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On the Scooping Property of Measures by Means of Disjoint Balls

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Let X be a Banach space. The measures in X considered below are assumed to be Borel finite measures, and all balls are assumed to be open.

Definition 1. We say that a collection G of subsets of X has the *scooping property* $(G \in (\text{s.p.}))$ if for every measure μ on X and for every $\varepsilon > 0$ there exists a finite family $\{F_n\}_1^N \subset G$ of pairwise disjoint sets such that

$$\mu\left(X \setminus \bigcup_{i=1}^{N} F_n\right) < \varepsilon .$$

This property is related to what is known as the positivity principle (p.p.) introduced by Christensen in [1]; the p.p. holds for a family G of subsets of a Banach space X if for arbitrary measures μ and ν on X, the relation $\mu(B) \geq \nu(B)$ for all $B \in G$ implies $\mu \geq \nu$.

The positivity principle admits an interesting geometric interpretation (see [1]): for a family G in a Banach space X the positivity principle is fulfilled if and only if for every measure μ on X and every Borel set $M \subset X$ we have

$$\mu(M) = \inf\left\{\sum_i \mu(B_i) : B_i \in G,\ \mu\left(M \Big\backslash \bigcup_i B_i\right) = 0\right\};$$

this means that the measure of any set M can be "scooped" by means of sets that belong to G, are almost pairwise disjoint, and almost lie in M.

It is known that the positivity principle is not fulfilled in l_2 for the family $I_{r<1}$ of balls of radius $r \in (0, 1)$ (we call such balls "small balls"); the counterexample is constructed by D. Preiss in [2]. For more information see [3, 4]. On the other hand, the p.p. holds true in $C[0, 1]$ for the family $I_{r=1}$ of balls of radius 1 ([5]). This fact allows us to simplify the above-mentioned interpretation of the p.p. in X as follows (for simplicity, we consider the family $I_{r<1}$):

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the p.p. is fulfilled in X for $I_{r<1}$ if and only if for every measure μ on X and every $r < 1$, $\varepsilon > 0$ there exists a family $\{B_k\} \subset I_{r<r_0}$ such that

$$\mu\left(X \setminus \bigcup_k B_k\right) + \left|\sum_k \mu(B_k) - \mu(X)\right| < \varepsilon ,$$

i.e., the measure of the entire space X (not of an arbitrary Borel set!) can be almost scooped by means of almost disjoint small balls.

This interpretation is a consequence of the following lemma, which can be deduced from the results of [5] by embedding the space X in $C[0, 1]$.

Lemma. *Let μ and ν be two measures on a separable Banach space X. If $\mu(B) \geq \nu(B)$ for every convex open set $B \subset X$ with* $\operatorname{diam} B \leq 1$, *then $\mu \geq \nu$.*

Thus, we see that the scooping property is not weaker than the positivity principle.

Here we discuss some results related to the scooping property.

This Preiss' example mentioned above shows that the family of small balls in the Hilbert space does not posess the scooping property. It seems that such a situation is typical for "good" spaces. Below we list some properties of the spaces that fail to have the scooping property for $I_{r<1}$.

Property 1. *Let X be a Banach space; we assume that $I_{r<1} \notin$ (s.p.). There is a probability measure μ on X such that*

$$\lim_{\varepsilon \to 0} \sup_{\substack{\{B_n\} \subset I_{r_n < \varepsilon} \\ B_i \cap B_j = \emptyset}} \mu\left(\bigcup B_n\right) = 0 .$$

In other words, there is a measure that cannot be scooped essentially by means of very small disjoint balls.

Proof. For a measure ν on X, we denote

$$F_\nu(\delta) = \frac{1}{\nu(X)} \sup_{\substack{\{B_n\} \subset I_{r<\delta} \\ B_i \cap B_j = \emptyset}} \nu\left(\bigcup B_n\right) .$$

We construct μ as the week limit of a sequence of measures $\{\mu_n\}$. This sequence $\{\mu_n\}$ and a sequence of radii $\{\delta_n\}$ will be chosen in such a way that

$$F_{\mu_n}(\delta_n) < \frac{1}{2^n} . \tag{1}$$

Since the family $I_{r<1}$ does not possess the scooping property, for some measure ν we have $F_\nu = l < 1$. Let $\tau = 1 - l$. There exists a family $\{B_n^{(1)}\}_1^{N_1} \subset I_{r<1}$ of disjoint balls such that

$$\frac{\nu\left(\bigcup_{n=1}^{N_1} B_n^{(1)}\right)}{\nu(X)} > l - \frac{\tau}{4} .$$

We may assume that $\nu(\partial B_n^{(1)}) = 0$ for $n = 1, 2, \ldots, N_1$. So, for some $\delta_1 > 0$ and $n = 1, 2, \ldots, N_1$ we have $\nu\left(\bigcup_{n=1}^{N_1}[(W_{2\delta_1}(B_n^{(1)}))\setminus B_n^{(1)}]\right) < \frac{\tau}{4}\cdot\nu(X)$, where $W_\delta(B) = \{x \in X : \operatorname{dist}(x, B) < \delta\}$.

