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On Some Descriptive Characterizations of the N^{-1}-Property

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Let E be a nonempty subset of $\mathbb{R}^n$. A mapping (i.e. function) $f: E \to \mathbb{R}^k$ is said to have the N^{-1}-property [1] if

$$\forall A \subset \mathbb{R}^k \{|A| = 0 \Rightarrow |f^{-1}(A)| = 0\}, \tag{1}$$

where $|.|$ denotes the outer Lebesgue measure. Being in a certain sense "inverse" with respect to the widely known Lusin's N-property, the N^{-1}-property turns out to be interesting in some settings particularly when one studies properties of a composite mapping $g \circ f$ via properties of g and f which are fulfilled almost everywhere (a.e.). For instance if g and f are a.e. differentiable, each on its domain, and $g \circ f$ is defined, then $g \circ f$ is not in general a.e. differentiable, but it certainly is if f has the N^{-1}-property. Some systematic study of the N^{-1}-property for smooth and for a.e. differentiable functions can be found in [3] (where the N^{-1}-property was termed as the "0"-property), [4].

In what follows we deal with continuous (or continuous a.e. differentiable) functions $f: M \to \mathbb{R}^k$ where M is a compact subset of $\mathbb{R}^n$, $|M| > 0$.

It can be easily checked [3] that the N^{-1}-property is equivalent to each of the following properties:

$$(N_1^{-1})\ \forall A \subset M, \quad A \text{ compact } \{|A| > 0 \Rightarrow |f(A)| > 0\};$$

$$(N_1^{-1})\ \forall B \subset \mathbb{R}^k, \quad B \text{ compact } \{|B| = 0 \Rightarrow |f(B)| = 0\};$$

1. Our first result concerns continuous mappings

Let M be a compact subset of $\mathbb{R}^n$, $|M| > 0$. Let $X = C(M, \mathbb{R}^k)$ i.e. the space of all continuous mappings $M \to \mathbb{R}^k$ with the norm $\|f\| = \sup_{x \in M} \|f(x)\|$. We

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denote by $N^{-1} = N^{-1}(X)$ the subset of X consisting of all mappings with the N^{-1}-property.

Theorem 1. *N^{-1} is the 1st category $\mathscr{F}_{\sigma\delta}$-subset of X.*

Proof. Clearly we may assume thet $M \subset \mathscr{Q} := [0, 1]^n$. For natural numbers m, v we let

$$G_{m,v} = \{f \in X : \exists A \subset M,\ A \text{ compact},\ |A| \geq |M|/2m,\ |f(A)| < v^{-2}\}. \qquad (2)$$

It is almost immediate that $G_{m,v}$ is open.

For if $f \in G_{m,v}$ then there exists a compact set $A \subset M$ having properties indicated in (2). We can therefore take an open set $\Omega \supset f(A)$ such that $|\Omega| < v^{-2}$. Since $f(A)$ is compact, we have

$$\varepsilon := \operatorname{dist}(f(A), \mathbb{R}^k \setminus \Omega) > 0\,,$$

whence it follows that the ball $\{f \in X : \|f - f_0\| < \varepsilon\}$ is contained in $G_{m,v}$. Next we will show that $G_{m,v}$ is dense in X. To this end we subdivide [0, 1], for each natural $p > 1$, into p equal intervals. This given us the partition of the cube $Q = [0, 1]^n$ into p^n equal cubes Q_{pi}, $1 \leq i \leq p^n$.

Fix a number α, $0 < \alpha < 1$, sufficiently close to 1, such that we have

$$\sum_{i=1}^{p^n} |Q'_{pi} \cap M| < |M|/2m\,, \qquad (3)$$

where Q'_{pi} is the cube, concentric with Q_{pi}, defined by the relation $\operatorname{diam} Q'_{pi} = \alpha Q_{pi}$. Next for each $i \in \{1, \ldots, p^n\}$ consider one more cube Q''_{pi} concentric with Q_{pi}, and defined by $\operatorname{diam} Q''_{pi} = (2 - \alpha)\, Q_{pi}$. It follows from this construction that

$$\forall i \in \{1, \ldots, p^n\} : Q'_{pi} \cap \bigcup_{j \neq i} \operatorname{Int} Q''_{pj} = \emptyset \qquad (4)$$

