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A Note on Almost Disjoint Refinement

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We shall consider the old problem whether every nowhere dense subset of ω^* is a $\mathfrak{c}$-set. It is known that both of $\mathfrak{a} = \mathfrak{c}$ and $\mathfrak{b} = \mathfrak{d}$ solve the problem in affirmative. We show that the assumption $\mathfrak{d} \leqslant \mathfrak{a}$ is sufficient for the positive answer, too.

In 1978, S. H. Hechler asked whether every nowhere dense subset D of the space $\beta\omega\setminus\omega$ admits a pairwise disjoint family $\mathscr{U}$ of open subsets of $\beta\omega\setminus\omega$ such that $|\mathscr{U}| = \mathfrak{c}$ and $D \subseteq \bar{U}$ for every $U \in \mathscr{U}$ (i.e., D is a $\mathfrak{c}$-set of ω^*) [He]. Since then, many partial results were obtained [CH, Ro, BV, BS], but the definitive solution is still missing. The aim of the present paper is to prove Hechler's conjecture under a set-theoretical assumption $\mathfrak{d} \leqslant \mathfrak{a}$ and to give another equivalent formulation of it.

The notation used in the paper is the standard one. If A is a set and κ is a cardinal, then $[A]^\kappa = \{M \subseteq A : |M| = \kappa\}$, similarly for $[A]^{<\kappa}$, $[A]^{\leqslant\kappa}$. A family $\mathscr{C} \subseteq [\omega]^\omega$ is called *almost disjoint*, if for every two distinct $A, B \in \mathscr{C}$ one has $A \cap B$ finite. A MAD family on ω is an almost disjoint family, which is maximal with respect to inclusion. To avoid trivialities, we always assume that a MAD family is infinite. As usually adopted, $A \subseteq^* B$ means $|A\setminus B| < \omega$, $A =^* B$ means $|(A \cup B)\setminus(A \cap B)| < \omega$. For two MAD families $\mathscr{A}$, $\mathscr{B}$ we shall write $\mathscr{B} \prec \mathscr{A}$ if for every $B \in \mathscr{B}$ there is some $A \in \mathscr{A}$ with $B \subseteq^* A$. The set ${}^\omega\omega$ of all mappings from ω to ω will be considered with the order $\leqslant^*$ defined by $f \leqslant^* g$ if the set $\{n \in \omega : f(n) > g(n)\}$ is finite.

For the reader's convenience, let us remind a few of so called small cardinals (cf. [vD], [Va]). A *tree π-base of* ω^* is a family Θ so that every member of Θ is a MAD family on ω, Θ is well-ordered by $\succ$ and for every $M \in [\omega]^\omega$ there is some $Q \in \bigcup\Theta$ with $Q \subseteq M$. For the existence, see [BPS].

$\mathfrak{h} = \min\{|\Theta| : \Theta \text{ is a tree } \pi\text{-base for } \omega^*\}$;

$\mathfrak{s} = \min\{|\mathscr{X}| : \mathscr{X} \subseteq [\omega]^\omega \;\&\; (\forall M \in [\omega]^\omega)(\exists X \in \mathscr{X})\, |M \cap X| = \omega = |M\setminus X|\}$;

$\mathfrak{b} = \min\{|F| : F \subseteq {}^\omega\omega \;\&\; (\forall g \in {}^\omega\omega)(\exists f \in F)\, f \not\leqslant^* g\}$;

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$\mathfrak{d} = \min\{|D| : D \subseteq {}^{\omega}\omega \;\&\; (\forall g \in {}^{\omega}\omega)(\exists f \in D)\, g \leqslant^* f\}$;
$\mathfrak{a} = \min\{|\mathscr{A}| : \mathscr{A}$ is an infinite MAD family on $\omega\}$.

It is well-known that the following inequalities can be proved: $\mathfrak{h} \leqslant \mathfrak{b} \leqslant \mathfrak{a} \leqslant \mathfrak{c}$, $\mathfrak{h} \leqslant \mathfrak{s} \leqslant \mathfrak{d} \leqslant \mathfrak{c}$, $\mathfrak{b} \leqslant \mathfrak{d}$, and any sharp inequality as well as equality is consistent with ZFC. For the details and proofs, see [vD] and [Va].

Notation. Let $\mathscr{C}$ be an infinite almost disjoint family. Let us denote by $\mathscr{I}^+(\mathscr{C})$ the family $\{M \in [\omega]^\omega : |\{C \in \mathscr{C} : |M \cap C| = \omega\}| \geqslant \omega\}$.

Let $\mathscr{F} \subseteq [\omega]^\omega$ be a $\supseteq^*$-decreasing infinite family. We shall denote $bd(\mathscr{F}) = \{M \in [\omega]^\omega : (\forall F \in \mathscr{F})(\exists H \in \mathscr{F})\, |M \cap F \setminus H| = \omega\}$.

Definition. [ES] An almost disjoint family $\mathscr{C}$ on ω is called *completely separable*, if for every $M \in \mathscr{I}^+(\mathscr{C})$ there is some $C \in \mathscr{C}$ with $C \subseteq M$.

Definition. Let $\mathscr{W} \subseteq [\omega]^\omega$. We shall say that $\mathscr{W}$ *has an almost disjoint refinement*, if there is an almost disjoint family $\mathscr{B}$ on ω such that for every $W \in \mathscr{W}$ there is some $B \in \mathscr{B}$ with $B \subseteq W$.

It turns out that the notions just introduced allow one to reformulate the above mentioned topological statement concerning nowhere dense subsets of $\beta\omega \setminus \omega$ to a purely combinatorial statement: Every nowhere dense subset of ω^* is a $\mathfrak{c}$-set if and only if for every MAD family $\mathscr{A}$ on ω, $\mathscr{I}^+(\mathscr{A})$ has an almost disjoint refinement.

Our aim is to show first that assuming $\mathfrak{d} \leqslant \mathfrak{a}$, Hechler's conjecture holds. Before doing so, we shall prove two auxiliary lemmas. The forthcoming Lemma 1 is slightly more complicated that the similar one proved in [BDS] or [BS].

Lemma 1. *Let $\mathscr{A}$ be an infinite MAD family on ω, let $\mathscr{F} \subseteq [\omega]^\omega$ be a countable decreasing family of sets such that $\mathscr{F} \subseteq \mathscr{I}^+(\mathscr{A})$, let $M \in bd(\mathscr{F})$.*

Then there is a family $\{H_\alpha : \alpha < \mathfrak{b}\} \subseteq [\omega]^\omega$ such that:

(i) *For each $\alpha < \mathfrak{b}$ and each $F \in \mathscr{F}$, $H_\alpha \subseteq^* F$;*

(ii) *whenever $\alpha < \beta < \mathfrak{b}$, then $H_\alpha \subseteq^* H_\beta$;*

(iii) *for every $0 \leqslant \alpha < \beta < \mathfrak{b}$, $H_\beta \setminus H_\alpha \in \mathscr{I}^+(\mathscr{A})$;*

(iv) *if $K \in bd(\mathscr{F})$, then the set $\{\alpha < \mathfrak{b} : K \in bd(\{H_\alpha \setminus H_\gamma : \gamma < \alpha\})\}$ is closed unbounded in $\mathfrak{b}$;*