Now we define μ_1 as follows: $\mu_1 \equiv \nu$ on $X \setminus \bigcup_{n=1}^{N_1} B_n^{(1)}$ and $\mu_1 \equiv 0$ on $\bigcup_{n=1}^{N_1} B_n^{(1)}$. It is easy to check that $F_{\mu_1}(\delta_1) < \frac{1}{2}$.

Repeating the above argument, by induction we construct a sequence $\{\mu_n\}$ of measures and a sequence $\{\delta_n\}$ of radii satisfying inequality (1). Namely, if μ_{n-1} and δ_{n-1} have already been constructed, then we take a family $\{B_k^{(n)}\}_1^{N_n} \subset I_{r<\delta_{n-1}}$ of disjoint balls such that

$$\frac{\mu_{n-1}\left(\bigcup_{k=1}^{N_n} B_k^{(n)}\right)}{\mu_{n-1}(X)} > F_{\mu_{n-1}}(\delta_{n-1}) - \frac{\mu_{n-1}(X)}{2^{n-1}}\cdot\left(1 - F_{\mu_{n-1}}(\delta_{n-1})\right)$$

and find $\delta_n > 0$ such that

$$\mu_{n-1}\left(\bigcup_{k=1}^{N_n}(W_{2\delta_{n+1}}(B_k^{(n)}))\setminus B_k^{(n)}\right) < \frac{1 - F\mu_{n-1}(\delta_{n-1})}{4}\cdot\mu_{n-1}(X)\,.$$

Now we define μ_n as follows: $\mu_n \equiv \mu_{n-1}$ on $X \setminus \bigcup_{k=1}^{N_n} B_k^{(n)}$, $\mu_n \equiv 0$ on $\bigcup_{k=1}^{N_n} B_k^{(n)}$. Again we have $F\mu_n(\delta_n) < \frac{1}{2^n}$. Now we obtain

$$\mu_n(X) > \mu_{n-1}(X)\left(1 - \frac{1}{2^{n-1}}\right),$$

whence

$$\mu_2(X) > \tau\left(1 - \frac{1}{2}\right) = \frac{\tau}{2},$$

$$\mu_3(X) > \frac{3}{8}\tau\,,$$

$$\mu_n(X) > \mu_{n-1}(X) - \frac{1}{2^{n-1}}\tau > \mu_{n-2}(X) - \frac{1}{2^{n-2}}\tau - \frac{1}{2^{n-2}}\tau > \ldots > \mu_3(X) - \frac{\tau}{8} > \frac{\tau}{4}.$$

So, $\mu_n(X) > \frac{\tau}{4}$ for every $n \in \mathbb{N}$.

From (1) it follows that the measures μ_n and μ_{n+k} differ only on a set $D_{n,k}$ of the from $\bigcup_{l=1}^{k}\bigcup_{m=1}^{N_{n+1}} B_m^{(n+1)}$ satisfying $\mu_n(D_{n,k}) \underset{n\to\infty}{\longrightarrow} 0$ for all $k \in \mathbb{N}$. So, the sequence $\{\mu_n\}$ has a weak limit μ, $\mu \neq 0$, and the measures μ_n, μ differ only on a set D_n of the form $\bigcup_{l=1}^{\infty} \bigcup_{m=1}^{N_{n+1}} B_m^{(n+1)}$ with $\mu_n(D_n) \xrightarrow[n\to\infty]{} 0$. Also, we can assume that $\mu(X) = 1$ taking $\frac{\mu}{\mu(X)}$ in place of μ.

