Acta Universitatis Carolinae. Mathematica et Physica

Reinhard Börger

Connectivity properties of sequential Boolean algebras

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 36 (1995), No. 2, 5--18

Persistent URL: http://dml.cz/dmlcz/702020

Terms of use:

© Univerzita Karlova v Praze, 1995

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Connectivity Properties of Sequential Boolean Algebras

R. BÖRGER

Hagen*)

Received 15. March 1995

Dedicated to the memory of Václav Koutník

Introduction

In this paper we study connectivity properties of a Boolean algebra with a sequential topology such that the finitary Boolean operations are sequentially continuous. On the first glance it may be surprising that such Boolean algebras can be connected at all, in contrast to the well-known result that compact Boolean algebras are (topological and algebraic) powers of discrete two-element Boolean algebras and therefore zero-dimensional. The second surprise is the close relationship between the purely topological property of connectedness and the purely algebraic notion of atom. The third surprise might be the relationship between path-connectedness and Souslin's Hypothesis. I am indebted to my deceased friend Václav Koutník and also to my friend Roman Frič for valuable discussions about sequential spaces, moreover to Bohuslav Balcar for discussions about Boolean algebras. Some of the results are taken from the unpublished paper [2].

1 Terminology and Basic Properties

A subset A of a topological space X is called *sequentially closed* if for every sequence $(x_n)_{n \in \mathbb{N}}$ in A which converges to some $x \in X$ in X it follows that $x \in A$. Every closed set is sequentially closed, and the collection of all sequentially closed sets is closed under finite unions and arbitrary intersections. Therefore we can

*) Fachbereich Mathematik, Fernuniversität, 58084 Hagen, Germany

define a topological space sX with the same points as X, such that the closed subsets of sX are the sequentially closed subsets of X. Then sX is called the *sequential modification* of X, and X is called *sequential* if s$X = X$, i.e. if every sequentially closed subset of X is closed. It is easy to see that sX is always sequential. Categorically speaking, sequential spaces form a coreflective subcategory of the category of all topological spaces with coreflection s. Therefore the class of sequential spaces is closed under topological sums (= coproducts) and quotients. But subspaces, binary products and continuous images of sequential spaces need not be sequential.

For sequential spaces X, Y the sequential modification $X \sqcap Y := \mathrm{s}(X \times Y)$ of the topological product is a product in the category of sequential spaces. A ***sequential Boolean algebra*** is a Boolean algebra B with a sequential topology such that complementation $B \to B$, $x \mapsto x^*$ and binary meet $B \sqcap B \to B$, $(x, y) \mapsto x \wedge y$ are continuous. The latter condition means that if $(x_n)_{n \in \mathbb{N}}$ converges to x and $(y_n)_{n \in \mathbb{N}}$ converges to y in B, then $(x_n \wedge y_n)_{n \in \mathbb{N}}$ converges to $x \wedge y$. By an unpublished result due to B. Balcar this is strictly weaker than continuity of "$\wedge$" as a map $B \times B \to B$ (on the topological product). Note that these conditions imply that all finitary Boolean operations (like the binary join "$\vee$") are sequentially continuous and thus in particular separately continuous (i.e. continuous in each variable).

In every sequential Boolean algebra B, the intersection of all 0-neighbourhoods B is an ideal I, which can be equivalently described as the set of all limits of the constant sequence $(0)_{n \in \mathbb{N}}$ or of all $x \in B$ such that the constant sequence $(x)_{n \in \mathbb{N}}$ converges to 0. Then B/I is a sequential Boolean algebra in the quotient topology and for the canonical projection $p: B \to B/I$ we see that a sequence $(x_n)_{n \in \mathbb{N}}$ in B converges to $x \in B$ if and only if $(p(x_n))_{n \in \mathbb{N}}$ converges to $p(x)$ in B/I. It is easily seen that B/I is the T_0-reflection of B; in particular we have $I = \{0\}$ if B is T_0 as a topological space. On the other hand sequential limits in B/I are always unique because the symmetric difference $B \sqcap B \to B$ $(x, y) \mapsto x \triangle y := (x^* \wedge y) \vee \vee (x \wedge y^*) = (x \vee y) \wedge (x \vee y)^* = (x \vee y) \wedge (x \wedge y)^*$ is (sequentially) continuous and B is an abelian group under "$\triangle$" hence if $(x_n)_n$ converges to both x and y, then $(0)_{n \in \mathbb{N}}$ converges to $x \triangle y \in I$. In particular, B/I is T_1, because limits of constant sequences are unique. On the other hand, B is not Hausdorff in general (B. Balcar, unpublished).