and that the family

$$\{\operatorname{Int} Q''_{pi},\ 1 \leq i \leq p^n\} \qquad (5)$$

forms an open covering of Q. Let $\{\theta_{pi}, 1 \leq i \leq p^n\}$ be a continuous partition of unity subordinated to the covering (5) ($\operatorname{supp} \theta_{pi} \subset \operatorname{Int} Q''_{pi}$). Then by (4) we have

$$\forall i \in \{1, \ldots, p^n\}\ \forall x \in Q'_{pi} : \sum_{j=1}^{p^n} \theta_{pj}(x) = \theta_{pi}(x) = 1\,. \qquad (6)$$

Let $I_p = \{i \in \{1, \ldots, p^n\} : Q_{pi} \cap M \neq 0\}$. For each $i \in I_p$ fix a point $x_{pi} \in Q_{pi} \cap M$. Now let f be any element of the space X. For each natural $p > 1$ consider the mapping $f_p : M \to \mathbb{R}^k$ defined by

$$f_p(x) = \sum_{i \in I_p} f(x_{pi})\, \theta_{pi}(x),\ \ x \in M\,. \qquad (7)$$

It is trivial that $f_p \in X$. We will show that $f_p \in G_{m,v}$. Let

$$A_p = \bigcup_{i \in I_p} Q'_{pi} \cap M\,. \tag{8}$$

By (3) we have $|A_p| \geq |M|/2m$, whereas (6) yields that d_p is constant on each set $Q'_{pi} \cap M$ whence $|f_p(A_p)| = 0$ and consequently, $f_p \in G_{m,\nu}$ for each $p > 1$ (and each ν). Before showing that $f_p \to f$ we will make the following remark. Let $x \in Q$. Then $x \in Q_{pi}$ for some i. Let

$$\mathscr{I}(p, i) = \{j \in \{1, \dots, p^n\} : Q''_{pj} \cap Q_{pi} \neq \emptyset\}\,.$$

An easy consideration shows that there exist at most 3^n cubes Q''_{pj} intersecting Q_{pi}. These are exactly those cubes for which $Q_{pj} \cap Q_{pi} \neq \emptyset$. Hence card $\mathscr{I}(p, i) \leq 3^n$. Furthermore all such cubes Q''_{pi} are contained in the cube Q^*_{pi} concentricwith Q_{pi} and whose diameter is five times bigger then that of the cube Q_{pi} (clearly diam $Q^*_{pi} = 5\sqrt{n}/p$). Now let $x \in M$. Then $x \in Q_{pi}$ for some i and, taking into consideration the above remark, we obtain that

$$\|f_p(x) - f(x)\| \leq \sum_{j \in \mathscr{I}(p, i)} \|f(x_{pj}) - f(x)\|\, \theta_{pj}(x) \leq 3^n \omega(f, 5\sqrt{n}/p)\,,$$

where $\omega(f, \delta)$ is the modulus of continuity of f. Thus $f_p \to f$ in X as $p \to \infty$ which shows that $G_{m,\nu}$ is dense in X.

Next we wish to prove that

$$X \setminus N^{-1} = \bigcup_{m=1}^{\infty} G_m\,, \tag{9}$$

where

$$G_m := \bigcap_{\nu=1}^{\infty} G_{m,\nu}\,.$$

Let $f \in X \setminus N^{-1}$. There exists a (compact) set $A \subset M$, $|A| > 0$ such that $|f(A)| = 0$. It follows immediately that $f \in G_{m,\nu}$ for each $m \geq |M|/2|A|$ and each ν, i.e. f is in the right hand part of (9). Conversely, let $f \in G_{m,\nu}$ for some m, and let $\{A_\nu\}$ be the corresponding sequence of (compact) subsets of M such that

$$|A_\nu| \geq |M|/2m,\ |f(A_\nu)| < \nu^{-2}\,. \tag{10}$$

Consider the set $B := \limsup A_\nu$. The first relation in (10) implies that $|B| \geq |M|/2m$, and from the second one we get that for each i

$$|f(B)| \leq \sum_{\nu \geq i} |f(A_\nu)| \leq \sum_{\nu \geq i} \nu^{-2}\,,$$

whence $|f(B)| = 0$, which means that $f \in X \setminus N^{-1}$. Thus we have shown that (9) holds.