The measure μ is what we wished to construct, i.e., $\lim_{n\to\infty} F_\mu(\delta_n) = 0$. Indeed, if

$$\overline{\lim_{n\to\infty}}\, F_\mu(\delta_n) = \gamma > 0\,, \tag{2}$$

then for some $n \in \mathbb{N}$ we have $\mu_n(D_n) < \frac{\gamma}{3}$, $F_{\mu_n}(\delta_n) < \frac{\gamma}{3}$. Hence, for every family $\{B_k\}_{k=1}^N \subset I_{r<\delta_n}$ of disjoint balls we can write

$$\mu\left(\bigcup_{k=1}^N B_k\right) \le \mu_n\left(\bigcup_{k=1}^N B_k\right) + \mu_n(D_n) < F_{\mu_n}(\delta_n) + \frac{\gamma}{3} < \frac{2\gamma}{3}\,,$$

which implies that $F_\mu(\delta_n) < \frac{2\gamma}{3}$. This contradicts (2). □

Corollary 1. *If $I_{r<1} \notin$ (s.p.) in X, then there exists a sequence $\{\nu_n\}$ of probability measures on X such that*

$$\lim_{n\to\infty} \sup_{\substack{\{B_k\}\subset I_{r<1}\\ B_i\cap B_j=\emptyset}} \nu\left(\bigcup B_k\right) = 0\,.$$

Proof. We define $\nu_n(A) = \mu\{x : nx \in A\}$, where μ is any measure having Property 1. Then

$$\nu_n(B(x_0, 1)) = \mu\{x : \|xn - x_0\| < 1\} = \mu\left(B\left(\frac{x_0}{n}, \frac{1}{n}\right)\right),$$

and Property 1 implies the required equality. □

Property 2. *Let $I_{r<1} \notin$ (s.p.) in X. If ν is a measure on X and $\{B_n\}_1^\infty$ is a sequence of pairwise disjoint balls of class $I_{r<\alpha}$ for some $\alpha > 0$, then*

$$\inf_{x\in X} \nu\left(\bigcup B_n + x\right) = 0\,. \tag{3}$$

In other words, if there exists a "bad" measure μ on X, then the family $I_{r<1}$ is "bad" for every measure ν: each family of pairwise disjoint balls can be moved so that to become "almost disjoint" with supp ν.

Proof. We may assume that $\alpha = 1$. Corollary 1 implies the existence of a sequence $\{\nu_k\}$ of probability measures such that

$$\lim_{n\to\infty} \sup_{x\in X} \nu_n\left(\bigcup B_k + x\right) = 0\,. \tag{4}$$

If relation (3) is not fulfilled, then

$$\inf_{x\in X} \nu\left(\bigcup_{k=1}^\infty B_k + x\right) = \tau > 0\,, \tag{5}$$

and we can find an integer n such that

(6) $$\sup_x \nu_n\left(\bigcup B_k + x\right) \leq \frac{\tau}{2\nu(X)}.$$

For the convolution $\nu * \nu_n$, (5) yeilds

$$(\nu * \nu_n)\left(\bigcup B_k\right) = \int_X \nu\left(\bigcup B_k + x\right) \mathrm{d}\nu_n(x) \geq \tau .$$

On the other hand, relations (6) and (5) imply

$$(\nu * \nu_n)\left(\bigcup B_k\right) = \int_X \nu_n\left(\bigcup B_k + x\right) \mathrm{d}\nu(x) \leq \frac{\tau}{2},$$

and this contradiction completes the proof. □

The facts proved above are of negative nature. Next we present some positive results concerning the scooping property.

Proposition 1. *The scooping property holds for the family* $I_{t=1}$ *in the real space* $C[0, 1]$.

Proof. Fix a probability measure μ on $C[0, 1]$, and let $\varepsilon > 0$. We wish to construct a family $\{B(x_n, 1)\}_{n=1}^N$ of disjoint balls centered at x_n and of radius 1 such that

(7) $$\mu\left(\bigcup_{n=1}^{N} B(x_n, 1)\right) > 1 - \varepsilon .$$

We choose a compact set K satisfying

(8) $$\mu(K) > 1 - \frac{\varepsilon}{4}$$

and fix numbers $M > 0$ and $n_0 \in \mathbb{N}$ such that, for every $f \in K$ and every $t_1, t_2 \in [0, 1]$,

(9) $$\|f\| \leq M,$$

(10) $$|t_1 - t_2| \leq \frac{1}{n_0} \Rightarrow |f(t_1) - f(t_2)| < \frac{1}{2}.$$

For each $i = 0, 1, \ldots, n_0$ we can find a family of disjoint intervals $(\alpha_k^{(i)}, \beta_k^{(i)})$, $k = 1, \ldots, N_i$, lying on the segment $\{\frac{i}{n_0}\} \times [-M, M]$ and such that