(v) *if $L \in [\omega]^\omega$ is such that for every finite $\mathscr{A}' \subseteq \mathscr{A}$, $L \setminus \bigcup \mathscr{A}' \in bd(\mathscr{F})$, then the set $\{\alpha \in \mathfrak{b} :$ for every finite $\mathscr{A}' \subseteq \mathscr{A}$, $L \setminus \bigcup \mathscr{A}' \in bd(\{H_\alpha \setminus H_\gamma : \gamma < \alpha\})\}$ is closed unbounded in $\mathfrak{b}$;*

(vi) *$H_0 \subseteq^* M$ and, if moreover M satisfies that $M \setminus \bigcup \mathscr{A}' \in bd(\mathscr{F})$ for every finite $\mathscr{A}' \subseteq \mathscr{A}$, then $H_0 \in \mathscr{I}^+(\mathscr{A})$.*

Proof. Fix an unbounded family $\{f_\alpha : \alpha < \mathfrak{b}\} \subseteq {}^{\omega}\omega$. Since $\mathscr{F}$ is countable and decreasing, we may without loss of generality (pass to a cofinal part of $\mathscr{F}$, if necessary) assume that $\mathscr{F} = \{F_n : n \in \omega\}$ and for every $n \in \omega$, the set $F_n \setminus F_{n+1}$ is infinite.

First, we shall find a set H_0: Since $M \in bd(\mathscr{F})$, there is an infinite set $I_0 \subseteq \omega$ and a mapping $g_0 \in {}^{I_0}\omega$ such that for every $n \in I_0$, $g_0(n) \in M \cap F_n \setminus F_{n+1}$. If the set M does not satisfy the "moreover" assumption from (vi), we shall find some strictly increasing mapping $h_0 \in {}^{\omega}\omega$, $h_0 > f_0$, $h_0(n) > g_0(n)$ for all $n \in I_0$ and define $H_0 = \bigcup_{n\in\omega}\{i \in F_n \setminus F_{n+1} : i \leqslant h_0(n)\}$.

If, on the other hand, $M \setminus \bigcup \mathscr{A}' \in bd(\mathscr{F})$ for every finite $\mathscr{A}' \subseteq \mathscr{A}$, then we shall continue by a simple induction. Since $\mathscr{A}$ is a MAD family, one may choose some $A_0 \in \mathscr{A}$ such that the set $\{n \in I_0 : g_0(n) \in A_0\}$ is infinite. Clearly, the set $M \setminus A_0$ belongs to $\mathscr{I}^+(\mathscr{A})$ and by our assumption on the set M we have that $M \setminus A_0 \in bd(\mathscr{F})$, too. Set $M_0 = M \setminus A_0$.

Repeat the argument starting with M_0 to obtain I_1, g_1, A_1 and M_1 and proceed further. Finally, choose a strictly incerasing mapping h_0 so that $h_0 > f_0$, $h_0^* \geqslant g_n$ for all $n \in \omega$ and let $H_0 = \bigcup_{n\in\omega}\{i \in F_n \setminus F_{n+1} : i \leqslant h_0(n)\}$. The set H_0 obviously satisfies (vi) and the respective part of (i).

For the remaining, the transfinite induction follows. Suppose $h_\beta \in {}^{\omega}\omega$ is known and $H_\beta = \bigcup_{n\in\omega}\{i \in F_n \setminus F_{n+1} : i \leqslant h_\beta(n)\}$ for all $\beta < \alpha < \mathfrak{b}$. If α is a limit ordinal, choose $h_\alpha \in {}^{\omega}\omega$ to be an arbitrary function satisfying $h_\alpha^* \geqslant h_\beta$ for all $\beta < \alpha$, $h_\alpha^* \geqslant f_\alpha$, h_α is strictly increasing. If $\alpha = \beta + 1$, similarly as in the step 0 choose a strictly increasing $h_\alpha > h_\beta$ so that $\bigcup_{n\in\omega}(i \in F_n \setminus F_{n+1} : h_\beta(n) < i \leqslant h_\alpha(n)\} \in \mathscr{I}^+(\mathscr{A})$ and $h_\alpha^* \geqslant f_\alpha$. Then define $H_\alpha = \bigcup_{n\in\omega}\{i \in F_n \setminus F_{n+1} : i \leqslant h_\alpha(n)\}$.

Our definition of sets H_α immediately implies (i) and the inequality $h_\alpha \leqslant^* h_\beta$ for $\alpha < \beta < \mathfrak{b}$ gives (ii). Since we took care to ensure $H_{\beta+1} \setminus H_\beta \in \mathscr{I}^+(\mathscr{A})$ on each successor step of the induction, (iii) follows.

In order to verify (v), choose a set $L \in [\omega]^{\omega}$ such that $L \setminus \bigcup \mathscr{A}' \in bd(\mathscr{F})$ for every finite $\mathscr{A}' \subseteq \mathscr{A}$ and let $\beta < \mathfrak{b}$ be arbitrary. The set $L_0 = L \setminus H_\beta$ belongs to $\mathscr{I}^+(\mathscr{A})$: Indeed, suppose not, then for some finite $\mathscr{A}' \subseteq \mathscr{A}$, $L_0 \subseteq^* \bigcup \mathscr{A}'$, so $L \setminus \bigcup \mathscr{A}' \subseteq^* H_\beta$. But this contradicts the assumption that $L \setminus \bigcup \mathscr{A}' \in bd(\mathscr{F})$, as $H_\beta \notin bd(\mathscr{F})$.

There is a set $A_0 \in \mathscr{A}$ such that the set $\{n \in \omega : A_0 \cap L_0 \cap F_n \setminus F_{n+1} \neq \emptyset\}$ is infinite, because $L_0 \in bd(\mathscr{F})$ and because of the fact that $\mathscr{A}$ is a MAD family. Define a mapping $g \in {}^{\omega}\omega$ by the rule $g(n) = \min\{i : i \in A_0 \cap L_0 \cap F_{k(n)} \setminus F_{k(n)+1}\}$, where $k(n) = \min\{k : n \leqslant k \;\&\; A_0 \cap L_0 \cap F_k \setminus F_{k+1} \neq \emptyset\}$. There is some $\gamma > \beta$ such that $\{n \in \omega : g(n) < f_\gamma(n)\}$ is infinite, since the family $\{f_\alpha : \alpha < \mathfrak{b}\}$ has no upper bound. Because of $f_\gamma <^* h_\gamma$, the set $\{n \in \omega : g(n) < h_\gamma(n)\}$ is infinite, too. If $g(n) < h_\gamma(n)$, then $g(k(n)) = g(n) < h_\gamma(n) < h_\gamma(k(n))$, because the mapping h_γ is strictly increasing. Therefore the set $H_\gamma \cap L_0 \cap A_0$ is infinite.

Denote $\beta(0) = \gamma$, put $L_1 = L_0 \setminus (A_0 \cup H_{\beta(0)})$, apply the same reasoning to get $\beta(1), A_1, L_2$ and continue. Denote $\beta^* = \sup\{\beta(n) : n \in \omega\}$. We get that $L \cap H_{\beta^*} \setminus H_\beta \in \mathscr{I}^+(\mathscr{A})$.