So in the sequel we shall restrict our attention to T_0 sequential Boolean algebras; then sequential limits are unique. When we work with sequential topologies, it is often convenient to think in terms of convergent sequences rather than open or closed sets. Obviously, in a sequential space closed sets and therefore open sets are determined by convergent sequences, but in general it is not easy to decide whether a notion of convergence (for sequences) is induced by a sequential topology, some complicated conditions were given by V. Koutník [7]. If sequential limits are unique, there is the following well-known criterion, which was independently proven by Dolcher [4] and Kisyński [6]; An abstract notion of sequential convergence is induced by a sequential topology with unique sequential limits if and only if it satisfies the following conditions:

(CS1) Every constant sequence $(x)_{n\in\mathbb{N}}$ converges to x.

(CS2) Sequential limits are unique.

(CS3) If a sequence $(x)_{n\in\mathbb{N}}$ converges to x, then every subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to x.

(CS4) If every subsequence of $(x_n)_{n\in\mathbb{N}}$ contains a subsequence that converges to x, then $(x_n)_{n\in\mathbb{N}}$ itself converges to x.

Condition (CS4) is usually called "*Urysohn's axiom*". In every topological space, sequential convergence satisfies (CS1,3,4) but in general (CS1,3,4) do not imply that the convergence is induced by a topology [4].

Since different authors use different topological terminologies, we shall fix our notations. A topological space is called "*totally disconnected*" if different points can be separated by clopen sets; X is called "*hereditarily disconnected*" if the connected components of X are single points. Then every totally disconnected space is hereditarily disconnected and Hausdorff, and every hereditarily disconnected space is T_1. In particular, a sequential Boolean algebra B can only be hereditarily disconnected, if the above ideal I is zero. Moreover, B is connected if and only if B/I is connected. This justifies our restriction to the case $I = \{0\}$.

Our first observation is that *atoms* are obstacles against connectedness.

1.1 Proposition

For every T_0 *sequential Boolean algebra B the following assertions hold:*

(i) *If* $a \in B$ *is an atom, then* $\{x \in B : a \wedge x = 0\}$ *is open and closed.*

(ii) *If B is connected, then B is atomless.*

(iii) *If B is atomic, then B is totally disconnected.*

Proof. (i) The map $B \to B$, $x \mapsto a \wedge x$ is continuous, hence the inverse images $Z_0 := \{x \in B |\, a \wedge x = 0\}$ and $Z_1 := \{x \in B |\, a \wedge x = a\}$ of 0 and a are closed, because B is weakly Hausdorff and therefore T_1. But since a is an atom and $0 \leq a \wedge x \leq a$ for all $x \in B$, we have $Z_0 \cup Z_1 = B$, $Z_0 \cap Z_1 = \emptyset$, thus $Z_0 = B \backslash Z_1$ is also open.

(ii) If $a \in B$ were an atom, for Z_0 as above we should get $0 \in Z_0$, $1 \notin Z_0$, hence Z_0 were open and closed with $\emptyset \subsetneqq Z_0 \subsetneqq B$.