(11a) $$|\beta_k^{(i)} - \alpha_{k+1}^{(i)}| < 1, \quad |\alpha_1^{(i)} + M| < 1, \quad |\beta_{N_i}^{(i)} - M| < 1,$$

(11b) $$\mu\left\{f \in K \mid \exists k : f\left(\frac{i}{n_0}\right) \in [\alpha_k^{(i)}, \beta_k^{(i)}]\right\} < \frac{\varepsilon}{4(n_0 + 1)}.$$

Let

$$\gamma_0^{(i)} = \left(\frac{i}{n_0}, \frac{\alpha_1^{(i)} - M}{2}\right), \gamma_k^{(i)} = \left(\frac{i}{n_0}, \frac{\beta_k^{(i)} + \alpha_{k+1}^{(i)}}{2}\right), k = 1, \dots, N_i - 1, \gamma_{N_i}^{(i)} = \left(\frac{i}{n_0}, \frac{\beta_N^{(i)} + M}{2}\right)$$

be the midpoints of the corresponding intervals (see Figure 1).

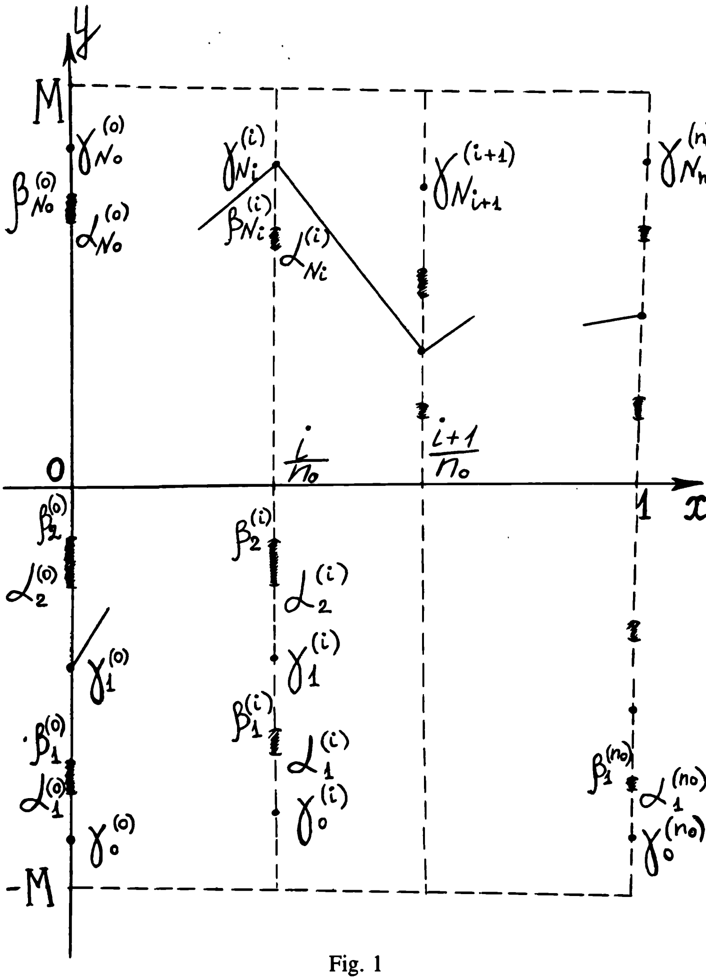

Fig. 1

We look at all polygonal lines with vertices $(\frac{i}{n_0}, \gamma_k^{(i)})$, $i = 0, \dots, n_0$, $k = 0, \dots, N_i$ (with the natural order of vertices), and denote the corresponding piecewise linear functions by $\varphi_1, \dots, \varphi_N$. It is not hard to show that

$$\mu\left(\bigcup_{k=1}^{N} B(\varphi_k, 1)\right) > 1 - \frac{\varepsilon}{2}$$

(indeed, by (8) and (11b), it suffices to verify that

(12) $$\bigcup_{k=1}^{N} B(\varphi_k, 1) \supset G\,,$$

where $G \text{ def} \to \;=\left\{f \in K : f\left(\frac{i}{n_0}\right) \notin [\alpha_k^{(i)}, \beta_k^{(i)}] \text{ if } k = 1, \ldots, N_i, i = 0, \ldots, n_0\right\}$).

However, the balls of the family $\{B(\varphi_k, 1)\}_{k=1}^N$ are not disjoint; we are going to remove this difficulty.