Now to finish the proof of the theorem it remains to observe that $G_{m,\nu}$ being open and dense in the complete space X, each G_m is a dense $\mathscr{G}_\delta$-set. And we conclude finally from (9) that N^{-1} is the 1st category $\mathscr{F}_{\sigma\delta}$-set.

2. N^{-1}-property for continuous almost everywhere differentiable mappings

In this section we restrict ourselves to considering mappings $Q = [0, 1]^n \to \mathbb{R}^n$ and we weill make use of.

Theorem 2. *[3] Let $f: \Omega \to \mathbb{R}^n$ be a continuous a.e. differentiable mapping, where Ω is an open subset of $\mathbb{R}^n$. Then the following equivalence holds*

$$f \text{ has the } N^{-1}\text{-property} \Leftrightarrow \det f' \neq 0 \text{ a.e. on } \Omega. \tag{11}$$

Now let X be the space of all continuous a.e. differentiable mappings $f = (f^1, \ldots, f^n): Q \to \mathbb{R}^n$ such that $D_j f^i \in L(Q)$, $1 \leq i, j \leq n$, where $D_j f^i = \partial f^i / \partial x_j$. The norm on X is defined by

$$\|f\| = \sup_{x \in Q} \|f(x)\| + \sum_{1 \leq i,j \leq n} \int_Q |D_j f^i| \, dx \tag{12}$$

and again we denote by $N^{-1} = N^{-1}(X)$ the set of all elements of X having the N^{-1}-property.

Theorem 3. *N^{-1} is a $\mathcal{G}_\delta$-subset of X.*

Proof. It is obvious that Theorem 2 works if an open set Ω is replaced by the closed cube Q. Applying (11) we may write at once

$$X \setminus N^{-1} = \bigcup_{m=1}^{\infty} F_m, \tag{13}$$

where

$$F_m = \left\{ f \in X : \exists A \subset Q \text{ measurable, } |A| \geq \frac{1}{m}, \det f' \mid A = 0 \right\}. \tag{14}$$

This reduces the proof to showing that the sets F_m are closed. Let $f_k \to f$ in X, $f_k = (f_k^1, \ldots, f_k^n) \in F_m$, $f = (f^1, \ldots, f^n)$. By (12) it follows that $D_j f_k^i \to D_j f^i$ in $L^1(Q)$, $1 \leq i, j \leq n$. Hence there exists a subsequence $\{f_{k_\nu}\}$ such that $D_j f_{k_\nu}^i \to D_j f^i$ a.e. on Q, $1 \leq i, j \leq n$. Denote by $\{A_{k_\nu}\}$ the sequence of corresponding measurable sets determined by (14) i.e. such that for each ν

$$|A_{k_\nu}| \geq \frac{1}{m} \quad \text{and} \quad \det f'_{k_\nu} \mid A_{k_\nu} = 0, \tag{15}$$

and let $A := \limsup A_{k_\nu}$. Rejecting, if necessary, sets of measure zero, we may assume with no loss of generality that $D_j f_{k_\nu}^i \to D_j f^i$ (pointwise) on A and that f and f_{k_ν}, $\nu = 1, 2, \ldots$, are differentiable on A. Let x be any point of A. There exists a subsequence $\{A_{k_{\nu_s}}\}$ (depending on x) such that $x \in A_{k_{\nu_s}}$ for all s. Then by (15) we get

$$\det f'(x) = \lim_{s \to \infty} \det f'_{k_{\nu_s}}(x) = 0.$$

Hence $\det f' = 0$ on A, and since $|A| \geq \frac{1}{m}$ we have that $f = \lim f_k \in F_m$. Thus F_m is closed which concludes the proof.

Remark. It was observed in [3] that Theorem 2 remains valid, with minor modification in proof, if differentiability a.e. is replaced by existence a.e. of finite derivatives $D_j f^i$ (in which case $f'(x)$ in (11) is understood as a linear operator whose matrix in standard basis of $\mathbb{R}^n$ is $(D_j f^i(x))$). A brief examination of the proof then show that Theorem 3 is also valid under this weaker assumption.

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