(iii) If $x, y \in B \backslash \{0\}$ with $x \neq y$, we have $x \nleq y$ or $y \nleq x$, w.l.o.g. $x \nleq y$. Then we have $x \wedge y^* \neq 0$, hence there is an atom a with $a \leq x \wedge y^* \leq x$, thus

$a \wedge x = a \neq 0$ and $a \wedge y \leq x \wedge y^* \wedge y = 0$, thus $x \notin Z_0$, $y \in Z_0$ as above. In particular, no connected subset of B contains x and y.

In general, the converses of (ii) and (iii) above are not true; every non-trivial atomless Boolean algebra with discrete topology is a counterexample to both statements. But the converses are true under certain additional conditions. Our main tool is the following simple

1.2 Proposition

A (not necessarily weakly Hausdorff) sequential Boolean algebra is connected if 0 *and* 1 *belong to the same connected component.*

Proof. For any element a the map $b \mapsto a \wedge b$ is continuous, therefore $0 = a \wedge 0$ and $a = a \wedge 1$ lie in the same component.

We are also interested in the analogous question about path-connectedness. Recall that a topological space is called *path-connected* if for all $x, y \in X$ there exists a path from x to y, i.e. a continuous map $f : [0, 1] \to X$ with $f(0) = x$ and $f(1) = y$, where $[0, 1] \subset \mathbb{R}$ carries the usual topology.

1.3 Proposition

For B a sequential Boolean algebra, the following statements are equivalent:

(i) *B is contractible.*
(ii) *B is simply connected.*
(iii) *B is path-connected.*
(iv) 0 *and* 1 *lie in the same path-component of B.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is trivial. (iv) $\Rightarrow$ (i): Let $f : [0, 1] \to B$ be a path from 0 to 1 in B, where $[0, 1]$ is the real unit interval, i.e. $f(0) = 0$, $f(1) = 1$, and f is continuous. Then $F : B\amalg[0, 1] \to B$, $(x, t) \mapsto a \wedge f(t)$ is also continuous with $F(x, 0) = x \wedge f(0) = x \wedge 0 = 0$ and $F(x, 1) = x \wedge f(1) = x \wedge 1 = x$ for all $x \in B$. But the sequential product coincides wit the k-product (cf.[8]), and since $[0, 1]$ is compact we obtain $B\amalg[0, 1] = B \times [0, 1]$. Thus $F : B \times [0, 1] \to B$ is continuous and therefore a homotopy between the identity and the constant map with value 0, proving (i).

2 Sufficient Conditions for Connectedness

The guiding example of sequential Boolean algebras B are those whose topology is induced by a *strictly positive σ-additive measure* μ. This means that B is a *σ-complete* Boolean algebra (i.e. a Boolean algebra in which all countable joins exist) and $\mu : B \to \mathbb{R}$ satisfies the following conditions

(M1) $\mu(0) = 0, \quad \mu(x) > 0$ for $x \neq 0$.

(M2) $\mu(\bigvee_{n \in \mathbb{N}} x_n) = \sum_{n \in \mathbb{N}} \mu(x_n)$ for every disjoint sequence $(x_n)_{n \in \mathbb{N}}$, i.e. $(x_n \wedge x_m = 0$ for $n \neq m)$; in particular the sum on the right converges.

Here 0 denotes the least element of B; the largest element will be called 1. Condition (M1) is not so restrictive as it might look. If a map μ satisfies (M2) and $\mu(x) \geq 0$ for all $x \in B$, then $I := \{x \in B \mid \mu(x) = 0\}$ is a σ-ideal in B, and μ induces a strictly positive σ-additive measure on B. Moreover, (M1) and (M2) imply that B satisfies the *countable chain condition:*

(ccc) If $Z \subset X$ and $x \wedge y = 0$ for all $x, y \in Z$ with $x \neq y$, then Z is countable.

This is clear because for $n \in \mathbb{N}$, Z can contain at most n elements x with $\mu(x) \geq \dfrac{\mu(1)}{n}$. It is well-known that every σ-complete Boolean algebra with (ccc) is a *complete* Boolean algebra, i.e. every subset admits a join (=supremum) (cf. e.g. [5]). Whenever we say that a sequential Boolean algebra satisfies (ccc) we mean it satisfies (ccc) as Boolean algebra (not as a topological space).