Now, put $\alpha(0) = \beta^*$ and $\alpha(n+1) = \alpha(n)^*$, $\alpha = \sup\{\alpha(n) : n \in \omega\}$. We claim that for every finite $\mathscr{A}' \subseteq \mathscr{A}$, $L \setminus \bigcup \mathscr{A}' \in bd(\{H_\alpha \setminus H_\delta : \delta < \alpha\})$. To see this, let $\delta < \alpha$ be arbitrary. Pick some $n \in \omega$ with $\delta < \alpha(n) < \alpha$. Then $L \cap H_{\alpha(n+1)} \setminus H_{\alpha(n)} \in \mathscr{I}^+(\mathscr{A})$ and hence trivially also $(L \setminus \bigcup \mathscr{A}') \cap H_{\alpha(n+1)} \setminus H_{\alpha(n)} \in \mathscr{I}^+(\mathscr{A})$.

From the obvious inequality $H_{\alpha(n+1)} \setminus H_{\alpha(n)} \subseteq^* H_\alpha \setminus H_\delta$ we get that $(L \setminus \bigcup \mathscr{A}') \cap H_\alpha \setminus H_\delta \in \mathscr{I}^+(\mathscr{A})$. Moreover, the set $(L \setminus \bigcup \mathscr{A}') \cap H_{\alpha(n+1)} \setminus H_{\alpha(n)}$ is infinite, hence so is the set $(L \setminus \bigcup \mathscr{A}') \cap H_\alpha \setminus H_\delta$, which implies that $(L \setminus \bigcup \mathscr{A}') \in bd(\{H_\alpha \setminus H_\delta : \delta < \alpha\})$, too.

We have proved that the set $\{\alpha < \mathfrak{b} : \text{for every finite } \mathscr{A}' \subseteq \mathscr{A},\ L \setminus \bigcup \mathscr{A}' \in bd(\{H_\alpha \setminus H_\gamma : \gamma < \alpha\})\}$ is unbounded in $\mathfrak{b}$; the verification that it is also closed is easy and may be left to the reader. Now, (v) is shown.

The verification that (iv) is valid is analogous, but simpler. Let $K \in bd(\mathscr{F})$ be arbitrary. Again we show only that the set $\{\alpha < \mathfrak{b} : K \in bd(\{H_\alpha \setminus H_\gamma : \gamma < \alpha\})\}$ is unbounded in $\mathfrak{b}$. Fix $\beta < \mathfrak{b}$. Since $K \in bd(\mathscr{F})$ and $H_\beta \notin bd(\mathscr{F})$, we have that $K \setminus H_\beta \in bd(\mathscr{F})$. So there is some $\alpha(0) < \mathfrak{b}$, $\alpha(0) > \beta$ with $K \cap (H_{\alpha(0)} \setminus H_\beta)$ infinite. Next, knowing $\alpha(n)$, there is some $\alpha(n+1)$ so that $\alpha(n) < \alpha(n+1) < \mathfrak{b}$ and the set $K \cap (H_{\alpha(n+1)} \setminus H_{\alpha(n)})$ is infinite. Put $\alpha = \sup_{n<\omega} \alpha(n)$. Then $K \in bd(\{H_\alpha \setminus H_\delta : \delta < \alpha\})$, which was to be proved. □

Treating this lemma as an essential step in a transfinite induction, we may prove the following.

Lemma 2. *Let $\mathscr{A}$ be an infinite MAD family on ω, let $\mathscr{F} = \{F_n : n \in \omega\} \subseteq [\omega]^\omega$ be a countable decreasing family of sets such that each $F_n \setminus F_{n+1}$ is infinite and $\mathscr{F} \subseteq \mathscr{I}^+(\mathscr{A})$. Then there is a completely separable almost disjoint family $\mathscr{C}$ such that*

(i) *$bd(\mathscr{F}) \subseteq \mathscr{I}^+(\mathscr{C})$;*

(ii) *if $L \in [\omega]^\omega$ is such that $L \setminus \bigcup \mathscr{A}' \in \mathscr{I}^+(\mathscr{C})$ for every finite $\mathscr{A}' \subseteq \mathscr{A}$, then there is a set $C \in \mathscr{C}$, $C \subseteq L$ with $C \in \mathscr{I}^+(\mathscr{A})$.*

Proof. For each $\xi < \omega_1$ we shall first find a family Θ_ξ consisting of countable $\supseteq^*$-decreasing families, then an almost disjoint family $\mathscr{C}_{\xi+1}$.

$\Theta_0 = \{\mathscr{F}\}$.

Let $\xi < \omega_1$ be a limit ordinal and suppose that all Θ_η and $\mathscr{C}_{\eta+1}$ have been defined. There are two inductive assumptions:

$\bigcup\{\Theta_\eta : \eta < \xi\}$, when ordered by $\subseteq$, forms a tree of height ξ, and

for every $\eta < \xi$ and for every $\mathscr{T} \in \Theta_\eta$, $\mathscr{T} \subseteq \mathscr{I}^+(\mathscr{A})$.

Define $\Theta_\xi = \{\bigcup b : b \text{ is a branch in } \langle \bigcup\{\Theta_\eta : \eta < \xi\}, \subseteq \rangle\}$. Clearly $\mathscr{T} \in \mathscr{I}^+(\mathscr{A})$ for every $\mathscr{T} \in \Theta_\xi$, too.

Let $\xi < \omega_1$ and let Θ_ξ be known. We have to find $\mathscr{C}_{\xi+1}$ and $\Theta_{\xi+1}$. To do this, let $\mathscr{M}_\xi = \mathscr{L}_\xi \cup \mathscr{K}_\xi$, where $\mathscr{L}_\xi = \{L \in [\omega]^\omega : |\{\mathscr{T} \in \Theta_\xi : \text{for every finite } \mathscr{A}' \subseteq \mathscr{A},\ L \setminus \bigcup \mathscr{A}' \in bd(\mathscr{T})\}| = \mathfrak{c}\}$ and $\mathscr{K}_\xi = \{K \in [\omega]^\omega : |\{\mathscr{T} \in \Theta_\xi : K \in bd(\mathscr{T})\}| = \mathfrak{c}$ and $K \notin \mathscr{L}_\xi\}$.

Choose a one-to-one mapping $h : \mathscr{M}_\xi \to \Theta_\xi$ such that for every $L \in \mathscr{L}_\xi$ and for every finite $\mathscr{A}' \subseteq \mathscr{A}$, we have $L \setminus \bigcup \mathscr{A}' \in bd(h(L))$, and for each $K \in \mathscr{K}_\xi$, $K \in bd(h(K))$. This is clearly possible, since $|\mathscr{M}_\xi| \leqslant \mathfrak{c}$.

For $\mathscr{T}$ in Θ, and $M \in \mathscr{M}_\xi$ with $\mathscr{T} = h(M)$ apply Lemma 1 to $\mathscr{T}$ and the set M. Next, if $\mathscr{T} \notin \operatorname{rng} h$, then apply Lemma 1 to $\mathscr{T}$ and the set ω. This is always possible since $\mathcal{T} \subseteq \mathcal{I}^+(\mathcal{A})$ by the inductive assumption. Let $\{H_\alpha : \alpha < \mathfrak{b}\}$ be the result. (We omitted to express that H_α's depend on the $\mathcal{T}$ in question, hoping that it will not present confusions.) Denote then $\mathcal{C}(\mathcal{T})$ the set $\{H_0\} \cup \{H_{\alpha+1} \setminus H_\alpha : \alpha < \mathfrak{b}\}$. For every $\alpha < \mathfrak{b}$ with countable cofinality select an increasing sequence $\langle \alpha_n : n \in \omega \rangle$ of ordinals converging to α and put $\Theta(\mathcal{T}) = \{\mathcal{T} \cup \{H_\alpha \setminus H_{\alpha_n} : n \in \omega\} : \alpha < \mathfrak{b}, cf\alpha = \omega\}$.