For every $n \in \mathbb{N}$, we construct a family of functions $\{\varphi_n^{(k)}\}_{k=1}^N$. The construction will be one and the same on every segment $[\frac{i}{n_0}, \frac{i}{n_0} + \frac{i}{4nn_0}]$. Every $\varphi_n^{(k)}$ will coincide with φ_k outside of $[\frac{i}{n_0}, \frac{i}{n_0} + \frac{i}{4nn_0}]$. So, we shall describe the construction of $\varphi_n^{(1)}$ on the segment $[0, \frac{i}{4nn_0}]$.

Let $\varphi_k(0) = z^{(k)}$, $z^{(k)}$ being the midpoint of the corresponding segment $[\beta_i^{(0)}, \alpha_{i+1}^{(0)}]$. Let $2l_k$ be the length of this segment, and let Δ_0 be the smallest length of the segments $[\alpha_p^{(i)}, \beta_p^{(i)}]$, $p = 1, \ldots, N_i$, $i = 0, \ldots, n_0$. We set $R_k = z^{(k)} + 1 - l_k - \frac{\Delta_0}{2}$, $r_k = z^{(k)} - 1 + l_k + \frac{\Delta_0}{2}$.

In order to construct the family $\{\varphi_n^{(k)}\}_{k=1}^N$, we proceed by induction on k as follows:

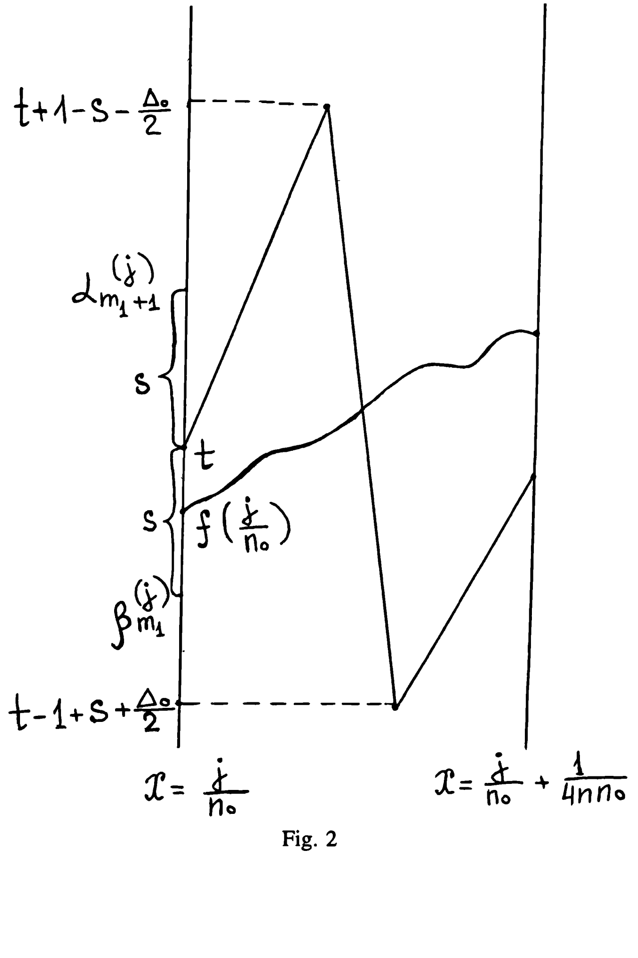

Fig. 2

1. With every function $\varphi_n^{(k)}$ we associate two nonempty disjoint finite subsets of $[0, \frac{1}{4nn_0}]$, namely, P_k — the "set of peak-point", and H_k — the "set of hollow-points" so that the following conditions be fulfilled: a) for every $k_1 < k_2 \le N$,

$$H_{k_2} \cap P_{k_1} \neq \emptyset \quad \text{and} \quad P_{k_2} \cap H_{k_1} \neq \emptyset\,;$$

b) every function $\varphi_n^{(k)}$ has at least one peak-point and at least one hollow-point which are not among the peak-points or hollow-points of $\varphi_n^{(m)}$, $m < k$.

2. For $p \in P_k$, $h \in H_k$ we set $\varphi_n^{(k)}(p) = R_k$, $\varphi_n^{(k)}(h) = r_k$; $\varphi_n^{(k)}(0) = z^{(k)}$; $\varphi_n^{(k)}(\frac{1}{4nn_0}) = \varphi_k(\frac{1}{4nn_0})$. It remains to extend $\varphi_n^{(k)}$ by linearity to the other points of $[0, \frac{1}{4nn_0}]$.