If μ is a strictly positive σ-additive measure on B, then we can define a metric d on B by $d(x, y) := \mu(x \triangle y)$ for all $x, y \in B$. This metric induces a topology, which is sequential (because every metrizable space is sequential), and B is a T_0 sequential Boolean algebra in this topology. By the well-known lemma of the connection between *stochastic* and *sure* (=certain) *convergence*, we see that a sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in B$ if and only if every subsequence of $(x_n)_{n \in \mathbb{N}}$ has a subsequence with limes superior x and limes inferior x. Here the *limes superior* and the *limes inferior* of a sequence $(x_n)_{n \in \mathbb{N}}$ are defined by $\limsup_{n \to \infty} x_n := \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} x_k$ and $\liminf_{n \to \infty} x_n := \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} x_k$. Since these definitions do not involve μ, the topology does not depend on μ. For arbitrary σ-complete Boolean algebra, we can define a topology in this way by limes superior and limes inferior, but in general this topology need not be Hausdorff, and the binary Boolean operations need not be continuous on the topological product $B \times B$ (though they are always sequentially continuous).

In the sequel we shall often use 1.2 and 1.3. Thus in order to prove path-connectedness of a T_0 sequential Boolean algebra it suffices to find a path-connected subset containing 0 and 1. By Zorn's Lemma, B contains a *maximal totally ordered subset*, and each such subset contains 0 and 1.

2.1 Theorem

Let B be a σ-complete Boolean algebra with the toplogy induced by a strictly positive σ-additive measure μ. Then the following assertions hold:

(i) *If B is atomless, then every maximal totally ordered subset of B (and therefore B itself) is path-connected.)*

(ii) *If every path-connected subset of B consists of only one point, then B is atomic.*

Proof. (i) Let $X \subset B$ be a maximal totally ordered subset X. For $x, y \in X$ with $x \leq y$ we have $x \triangle y = x^* \wedge y$ and $x \vee (x^* \wedge y) = (x \vee x^*) \wedge (x \vee y) = = 1 \wedge y = y$ and $x \wedge (x^* \wedge y) \leq x \wedge x^* = 0$, hence $\mu(y) = \mu(x) + \mu(x^* \wedge y)$, thus $\mu(x \triangle y) = \mu(x^* \wedge y) = \mu(y) - \mu(x)$. Since X is totally ordered, this yields $|\mu(x) - \mu(b)| = \mu(x \triangle y)$ for all $x,y \in X$, i.e. the restriction $\mu|_X^{[0,1]} : X \to [0,1]$ is an isometry and thus particularly injective.

We claim that $\mu|_X^{[0,1]}$ is also surjective. For $s \in [0,1]$ define $r := \sup\{\mu(x) | x \in I, \mu(x) \leq s\}$. Then there exists an increasing sequence $(y_n)_{n \in \mathbb{N}}$ in X with $r = \sup\{\mu(y_n)|\ n \in \mathbb{N}\}$. By σ-additivity we get $\mu(y) = r$ for $y := \bigvee_{n=1}^{\infty} y_n$. For every $x \in X$ we have either $x < y_n \leq y$ for some $n \in \mathbb{N}$, or $y_n \leq x$ for all $n \in \mathbb{N}$, hence $y = \bigvee_{n=1}^{\infty} y_n \leq x$. Thus $X \cup \{y\}$ is totally ordered, and by maximality of X we have $y \in X$. Analogously, we obtain $t := \inf\{\mu(x)|\ x \in I,\ \mu(x) \geq s = \mu(w)\}$ for some $w \in X$: then we must have $y \leq w$. Assume $y < w$. Since y is atomless, $y^* \wedge w \neq 0$ is not an atom. Then there exists a $u \in B$ with $0 < u < y^* \wedge d$, and for $z := y \vee u$ we easily get $y < z < w$. But for every $x \in X$ we have either $\mu(x) \leq s$, hence $x \leq y$, or $\mu(x) > s$, hence $z < w \leq x$. Thus we have $z \notin X$, but $X \cup \{z\}$ is totally ordered, contradicting the maximality of X.