Now we are ready to define $\mathcal{C}_{\xi+1} = \bigcup\{\mathcal{C}(\mathcal{T}) : \mathcal{T} \in \Theta_\xi\}$ and $\Theta_{\xi+1} = \bigcup\{\Theta(\mathcal{T}) : \mathcal{T} \in \Theta_\xi\}$. This completes the inductive definitions. Notice that both inductive assumptions remain satisfied, the second one by the item (ii) from Lemma 1.

It remains to show that $\mathcal{C} = \bigcup\{\mathcal{C}_{\xi+1} : \xi < \omega_1\}$ is a completely separable almost disjoint family having the properties as required.

Notice first that $\mathcal{C}$ is almost disjoint. If $C \in \mathcal{C}_{\xi+1}$, $C' \in \mathcal{C}_{\zeta+1}$, then there is some $\mathcal{T} \in \Theta_\xi$ and $\mathcal{T}' \in \Theta_\zeta$ with $C \in \mathcal{C}(\mathcal{T})$ and $C' \in \mathcal{C}(\mathcal{T}')$. Four cases are possible: If $\mathcal{T} = \mathcal{T}'$, then $C = H_{\beta+1} \setminus H_\beta$ and $C' = H_{\alpha+1} \setminus H_\alpha$ with $\alpha \neq \beta$; by Lemma 1, (ii), the sets C and C' are almost disjoint. If $\mathcal{T} \subsetneq \mathcal{T}'$, then $C = H_{\beta+1} \setminus H_\beta$ and for some $\alpha < \mathfrak{b}$ with $cf\alpha = \omega$, $\mathcal{T}' \supseteq \{H_\alpha \setminus H_{\alpha_n} : n \in \omega\}$. Thus there is some α_n with $(H_{\beta+1} \setminus H_\beta) \cap (H_\alpha \setminus H_{\alpha_n})$ finite. However, $C' \subseteq^* T'$ for every $T' \in \mathcal{T}'$, which implies that C and C' are almost disjoint. The case $\mathcal{T}' \subsetneq \mathcal{T}$ is symmetrical. In the fourth case, there is some largest $\eta < \xi, \zeta$ and a $\mathcal{T}'' \in \Theta_\eta$ with $\mathcal{T}' \subseteq \mathcal{T}''$. In the η'th step of the induction, when we defined $\Theta(\mathcal{T}'')$, it was necessary to find distinct $\alpha, \alpha' < \mathfrak{b}$, both of countable cofinality and such that $\{H_\alpha \setminus H_{\alpha_n} : n \in \omega\} \subseteq \mathcal{T}$ and $\{H_{\alpha'} \setminus H_{\alpha'_n} : n \in \omega\} \subseteq \mathcal{T}'$, otherwise η would not be the largest one. Clearly there is some $k \in \omega$ such that the intersection $(H_\alpha \setminus H_{\alpha_k}) \cap (H_{\alpha'} \setminus H_{\alpha'_k})$ is finite. As $C \subseteq^* H_\alpha \setminus H_{\alpha_k}$ and $C' \subseteq^* H_{\alpha'} \setminus H_{\alpha'_k}$, the sets C and C' are almost disjoint in this case, too.

To show that $\mathcal{C}$ is completely separable and statements (i), (ii) hold, let us prove the following two claims.

Claim 1. Let $\mathcal{D} \in [\mathcal{C}]^\omega$. Then there is an infinite subcollection $\mathcal{D}' \subseteq \mathcal{D}$, an ordinal $\zeta < \omega_1$ and $\mathcal{T} \in \Theta_\zeta$ such that for every $T \in \mathcal{T}$, the set $\{D \in \mathcal{D}' : D \subseteq^* T\}$ is infinite, and for every $D \in \mathcal{D}'$ there is some $T \in \mathcal{T}$ with $D \cap T =^* \emptyset$.

Proceeding by induction, we shall find $\mathcal{T}_\xi \in \Theta_\xi$ such that for $\xi < \eta$, $\mathcal{T}_\xi \subseteq \mathcal{T}_\eta$ and so that for every ξ, the set $\{D \in \mathcal{D}$: for every $T \in \mathcal{T}_\xi, D \subseteq^* T\}$ is infinite. The family $\mathcal{T}_0 = \mathcal{F}$ is obviously the proper choice. If ξ is a limit ordinal and all $\mathcal{T}_\eta$ for $\eta < \xi$ are known, we select $\mathcal{T}_\xi$ to be the union $\bigcup_{\eta<\xi} \mathcal{T}_\eta$, if this union satisfies the condition, otherwise the induction stops here. If $\xi = \eta + 1$ and $\mathcal{T}_\eta$ is known, then we select as $\mathcal{T}_\xi$ arbitrary member from $\Theta(\mathcal{T}_\eta)$ which satisfies the condition, otherwise the induction stops. Notice that the induction always stops before ω_1, since $\mathcal{D}$ is countable.

Suppose that for some limit $\xi < \omega_1$ we were unable to continue. Therefore the set $\mathcal{E}$, consisting of all $D \in \mathcal{D}$ such that for every $\eta < \xi$ and for every $T \in \mathcal{T}_\eta$ we have $D \subseteq^* T$, is finite. Put $\xi(0) = 0$ and then, by induction, if $\xi(n)$ is known, choose

$D(n) \in \mathscr{D} \setminus \mathscr{E}$ such that for all $T \in \mathscr{T}_{\xi(n)}$, $D(n) \subseteq^* T$. According to the definition of $\mathscr{C}$, there must be some $\xi(n+1)$ such that for some $T \in \mathscr{T}_{\xi(n+1)}$ we have $|D(n) \cap T| < \omega$. Since $D(n) \notin \mathscr{E}$, we have also $\xi(n+1) < \xi$. Now it remains to put $\zeta = \sup_{n \in \omega} \xi(n)$, $\mathscr{D}' = \{D(n) : n \in \omega\}$ and $\mathscr{T} = \bigcup_{n \in \omega} T_{\xi(n)}$ to get the claim.

Now, suppose that $\mathscr{T}_\eta$ was found and then we were unable to continue. This means that when constructing $\Theta(\mathscr{T}_\eta)$, for every $\alpha < \mathfrak{b}$ with $cf\alpha = \omega$ there was some α_n so that $|\{D \in \mathscr{D} : D \subseteq^* H_\alpha \setminus H_{\alpha_n}\}| < \omega$, though the set $\{D \in \mathscr{D} : D \subseteq^* T\}$ was infinite. Thus there is some first $\alpha < \mathfrak{b}$ with $|\{D \in \mathscr{D} : D \subseteq^* H_\alpha\}| = \omega$. Clearly, $cf\alpha = \omega$ for this α and if we put $\zeta = \eta + 1$, $\mathscr{T} = \mathscr{T}_\eta \cup \{H_\alpha \setminus H_{\alpha_n} : n \in \omega\}$, and $\mathscr{D}' = \{D \in \mathscr{D} : D \subseteq^* H_\alpha\} \setminus \{D \in \mathscr{D} : (\forall n \in \omega)\, D \subseteq^* H_\alpha \setminus H_{\alpha(n)}\}$, then the claim is again verified.