Now for every pair of functions $\varphi_n^{(k_1)}$, $\varphi_n^{(k_2)}$ we have $\|\varphi_n^{(k_1)} - \varphi_n^{(k_2)}\| \ge 2$. Indeed, $\varphi_n^{(k_1)}(\frac{j}{n_0}) \neq \varphi_n^{(k_2)}(\frac{j}{n_0})$ for some j, $0 \le j \le n_0$. We may assume that $\varphi_n^{(k_1)}(\frac{j}{n_0}) < \varphi_n^{(k_2)}(\frac{j}{n_0})$, $k_2 > k_1$. Let $\varphi_n^{(k_1)}(\frac{j}{n_0}) = \varphi_{k_1}(\frac{j}{n_0}) = t_1$, $\varphi_n^{(k_2)}(\frac{j}{n_0}) = \varphi_{k_2}(\frac{j}{n_0}) = t_2$; then t_1, t_2 are the midpoints of the corresponding segment $[\beta_{m_1}^{(j)}, \alpha_{m_1+1}^{(j)}]$, $[\beta_{m_2}^{(j)}, \alpha_{m_2+2}^{(j)}]$ of length s_1 and s_2, respectively. Taking a point $u_0 \in [\frac{j}{n_0}, \frac{j}{n_0} + \frac{1}{4nn_0}]$ for $j < n_0$ (or $[1 - \frac{1}{4nn_0}, 1]$ for $j = n_0$) such that u_0 is a peak-point of $\varphi_n^{(k_2)}$ and a hollow-point of $\varphi_n^{(k_1)}$, we obtain

$$\varphi_n^{(k_2)}(u_0) - \varphi_n^{(k_1)}(u_0) = \left(t_2 + 1 - \frac{s_0}{2} - \frac{\Delta_0}{2}\right) - \left(t_1 - 1 + \frac{s_1}{2} - \frac{\Delta_0}{2}\right) =$$

$$2 + (t_2 - t_1) - \left(\frac{s_2}{2} + \frac{s_1}{2} + \Delta_0\right) \ge 2 + (t_2 - t_1) - \left(\frac{s_2}{2} + \frac{s_1}{2} + \beta_{m_2}^{(j)} - \alpha_{m_1+1}^{(j)}\right) = 2$$

(see Figure 3). Thus, we have $B(\varphi_n^{(k_1)}, 1) \cap B(\varphi_n^{(k_1)}, 1) = \emptyset$. It remains to prove that for large n

$$\mu\left(\bigcup_{k=1}^{N} N(\varphi_n^{(k)}, 1)\right) \ge 1 - \varepsilon\,,$$

or by (12), that $\bigcup_{k=1}^{N} B(\varphi_n^{(k)}, 1) \supset G$. To this end, we take a number $n \in \mathbb{N}$ such that, for every $f \in G$,

$$|v_1 - v_2| < \frac{1}{4nn_0} \Rightarrow |f(v_1) - f(v_2)| < \frac{\Delta_0}{4}$$

(we use the equicontinuity of the functions in G). For any $f \in G$ we can find a function φ_k that takes the value $\varphi_k(\frac{j}{n_0})$ on the same segment $[\beta_m^{(j)}, \alpha_{m+1}^{(j)}]$ with center t and of length $2s$ to which $f(\frac{j}{n_0})$ belongs, $j = 0, \dots, n_0$. Then for $t \in [\frac{j}{n_0}, \frac{j}{n_0} + \frac{1}{4nn_0}]$ we have

$$|f(t) - \varphi_n^{(k)}(t)| < \left|f\left(\frac{j}{n_0}\right) - \varphi_n^{(k)}(t)\right| + \frac{\Delta_0}{4} \le \max\left\{\left|f\left(\frac{j}{n_0}\right) - \left(t + 1 - s - \frac{\Delta_0}{2}\right)\right|,\right.$$

$$\left.\left|f\left(\frac{j}{n_0}\right) - \left(t - 1 + s + \frac{\Delta_0}{2}\right)\right|\right\} + \frac{\Delta_0}{4} \le \max\left\{1 - \frac{\Delta_0}{2}, \left|1 - 2s - \frac{\Delta_0}{2}\right|\right\} + \frac{\Delta_0}{4} < 1$$

(Δ_0 is small); see Figure 2. For $t \notin [\frac{j}{n_0}, \frac{j}{n_0} + \frac{1}{4nn_0}]$ the required inequality