Thus the isometry $\mu|_X^{[0,1]}$ is bijective, hence the inverse is also an isometry and therefore continuous. Then the map $f : [0,1] \to B$ with $f(t) := (\mu|_X^{[0,1]})^{-1}(t)$ is continuous with $\mu \circ f(0) = 0$ and $\mu \circ f(1) = 1$, hence $f(0) = 0$, $f(1) = 1$. Then for every $y \in B$ the map $g_y : [0,1] \to B$, $t \mapsto y \wedge f(t)$ is continuous with $g_y(0) = y \wedge f(0) = y \wedge 0 = 0$ and $g_y(1) = y \wedge f(1) = y \wedge 1 = y$. This proves that B is path-connected and thus in particular connected.

(ii) Assume that $y \in B$ and there exist no atom $x \leq y$. Then $B' := \{x \in B|\ x \leq y\}$ is a Boolean σ-algebra with 0, $\wedge$, $\vee$ as in B with new top element y and new complementation $x \mapsto x^* \wedge y$. Moreover, $\mu|_{B'}$ is a strictly positive σ-additive measure on B'. By our hypothesis about y, B' is atomless and thus path-connected. Since B is hereditarily pathwise disconnected (maybe even hereditarily disconnected) it follows that B' consists of only one point. Since 0, $y \in B'$ this gives $y = 0$, proving that B is atomic.

Note that the hypothesis of 2.1 can be formally generalized: Instead of (M2) assume $\mu(x \vee y) \leq \mu(x) + \mu(y)$ for all $x, y \in B$ and $\lim_{n \to \infty} \mu(x_n) = 0$ for every decreasing sequence $(x_n)_{n \in \mathbb{N}}$ with $\bigwedge_{n=1}^{\infty} x_n = 0$. These conditions follow easily from (M2), and together with (M1) they give rise to a metric. It is quite easy to find a complete Boolean algebra B and a strictly positive continuous submeasure satisfying (M1) and the above weaker condition, but not (M2). It is a well-known open problem due to D. Maharam whether the existence of a strictly positive continuous submeasure implies the existence of a strictly positive σ-additive measure. Moreover, in order to generalize 2.1, we have to replace the second part of (M1) by the stronger condition $(x < y \Rightarrow \mu(x) < \mu(y))$. This condition is equivalent to $(x > 0 \Rightarrow \mu(x) > 0)$ in the presence of (M2), but under the weakening of (M2) it is strictly stronger; just consider B finite with at least four elements and define $\mu(0) := 0$ and $\mu(x) := 1$ for $x > 0$. Under this weakening of (M2) and the strengthening of (M1) we do not get that $\mu|_X^{[0,1]}$ is an isometry, but it is strictly order preserving and also preserves countable joins and meets. This suffices to make its inverse continuous.