Claim 2. Let $M \in [\omega]^\omega$ and suppose that for some $\zeta < \omega_1$ there is some $\mathscr{T} \in \Theta_\zeta$ with $M \in bd(\mathscr{T})$. Then $M \in \mathscr{M}_{\zeta+\omega}$.

Indeed, observe that Lemma 1, (iv) as well as (v), guaranteed that there are at least two $\mathscr{T}(0), \mathscr{T}(1) \in \Theta(\mathscr{T}_\zeta)$ with $M \in bd(\mathscr{T}(0))$, $M \in bd(\mathscr{T}(1))$ and at least four $\mathscr{T}(00)$, $\mathscr{T}(01)$, $\mathscr{T}(10)$, $\mathscr{T}(11)$ such that both $\mathscr{T}(00)$, $\mathscr{T}(01)$ belong to $\Theta(\mathscr{T}(0))$ and both $\mathscr{T}(10)$, $\mathscr{T}(11)$ belong to $\Theta(\mathscr{T}(1))$ and $M \in bd(\mathscr{T}(\varphi))$ for all $\varphi \in^2 \{0,1\}$. After next ω steps we conclude, that in $\Theta_{\zeta+\omega}$, there is for every $f \in^\omega \{0,1\}$ a member $\mathscr{T}(f)$ with $M \in bd(\mathscr{T}(f))$. Therefore $M \in \mathscr{M}_{\zeta+\omega}$.

Let us complete the proof now. If $M \in \mathscr{I}^+(\mathscr{C})$, consider an arbitrary infinite family $\mathscr{D} \subseteq \{C \in \mathscr{C} : |M \cap C| = \omega\}$. If $\zeta < \omega_1$ and $\mathscr{T} \in \Theta_\zeta$ are as in Claim 1, then $m \in bd(\mathscr{T})$: Indeed, given an arbitrary $T \in \mathscr{T}$, choose $D \in \mathscr{D}$ so that $D \subseteq^* T$ and then find for this D a set $T' \in \mathscr{T}$ such that $D \cap T'$ is finite. Thus $D \subseteq^* T \setminus T'$, which in turn implies $|M \cap T \setminus T'| = \omega$.

By Claim 2, $M \in \mathscr{M}_{\zeta+\omega}$. So when $\mathscr{C}_{\zeta+\omega+1}$ was defined, it was obligatory to place a set $H_0 \subseteq M$ into $\mathscr{C}(h(M))$. This shows that $\mathscr{C}$ is completely separable.

The same argument shows that for an arbitrary $M \in bd(\mathscr{F})$ there is some $C \in \mathscr{C}_{\omega+1}$ with $C \subseteq M$, but observe moreover, that $|\{\mathscr{T} \in \Theta_\omega : M \in bd(\mathscr{T})| = \mathfrak{c}$ and Lemma 1, (iv), (v) implies that the same is true also for Θ_α whenever $\omega < \alpha$. So $M \in \mathscr{M}_\alpha$ for $\omega \leqslant \alpha < \omega_1$, hence $M \in \mathscr{I}^+(\mathscr{C})$, which shows (i).

In order to verify (ii), we shall argue similarly. Suppose $L \in [\omega]^\omega$ satisfies $L \setminus \bigcup \mathscr{A}' \in \mathscr{I}^+(\mathscr{C})$ for every $\mathscr{A}' \in [\mathscr{A}]^{<\omega}$. Since $\mathscr{C}$ is completely separable, pick some $D(0) \in \mathscr{C}$ with $D(0) \subseteq L$. If $D(0) \in \mathscr{I}^+(\mathscr{A})$, we are done, otherwise let $\mathscr{A}_0$ be a finite subcollection of $\mathscr{A}$ such that $D(0) \subseteq^* \bigcup \mathscr{A}(0)$. If the set $D(n)$ and a finite subfamily $\mathscr{A}(n) \subseteq \mathscr{A}$ are known, choose $D(n+1) \in \mathscr{C}$ such that $D(n+1) \subseteq L \setminus \bigcup \mathscr{A}(n)$ and, if our choice was unlucky again, i.e., $D(n+1) \notin \mathscr{I}^+(\mathscr{A})$, let $\mathscr{A}(n+1)$ be finite, $\mathscr{A}(n) \subseteq \mathscr{A}(n+1) \subseteq \mathscr{A}$ and such that $D(n+1) \subseteq^* \bigcup \mathscr{A}(n+1)$.

If we have missed to find the desired set C, we have got an infinite $\mathscr{D} = \{D(n) : n \in \omega\} \subseteq \mathscr{C}$. Let an ordinal $\zeta < \omega_1$, an infinite subset $\mathscr{D}' \subseteq \mathscr{D}$ and a decreasing centered family $\mathscr{T} \in \Theta_\zeta$ satisfy the conclusions of Claim 1. Pick an arbitrary $T \in \mathscr{T}$ and an arbitrary finite $\mathscr{A}' \subseteq \mathscr{A}$. Since $\mathscr{A}'$ is finite, there is some $k \in \omega$ such that for every $n > k$ and every $A \in \mathscr{A}(n) \setminus \mathscr{A}(k)$, $A \notin \mathscr{A}'$. Since the set $\{D \in \mathscr{D}' : D \subseteq^* T\}$ is infinite, there is some $n > k$ such that $D(n) \subseteq^* T$. Since $D(n) \subseteq L \setminus \bigcup \mathscr{A}(k)$, since $D(n) \subseteq^* \bigcup A(n)$, since $\mathscr{A}$ is almost disjoint and since $\mathscr{A}'$ is finite, we infer that $D(n) \cap \bigcup \mathscr{A}'$ is finite.

By Claim 1, there is some $T' \in \mathscr{T}$ with $D(n) \cap T'$ finite. So $D(n) \subseteq^* T \setminus T'$, but now our choice of $D(n)$ implies that also $(T \cap L \setminus \bigcup \mathscr{A}') \setminus T'$ is infinite. As $T \in \mathscr{T}$ was arbitrary, we have $L \setminus \bigcup \mathscr{A}' \in bd(\mathscr{T})$. As a finite $\mathscr{A}' \subseteq \mathscr{A}$ was arbitrary, we may see that all the assumptions of Lemma 1, (v), are satisfied.

Now it is clear how to continue: Starting with ζ and $\mathscr{T}$ and passing via Claim 2 to $\zeta + \omega$, we may assume that our choices of a branching family were always made in accordance with Lemma 1, (v). So $L \in \mathscr{L}_{\zeta+\omega}$ and the set H_0, the first member of $\mathscr{C}(h(L))$, satisfies both $H_0 \subseteq L$ and $H_0 \in \mathscr{I}^+(\mathscr{A})$.

