}(H)) = \{0\}$.

Theorem 5. *If* $\dim H \ge 2$ *then* $\operatorname{vertex} \mathsf{B}(\mathscr{N}(H)) = \emptyset$ *and* $\operatorname{vertex} \mathsf{B}_+(\mathscr{N}(H)) = \{0\}$.

Proof. Fix $\mathbf{x}, \mathbf{y} \in \mathsf{S}(H)$. Thus $\mathbf{x} \otimes \mathbf{y}$ is a general element of $\operatorname{ext} \mathsf{B}(\mathscr{N}(H))$. Choose $\mathbf{u} \in \mathsf{S}(H)$ with $\mathbf{u} \perp \mathbf{x}$. Suppose that a functional ξ supports $\mathsf{B}(\mathscr{N}(H))$ at $\mathbf{x} \otimes \mathbf{y}$. Then $(\cos\alpha\mathbf{x} + \sin\alpha\mathbf{u}) \otimes \mathbf{y} \in \mathsf{B}(\mathscr{N}(H))$, $\alpha \in [0, 2\pi)$. Hence $\xi[(\cos\alpha\mathbf{x} + \sin\alpha\mathbf{u}) \otimes \mathbf{y}] \le \xi(\mathbf{x} \otimes \mathbf{y})$. Thus $\sin\alpha\xi(\mathbf{u} \otimes \mathbf{y}) \le (1 - \cos\alpha)\,\xi(\mathbf{x} \otimes \mathbf{y})$. This implies that $\xi(\mathbf{u} \otimes \mathbf{y}) = 0$ and $(\mathbf{x} \otimes \mathbf{y}) \notin \operatorname{vertex} \mathsf{B}(\mathscr{N}(H))$.

By this same argument $(\mathbf{x} \otimes \mathbf{x}) \notin \text{vertex}\, \mathsf{B}_+(\mathcal{N}(H))$ and, by Theorem 2, $0 \in \text{vertex}\, \mathsf{B}_+(\mathcal{N}(H))$. □

Remark. Theorem 1 breaks down Example 6 (§13 D) in [3], p. 81.

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