Next we consider a more general situation. We say that a T_0 sequential Boolean algebra B is *monotonely complete* if every increasing sequence $(x_n)_{n \in \mathbb{N}}$ converges to some $x \in B$. In this case, for every $m \in \mathbb{N}$ the sequence $(x_n \wedge x_m)_{n \in \mathbb{N}}$ converges to $x \wedge x_m$. But $(x_n \wedge x_m)_{n \in \mathbb{N}}$ is eventually constant and thus convergent to x_m. By uniqueness of sequential limits we obtain $x \wedge x_m = x_m$, i.e. $x_m \leq x$. Now assume $y \in B$ with $x_n \leq y$ for all $n \in \mathbb{N}$. Then $(x_n \vee y)_{n \in \mathbb{N}} = (y)_{n \in \mathbb{N}}$ converges to $x \vee y = y$, hence $x \leq y$. This proves that x is a join of $\{x_n \mid n \in \mathbb{N}\}$. This means that every increasing sequence convergence to its join; in particular monotone convergence implies σ-completeness. Moreover by complementation we see that every decreasing sequence $(x_n)_{n \in \mathbb{N}}$ with $\bigwedge_{n=1}^{\infty} x_n = 0$ converges to 0. Conversely, assume that B is σ-complete and every decreasing sequence with meet 0 converges to 0. Then for every increasing sequence $(x_n)_{n \in \mathbb{N}}$ the join $x := \bigvee_{n=1}^{\infty} x_n$ exists and the decreasing sequence $(x \wedge x_n^*)_{n \in \mathbb{N}}$ converges to 0, therefore $(x_n)_{n \in \mathbb{N}} = (x \wedge (x \wedge x_n^*)^*)_{n \in \mathbb{N}}$ converges to $x \wedge (x \wedge x^*) = x$. This means that B is monotonely complete if and only if B is σ-complete and every decreasing sequence with meet 0 converges to 0. But note that these two conditions are independent. An infinite σ-complete Boolean algebra is not monotonely complete in the discrete topology. In 3.3(2) below we shall give an example of a non-σ-complete. Boolean algebra in which every decreasing sequence with meet 0 converges to 0. Note that every monotonely complete T_0 sequential Boolean algebra B with (ccc) is a complete Boolean algebra, because B is σ-complete and this together with (ccc) implies Boolean completeness.

2.2 Theorem

Let B be a monotonely complete T_0 sequential Boolean algebra with (ccc). *Then the following assertions hold:*

(i) *If B is atomless, then every totally ordered subset of B and hence B is itself connected.*

(ii) *If B is hereditarily disconnected then B is atomic.*

Proof. Let $X \subset B$ be a maximal totally ordered subset and let $Z \subset X$ be relatively clopen; without loss of generality assume $1 \notin Z$ (otherwise replace Z by $X \setminus Z$). For $s := \bigvee_{x \in X} x$ there exists a sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n \in X$ for all $n \in \mathbb{N}$ and $s = \bigvee_{n \in \mathbb{N}} x_n = \bigvee_{n \in \mathbb{N}} y_n$ for $y_n := \bigvee_{k=1}^{n} x_k \in \{x_1, \dots, x_n\} \subset X$. Now the increasing sequence $(y_n)_{n \in \mathbb{N}}$ converges to s, thus $s \in Z$, because Z is closed, in particular $s \neq 1$.

On the other hand for $t := \bigwedge_{z \in X, z > s} z$ we get $t = s$, because $s < t$ would imply that there exists a $w \in B$ with $s < w < t$, since $s^* \wedge t \neq 0$ is not an atom. But then $X \cup \{w\}$ would be totally ordered, thus $w \in X$ by maximality, leading to the contradiction $t = \bigwedge_{z \in X, s < z} z \leq w$.

So we have $s = t = \bigwedge_{z \in X, s < z} z$, hence s is a meet of countable many $z \in X$ with $s < z$, and as above there exists a decreasing sequence $(z_n)_{n \in \mathbb{N}}$ with $s = \bigwedge_{n \in \mathbb{N}} z_n$ and $s < z_n \in X$ for all $x \in \mathbb{N}$, and then $(z_n)_{n \in \mathbb{N}}$ converges to s. But for all $n \in \mathbb{N}$ we have $\bigwedge_{x \in X \cap Z} x = s < z_n \in X$, hence $z_n \notin Z$. Since Z is open, this implies $s \notin Z$, a contradiction. Thus B is connected.

(ii) follows from (i) as in 2.1.

2.3 Remarks

We obtain a stronger result under *Souslin's hypothesis* (SH). Usually (SH) is formulated as a characterization of the natural order of $\mathbb{R}$; we use an equivalent formulation characterizing the unit interval [0, 1], which is more convenient for our purpose:

A totally ordered set X is order-isomorphic to [0, 1] provided X has the following properties:

(i) X has at least two elements.