The proof is complete. □

Theorem. *Assume $\mathfrak{d} \leqslant \mathfrak{a}$. If $\mathscr{A}$ is a MAD family on ω, then $\mathscr{I}^+(\mathscr{A})$ has an almost disjoint refinement.*

Remark. The conclusion of this theorem was shown by J. Roitman under the assumption $\mathfrak{a} = \mathfrak{c}$ [Ro] and under $\mathfrak{b} = \mathfrak{d}$ by B. Balcar and the author in [BS]. Notice that each of the assumptions implies $\mathfrak{d} \leqslant \mathfrak{a}$. It must be however added that the consistency of a sharp inequality $\mathfrak{d} < \mathfrak{a}$ is still an open problem.

Proof. Let $f \in {}^\omega\omega$ be strictly increasing, associate with such an f a family $\mathscr{F}(f) = \{F_n : n \in \omega\}$, where $F_n = \{i \in \omega : (\exists k \geqslant n)(\exists m \in \omega) f(2^k(2m+1)) \leqslant i < f(2^k(2m+1)+1)\}$.

We leave the reader to verify the following fact (see [BS], 3.16): Let $X \in [\omega]^\omega$ be arbitrary, $X = \{x_0 < x_1 < x_2 < \dots\}$, let a strictly increasing $f \in {}^\omega\omega$ satisfy $f(n) > x_n$ for all but finitely many n's. Define $h = h(f) \in {}^\omega\omega$ by the rule $h(0) = 0$, $h(n+1) = f(h(n)+1)$. Then $X \in bd(\mathscr{F}(h))$.

Fix an arbitrary dominating family $\{f_\alpha : \alpha < \mathfrak{d}\} \subseteq {}^\omega\omega$ consisting of strictly increasing functions, denote by $h_\alpha = h(f_\alpha)$ and let $\mathscr{F}_\alpha = \mathscr{F}(h_\alpha)$. We are allowed to assume that for every $\alpha < \mathfrak{d}$, $\mathscr{F}_\alpha \subseteq \mathscr{I}^+(\mathscr{A})$: indeed, choose for every $k \in \omega$ a set $A_k = \{a_{k0} < a_{k1} < a_{k2} < \dots\} \in \mathscr{A}$, define then $\varphi(n) = \max\{\varphi(k), a_{kn} : k < n\} + 1$ and choose a dominating family in such a way that all f_α's satisfy $\varphi \leqslant^* f_\alpha$.

Apply Lemma 2 to each $\mathscr{F}_\alpha$ and denote by $\mathscr{C}_\alpha$ the resulting completely separable family.

Let us define an almost disjoint family $\mathscr{B}$ by an induction to $\mathfrak{d}$: For every $D \in \mathscr{D}_0 = \mathscr{C}_0 \cap \mathscr{I}^+(\mathscr{A})$ choose some $A(D) \in \mathscr{A}$ such that $A(D) \cap D$ is infinite. Denote $\mathscr{B}_0 = \{A(D) \cap D : D \in \mathscr{D}_0\}$.

Let $\alpha < \mathfrak{d}$ and suppose that for all $\gamma < \alpha$ the family $\mathscr{B}_\gamma$ has been defined. We assume that for $\beta < \gamma < \alpha$, $\mathscr{B}_\beta \subseteq \mathscr{B}_\gamma$, that the family $\bigcup_{\gamma<\alpha} \mathscr{B}_\gamma$ is almost disjoint and that for every $B \in \bigcup_{\gamma<\alpha} \mathscr{B}_\gamma$ there is some $A \in \mathscr{A}$ with $B \subseteq A$.

Let $\mathscr{D}_\alpha$ be the collection of all $C \in \mathscr{C}_\alpha \cap \mathscr{I}^+(\mathscr{A})$ such that for every $\gamma < \alpha$ there is some finite $\mathscr{A}' \subseteq \mathscr{A}$ such that $C \setminus \bigcup \mathscr{A}' \notin \mathscr{I}^+(\mathscr{C}_\gamma)$.

For $D \in \mathscr{D}_\alpha$ choose a set $A(D) \in \mathscr{A}$ such that the intersection $D \cap A(D)$ is infinite and almost disjoint with all members of $\bigcup_{\gamma<\alpha}\mathscr{B}_\gamma$. Let us show that such a choice is always possible. Given $D \in \mathscr{D}_\alpha$, for every $\gamma < \alpha$ there is a finite subfamily $\mathscr{A}_\gamma \subseteq \mathscr{A}$ and a finite subfamily $\mathscr{C}'_\gamma \subseteq \mathscr{C}_\gamma$ such that $C \cap D \setminus \bigcup \mathscr{A}_\gamma$ is finite for every $C \in \mathscr{C}_\gamma \setminus \mathscr{C}'_\gamma$. Denote by $\mathscr{D}'_\gamma$ the set $\mathscr{C}'_\gamma \cap \mathscr{D}_\gamma$. The set $\mathscr{D}'_\gamma$ is finite, thus so is also the set $\mathscr{A}'_\gamma = \{A(D) : D \in \mathscr{D}'_\gamma\}$. Since we assume that $\mathfrak{d} \leqslant \mathfrak{a}$ and since $|\bigcup\{\mathscr{A}_\gamma \cup \mathscr{A}'_\gamma : \gamma < \alpha\}| \leqslant \alpha \cdot \omega < \mathfrak{d}$, the family $\{A \cap D$: for some $\gamma < \alpha$, $A \in \mathscr{A}_\gamma \cup \mathscr{A}'_\gamma\}$ is not a MAD family on D. So there is some $A(D) \in \mathscr{A} \setminus \bigcup\{\mathscr{A}_\gamma \cup \mathscr{A}'_\gamma : \gamma < \alpha\}$ such that $D \cap A(D)$ is infinite.

Therefore we can define $\mathscr{B}_\alpha = \bigcup_{\gamma<\alpha}\mathscr{B}_\gamma \cup \{A(D) \cap D : D \in \mathscr{D}_\alpha\}$.

It remains to show that the family $\mathscr{B} = \bigcup_{\alpha<\mathfrak{d}}\mathscr{B}_\alpha$ is the desired almost disjoint refinement of $\mathscr{I}^+(\mathscr{A})$. It follows immediately from the construction that $\mathscr{B}$ is almost disjoint. Let $L \in \mathscr{I}^+(\mathscr{A})$ be arbitrary. Choose for every $k \in \omega$ a member $A_k \in \mathscr{A}$ with $L \cap A_k$ infinite. Let $\alpha < \mathfrak{d}$ be such that $f_\alpha(n) > m_{kn}$ for all but finitely many n's and for all $k \in \omega$; here $\{m_{k0} < m_{k1} < m_{k2} \ldots\} = L \cap A_k$. It is obvious that for every finite $\mathscr{A}' \subseteq \mathscr{A}$, $L \setminus \bigcup \mathscr{A}' \in bd(\mathscr{F}_\alpha)$, therefore $L \setminus \bigcup \mathscr{A}' \in \mathscr{I}^+(\mathscr{C}_\alpha)$, too. By Lemma 2, there is a $C \in \mathscr{C}_\alpha \cap \mathscr{I}^+(\mathscr{A})$ with $C \subseteq L$.