$|f(t) - \varphi_n^{(k)}(t)| < 1$ follows from (12) (for such t we have $\varphi_n^{(k)}(t) = \varphi_k(t)$ by definition). □

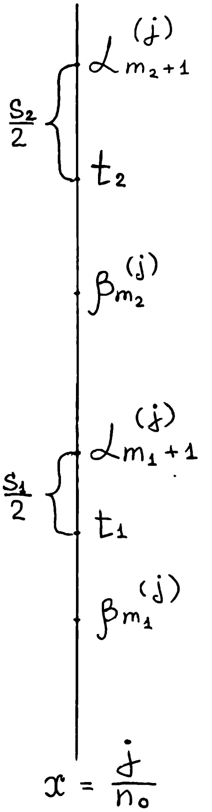

Fig. 3

Of course, $C[0, 1]$ cannot be viewed as the only example of infinitely dimensional separable Banach space with the scooping property (e.g., the space c_0 obviously has this property). In particular, it would be desirable to find an equivalent norm on l_2 such that the scooping property be true for $I_{r=1}$ (the Hilbert balls are too "round" for this). We can try "to square" the Hilbert balls adjusting them somewhat to each other. To this end we introduce the following construction.

Let $L = \{x_n\}$ be a lattice in l_2 with finite ratio of the covering number and the packing number. We recall that $L = \{x_n\}$ is a lattice if $x_i \pm x_j \in L$ for every $i, j \in \mathbb{N}$. The covering number of L in X is $c(L) = \min\{R: X = \bigcup_n B(x_n, R)\}$, the packing number is $p(L) = \max\{r: B(x_i, r) \cap B(x_j, r) = \emptyset \text{ for } i \neq j\}$. A lattice L with $\frac{c(L)}{p(L)} < \infty$ does exist (this fact follows from a theorem in [6]. We consider the following analog of the Voronoy polyhedron:

$$U_0 = \{x \in l_2 : \|x\| < \|x - x_n\| \text{ for all } x_n \in L \setminus \{0\}\}.$$

Then U_0 is a convex set equivalent to the unit ball U of l_2. Let $\|\cdot\|_1$ be the norm determined by the unit ball U_0. It is obvious that if $x_i, x_j \in L$, $x_i \neq x_j$, then

$$(U_0 + x_i) \cap (U_0 + x_j) = \emptyset .$$

In finite-dimensional spaces, the set $\bigcup(\bar{U}_0 + x_i)$ coincides with the entire space, but in the infinite-dimensional case this is not true (there are points in X for which there is no nearest point in the lattice).

Let $\bigcup_{i=1}^{\infty}(U_0 + x_i) = A$, and let μ be a probability measure on l_2. Is it true that for every $\varepsilon > 0$ there exist $\alpha \in (0, 1]$ and $x \in l_2$ such that

$$\mu\left(\alpha \bigcup_{i=1}^{\infty}(U_0 + x_i) + x\right) > 1 - \varepsilon ?$$

(If the answer is "yes", then we obtain the scooping property for $I_{r<1}$ in $X = (l_2, \|\cdot\|_1)$ and, of course, the positivity principle).

We do not know the answer. But we can perturb U_0 slightly so that the answer does become "yes". Moreover, the following assertions hold.

Proposition 2A. *Let H be the real separable Hilbert space. There is an equivalent norm on H with unit ball B such that for every probability measure μ on H and every positive ε, δ there exists a finite family $\{D_n\}_{n=1}^{N}$ of open convex sets*

1. $D_i \cap D_j = \emptyset$, $i \neq j$;
2. $D_i = D_1 + x_i$ *for some* $x_i \in H$, $i = 2, 3, \dots, N$;
3. $\mu\left(\bigcup_{n=1}^{N} D_n\right) > 1 - \varepsilon$;
4. $\varrho(D_1 - x, B) < \delta$ *for some* $x \in H$ *(ϱ is the Hausdorff metric).*

If we want to scoop all the measure μ, then we lose the congruence of the D_i's. As a substitute, we can use the following statement.

Proposition 2B. *There is an equivalent norm on H with unit ball B such that for every probability measure μ and every $\delta > 0$ there exists a family $\{\mathscr{E}_n\}_{n=1}^{\infty}$ of open convex sets*

1. $\mathscr{E}_i \cap \mathscr{E}_j = \emptyset$, $i \neq j$;
2. $\mu\left(\bigcup_{n=1}^{\infty} \mathscr{E}_n\right) = 1$;
3. $\forall n \in \mathbb{N}\ \exists x_n \in H : \varrho(\mathscr{E}_n - x_n, B) < \delta$.