Suppose that for every $\gamma < \alpha$ there is some finite $\mathscr{A}_\gamma \subseteq \mathscr{A}$ such that $C \setminus \bigcup \mathscr{A}_\gamma \notin \mathscr{I}^+(\mathscr{C}_\gamma)$. If this happens, then the construction provides some $B \in \mathscr{B}_\alpha$ with $B \subseteq C \subseteq L$.

In the opposite case, there is the first $\gamma(0)$ such that for every finite $\mathscr{A}' \subseteq \mathscr{A}$, $C \setminus \bigcup \mathscr{A}' \in \mathscr{I}^+(\mathscr{C}_{\gamma(0)})$. By Lemma 2, (ii), there is some $C_0 \subseteq C$ with $C_0 \in \mathscr{C}_{\gamma(0)} \cap \mathscr{I}^+(\mathscr{A})$. If $\gamma(0) = 0$, then $C_0 \in \mathscr{D}_0$ and hence there is some $B \in \mathscr{B}_0$ satisfying $B \subseteq C_0 \subseteq C \subseteq L$. If $\gamma(0) > 0$, then for every $\gamma < \gamma(0)$ there is some finite $\mathscr{A}_\gamma \subseteq \mathscr{A}$ such that $C_0 \setminus \bigcup \mathscr{A}_\gamma \notin \mathscr{I}^+(\mathscr{C}_\gamma)$. This follows by the choice of $\gamma(0)$ and by the fact that $C_0 \subseteq C$. Hence there is is some $B \in \mathscr{B}_{\gamma(0)}$ satisfying $B \subseteq C_0 \subseteq C \subseteq L$. Thus we have showed that the family $\mathscr{B}$ is the desired almost disjoint refinement of $\mathscr{I}^+(\mathscr{A})$. □

Theorem. *The following are equivalent:*

(i) *For every MAD family $\mathscr{A}$ on ω, $\mathscr{I}^+(\mathscr{A})$ has an almost disjoint refinement;*

(ii) *there exists some $\tau \leqslant \mathfrak{b}$ with $cf\tau > \omega$ and a collection $\{\mathscr{C}_\alpha : \alpha < \tau\}$ of completely separable almost disjoint families on ω such that for each $\alpha < \beta < \tau$, $\mathscr{I}^+(\mathscr{C}_\alpha) \subseteq \mathscr{I}^+(\mathscr{C}_\beta)$ and $\bigcup_{\alpha<\tau}\mathscr{I}^+(\mathscr{C}_\alpha) = [\omega]^\omega$.*

Proof. Suppose (i). By [BS, Theorem 4.19, (iii)], if $\mathscr{I}^+(\mathscr{A})$ has an almost disjoint refinement whenever $\mathscr{A}$ is a MAD family on ω, then there is a tree π-base for ω^* $\Theta = \{\mathscr{Q}_\alpha : \alpha < \mathfrak{h}\}$ with each $\mathscr{Q}_\alpha$ a completely separable MAD family. Thus in order to verify (ii), it is enough to put $\tau = \mathfrak{h}$ and $\mathscr{C}_\alpha = \mathscr{Q}_\alpha$. Since for $\alpha < \beta < \mathfrak{h}$, $\mathscr{Q}_\beta \prec \mathscr{Q}_\alpha$, we have $\mathscr{I}^+(\mathscr{Q}_\alpha) \subseteq \mathscr{I}^+(\mathscr{Q}_\beta)$ then.

Suppose (ii), let τ and $\{\mathscr{C}_\alpha : \alpha < \tau\}$ be as in (ii), let $\mathscr{A}$ be an arbitrary infinite

MAD family on ω. The reader is requested to verify (see also [BS], Prop. 4.9, (iv)) that if $\mathscr{C}$ is a completely separable almost disjoint family and if the set $D(C) \in [C]^\omega$ is chosen arbitrarily for each $C \in \mathscr{C}$, then the family $\{D(C): C \in \mathscr{C}\}$ is completely separable as well. Thus we may and shall assume that for every $\alpha < \tau$ and $C \in \mathscr{C}_\alpha$ there is some $A \in \mathscr{A}$ with $C \subseteq A$. Proceeding by induction, we put $\mathscr{B}_0 = \mathscr{C}_0$ and, knowing $\mathscr{B}_\beta$ for all $\beta < \alpha$, we define $\mathscr{B}_\alpha = \{C \in \mathscr{C}_\alpha : (\forall \beta < \alpha)\, (\forall B \in \mathscr{B}_\beta)\, |C \cap B| < \omega\}$.

The family $\mathscr{B} = \bigcup_{\alpha<\tau} \mathscr{B}_\alpha$ is obviously almost disjoint. We shall show that $\mathscr{B}$ refines $\mathscr{I}^+(\mathscr{A})$. Pick an $L \in \mathscr{I}^+(\mathscr{A})$ arbitrarily. Our aim is to find a subset $\tilde{L} \subseteq L$ and an $\alpha < \tau$ so that $\tilde{L} \in \mathscr{I}^+(\mathscr{C}_\alpha)$ and $\tilde{L} \cap C$ is finite for every $C \in \bigcup_{\beta<\alpha} \mathscr{C}_\beta$. This will clearly suffice, for if $C \in \mathscr{C}_\alpha$ satisfies $C \subseteq \tilde{L}$, then for this C we shall have $C \in \mathscr{B}_\alpha \subseteq \mathscr{B}$ as well as $C \subseteq \tilde{L} \subseteq L$.

Put $\alpha(0) = 0$ and $L_0 = L$. If $L_0 \in \mathscr{I}^+(\mathscr{C}_{\alpha(0)})$, then we are done. Otherwise we continue by an induction. Suppose that for an $n \in \omega$ the ordinal $\alpha(n)$ and the set $L_n \in \mathscr{I}^+(\mathscr{A})$ is known and that for every $C \in \bigcup_{\beta<\alpha(n)} \mathscr{C}_\beta$, $C \cap L_n$ is finite. If $L_n \in \mathscr{I}^+(\mathscr{C}_{\alpha(n)})$, then we succeeded. Otherwise let $\alpha(n+1)$ be the first ordinal $<\tau$ such that $L_n \in \mathscr{I}^+(\mathscr{C}_{\alpha(n+1)})$. So for every $\beta < \alpha(n+1)$, the family $\{C \in \mathscr{C}_\beta : |C \cap L_n| = \omega\}$ is finite. Let us denote by $\mathscr{A}'$ the family of all $A \in \mathscr{A}$ such that there is some $\beta < \alpha(n+1)$ and $C \in \mathscr{C}_\beta$ with $|C \cap L_n| = \omega$ and $C \subseteq A$. Then $|\mathscr{A}'| \leqslant |\alpha(n+1) \cdot \omega < \tau \leqslant \mathfrak{b} \leqslant \mathfrak{a}$ and hence we can select for each $i \in \omega$ a set $A_i \in \mathscr{A} \setminus \mathscr{A}'$ such that $A_i \cap L_n$ is infinite. We are allowed to assume that the sets A_i are pairwise disjoint – indeed, $A_i =^* A_i \setminus \bigcup_{j<i} A_j$. Whenever $C \in \bigcup_{\beta<\alpha(n+1)} \mathscr{C}_\beta$ satisfies $|C \cap L_n|$ is infinite, then $C \cap A_i$ is finite for all $i < \omega$. Thus using once more the fact that $\alpha(n+1) < \mathfrak{b}$, we can find a mapping $f \in {}^\omega\omega$ such that for every such C we have $|C \setminus \bigcup_{i<\omega} \{k \in A_i : k < f(i)\}| < \omega$. It remains to set $L_{n+1} = L_n \cap \bigcup_{i<\omega} \{k \in A_i : f(i) \leqslant k\}$. Clearly $L_{n+1} \in \mathscr{I}^+(\mathscr{A})$ and $L_{n+1} \cap C$ is finite whenever $C \in \mathscr{C}_\beta$, $\beta < \alpha(n+1)$.

If the induction proceeded till ω, then put $\alpha = \sup_{n\in\omega} \alpha(n)$. Since $\mathscr{I}^+(\mathscr{C}_{\alpha(n)}) \subseteq \mathscr{I}^+(\mathscr{C}_\alpha)$ by (ii), we have that $L_n \in \mathscr{I}^+(\mathscr{C}_\alpha)$ for all $n \in \omega$. Making use of the complete separability of $\mathscr{C}_\alpha$, select $C_0 \subseteq L_0$, $C_0 \in \mathscr{C}_\alpha$ and then, knowing C_i for $i < n$, choose $C_n \in \mathscr{C}_\alpha$ such that $C_n \subseteq L_n \setminus \bigcup_{i<n} C_i$. Let D be an arbitrary subset of $\bigcup_{n<\omega} C_n$ such that for every $n \in \omega$, both sets $D \cap C_n$, $C_n \setminus D$ are infinite. Clearly $D \cap L_n \in \mathscr{I}^+(\mathscr{C}_\alpha)$ and $D \setminus L_n \subseteq C_0 \cup C_1 \cup \ldots \cup C_{n-1}$ for all $n \in \omega$.

Repeating the reasoning once more, choose $\tilde{C}_n \in \mathscr{C}_\alpha$ such that $\tilde{C}_n \subseteq D \cap L_n \setminus \bigcup_{i<n} \tilde{C}_i$ and put $\tilde{L} = \bigcup_{n<\omega} \tilde{C}_n$. Since $C_n \setminus D$ is always infinite, we have $\tilde{C}_n = C_j$ for no pair n, j. Again, $\tilde{L} \in \mathscr{I}^+(\mathscr{C}_\alpha)$. Moreover, for every $n \in \omega$, $\tilde{L} \subseteq^* L_n$: Indeed, if $n \leqslant i$, then $\tilde{C}_i \subseteq \bigcup_{n \leqslant j < \omega} C_j \subseteq L_n$; if $i < n$, then $\tilde{C}_i \cap C_j$ is finite for every $j \in \omega$ and $\tilde{C}_i \subseteq D$, so $\tilde{L} \setminus L_n = \bigcup_{i<\omega} \tilde{C}_i \setminus L_n \subseteq \bigcup_{i<n} \tilde{C}_i \cap \bigcup_{j<n} C_j = \bigcup_{i,j<n} \tilde{C}_i \cap C_j$. So the set $\tilde{L} \setminus L_n$ is a subset of a finite union of finite sets, hence finite.

If $\beta < \alpha$ and $C \in \mathscr{C}_\beta$, then there is some $n < \omega$ with $\beta < \alpha(n)$. By the choice of L_n, $L_n \cap C$ is finite, so $\tilde{L} \cap C$ is finite, too. Thus the set $\tilde{L}$ is as required, which completes the proof. □

Remark. Let us briefly sketch that our last Theorem generalizes the result from [BS], namely: If $\mathfrak{s} = \omega_1$, then for every MAD family $\mathscr{A}$ on ω, $\mathscr{I}^+(\mathscr{A})$ has an almost disjoint refinement.

Indeed, assume $\mathfrak{s} = \omega_1$ and select a splitting family $\mathscr{X} = \{X_\alpha : \alpha < \omega_1\}$. Putting $X_{\alpha,0} = X_\alpha$, $X_{\alpha,1} = \omega \setminus X_\alpha$ we obtain a family of partitions of ω, $\{\{X_{\alpha,0}, X_{\alpha,1}\} : \alpha < \omega_1\}$. Given $\alpha \geqslant \omega$, $\alpha < \omega_1$ and $f : \alpha \to \{0,1\}$, if the family $\{X_{\beta,f(\beta)} : \beta < \alpha\}$ has a finite intersection property, then reenumerate it as $\{X_{\beta,f(\beta)} : \beta < \alpha\} = \{Y_n : n \in \omega\}$ and denote by $\mathscr{F}(f) = \{\bigcap_{n=0}^k Y_n : k \in \omega\}$. Let $\Xi(\alpha)$ be the family of all $\mathscr{F}(f)$ such that $f \in {}^\alpha\{0,1\}$, the family $\{X_{\beta,f(\beta)} : \beta < \alpha\}$ has a finite intersection property and $\mathscr{F}(f)$ is $\supset^*$-decreasing.

Thus, applying Lemma 2 on each $\mathscr{F}(f) \in \Xi(\alpha)$ (the MAD family $\mathscr{A}$ assumed there may be taken arbitrarily), we obtain a completely separable almost disjoint family $\mathscr{C}(f)$ and it is enough to set $\mathscr{C}_\alpha = \bigcup\{\mathscr{C}(f) : f \in {}^\alpha\{0,1\}\ \&\ \mathscr{F}(f) \in \Xi(\alpha)\}$. The family $\{\mathscr{C}_\alpha : \alpha < \omega_1\}$ has all the properties required in (ii) from the theorem, hence (i) follows.

Indeed, the family $\mathscr{C}_\alpha$ is almost disjoint. Though we did not mention it explicitly, the reader undoubtedly noticed that there in Lemma 2, for each member of the resulting completely separable family $\mathscr{C}$ and for every member F from the given $\supset^*$-decreasing family $\mathscr{F}$, $C \subseteq^* F$ holds. Thus if $C \in \mathscr{C}(f)$, $C' \in \mathscr{C}(g)$, then $C \cap C'$ is finite, because for some $\beta < \alpha$, $X_{\beta,f(\beta)} \cap X_{\beta,g(\beta)} = \emptyset$.

Next, given $M \in [\omega]^\omega$, applying repeatedly the fact that the starting family $\mathscr{X}$ is splitting, it is easy to find an $\alpha < \omega_1$ and $f \in {}^\alpha\{0,1\}$ such that $M \in bd(\mathscr{F}(f))$, so $M \in \mathscr{I}^+(\mathscr{C}_\alpha)$ and obviously $M \in \mathscr{I}^+(\mathscr{C}_\gamma)$ whenever $\alpha \leqslant \gamma < \omega_1$. In order to see that the family $\mathscr{C}_\alpha$ is completely separable, notice that if $M \in [\omega]^\omega$ is such that the family $\{\mathscr{F}(f) : \mathscr{F}(f) \in \Xi(\alpha)$, for every $F \in \mathscr{F}(f)$, $|M \cap F| = \omega\}$ is infinite, then it is either countable and for some $\mathscr{F}(f)$, $M \in bd(\mathscr{F}(f))$, or it is of size $\mathfrak{c}$. We leave this statement to the reader, because it simply mimicks the well-known proof that an infinite closed set of reals is either countable, and then contains a convergent sequence together with its limit, or of size $\mathfrak{c}$.

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