Finally, we mention that not only the measure but also an arbitrary compact set can be scooped by means of disjoint equal convex sets.

Proposition 2C. *There is an equivalent norm on H with unit ball B such that for every compact set $K \subset H$ and every $\varepsilon > 0$ there exists a finite family $\{G_n\}_1^N$ of convex open sets*

1. $G_i \cap G_j = \emptyset,\ i \neq j$;
2. $K \subset \bigcup_{i=1}^{N} \bar{G}_i$;
3. $G_i = G_1 + y_i$ *for some* $y_i \in H,\ i = 1, 2, \ldots, N$;
4. $\varrho(G_1 - x, B) < \varepsilon$ *for some* $x \in H$.

Propositions 2A, 2B, 2C can be generalized.

Proposition 2A'. *Let* $(X, \|\cdot\|)$ *be a uniformly convex Banach space with basis. Then there is an equivalent pseudonorm* $\|\cdot\|_1$ *on* X, $\|\cdot\| \leq \|\cdot\|_1 \leq 3\|\cdot\|$, *with "unit ball"* B *(a star-shaped open set which is bounded and symmetric) such that for every measure* μ *on* X *and every positive* ε, δ *there exists a finite family* $\{D_n\}_1^N$ *of star-shaped open bounded sets satisfying conditions* 1.–4. *of Proposition 2A.*

Propositions 2B and 2C have similar generalizations.

We prove one of the above propositions (the other statements can be proved similarly).

Proof of Proposition 2A. We consider the Banach space $X = \mathbb{R}_1^1 \oplus_1 H = \{(a, x_1, x_2, \ldots) : a \in \mathbb{R}, (x_1, x_2, \ldots) \in H\}$ with the norm $\|(a, x_1, \ldots)\| = |a| + \|(x_1, x_2, \ldots)\|_H$. We identify H with the hyperplane $\tilde{H} \subset X : \tilde{H} = \{(1, x_1, x_2, \ldots) : (x_1, x_2, \ldots) \in H\}$. Let Y denote the subspace $\{0, x_1, \ldots) : (x_1, \ldots) \in H\}$. We construct a lattice L in Y such that $\overline{\operatorname{span}} L = Y$ and $r(L) \leq 3$, $p(L) = 1$ (this is possible; see [6]). Consider the subset

$$\tilde{B}_0 = \left\{x \in \tilde{H} : \inf_{\substack{y \in L \\ y \neq 0}} \|x - y\|_X > \|x\|\right\}$$

of the hyperplane $\tilde{H}$. It is obvious that every set $\tilde{B}_z = \{x \in \tilde{H} : \inf_{\substack{y \in L \\ y \neq z}} \|x - y\|_X > \|x - z\|\}, z \in L$, can be obtained from $\tilde{B}_0$ by translation by z. The sets $\tilde{B}_z$ are some analogs of the Voronoy polyhedrons. They are open and convex, and for the unit ball U of H we have $U \subset \tilde{B}_0 \subset 3U$.

Now it will be shown that we can take $B = \tilde{B}_0$ (B is the required unit ball mentioned in Proposition 2A). Let μ be a measure on H (hence, on $\tilde{H}$). We fix $\varepsilon > 0$ and $\delta > 0$. Let $\varphi(x) = \inf_{a \in L} \|x - a\|$, $x \in \tilde{H}$. We can find a finite set $\{a_1, a_2, \ldots, a_N\}$ in L such that

$$\mu\left\{x \in \tilde{H} : \min_{i = 1, \ldots, N} \|x - a_i\| - \varphi(x) > \frac{\delta}{2}\right\} < \varepsilon\,. \tag{13}$$

Assume that $a_1 = 0$; replacing the set $\{a_1, \ldots, a_N\}$ in L by the set $\{b_1, \ldots, b_N\}$ with $b_i = a_i + (\delta, 0, \ldots)$, we obtain L_1 in place of L.

Now we consider the following analogs of the Voronoy polyhedron for L_1:

$$C_b = \{x \in \tilde{H} : \|x - b\| < \inf_{\substack{y \in L \\ y \neq 0}} \|x - c\|\}, \quad b \in L_1\,.$$

It is easy to check that we can take $D_i = C_b$, $i = 1, 2, \ldots, N$. Indeed, the inequality $\mu(\bigcup C_{b_i}) > 1 - \varepsilon$ follows from (13), and the condition $\varrho(C_{b_1}, U) < \delta$ can be proved by contradiction. □

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