(ii) Every subset of X has a supremum.

(iii) For all $x, z \in X$ with $x < z$ there exists a $y \in X$ with $x < y < z$.

(iv) If $(]x_i, z_i[)_{i \in I}$ is a family of pairwise disjoint non-empty open intervals of X i.e. $]x_i, z_i[:= \{y \in X : x_i < y < z_i\}$ and $]x_i, z_i[\cap]x_j, z_j[= 0$ for $j \neq i$, then I is (at most) countable.

Note that both (SH) and its negation are relatively consistent in ZFC; for further discussions see [3].

2.4 Theorem

The following statement is equivalent to (SH):

Let B be an atomless monotonely complete weakly Hausdorff sequential Boolean algebra with (ccc). *Then B is path-connected.*

Proof. First assume (SH), let $a \in X$ and let $X \subset B$ be a maximal totally ordered subset with $a \in X$, which exists by Zorn's Lemma. Then we have $\overline{0}, \overline{1} \in X$ for $\overline{0}$ the least and $\overline{1}$ the largest element of B, and we assume $\overline{0} \neq \overline{1}$, because otherwise $B = \{\overline{0}\}$ is trivially path-connected. As above, every $Z \subset X$ has a join, and if $Z \neq \emptyset$ there exists an increasing sequence $(z_n)_{n \in \mathbb{N}}$ in Z with $s := \bigvee_{n \in \mathbb{N}} z_n = \bigvee_{z \in Z} z$. Now $Z \cup \{s\}$ is also totally ordered, hence $s \in Z$ by maximality, i.e. Z is closed under joins. Moreover $(z_n)_n$ converges to s in B by monotone completeness. Similarly, X is closed under meets; non-empty meets can be represented by meets of decreasing sequences, and every decreasing sequence converges to its meet.

For $r, s \in X$ with $r < s$ there exists a $w \in X$ with $r < w < s$ because $r^* \wedge s$ is not an atom. In particular, every open interval of X is non-empty. If $(]r_i, s_i[)_{i \in I}$ is a family of pairwise disjoint intervals in X, then for every $i \neq j$ in I we get either $s_i \leq r_j$ or $s_j \leq r_i$ by disjointness. In the first case we get $r_i^* \wedge s_i \wedge r_j^* \wedge s_j \leq s_i \wedge r_j^* = \overline{0}$, and in the second case we have $r_i^* \wedge s_i \wedge r_j^* \wedge s_j \leq r_i^* \wedge s_j = \overline{0}$. Thus $(r_i^* \wedge s_i)_{i \in I}$ is a disjoint family of non-zero elements. Thus I is countable because B satisfies (ccc).

Now from (SH) we see that there exists an order-isomorphism $\overline{f}\colon [0, 1] \to X$, where $[0, 1] \subset \mathbb{R}$ is the real unit interval. By convergence of increasing and decreasing sequences, the composite $f\colon [0, 1] \to X \hookrightarrow B$ is even continuous, and we have $f(0) = \overline{0}$, $f(1) = \overline{1}$. From $\overline{0}, a \in X$ we see that $\overline{0}$ and a belong to the same path-component. Since $a \in B$ was arbitrary, X is path-connected.

The converse will be proved indirectly. Let X be a Souslin continuum, i.e. a counterexample to (SH). Then X has a least element $\overline{0}$ and a largest element $\overline{1}$. Consider the (Hausdorff) topology on X, one of whose bases consist of all open intervals $]x, y[$ $(x, y \in X, x < y)$ and all semi-open intervals of the form $[\overline{0}, x[$, $]x, \overline{1}]$ $(x \in X)$. Recall that a subset $U \subset X$ is called regularly open, if $U = \operatorname{int} \operatorname{cl} U$, i.e. U is the interior of its closure. Let B be the set of all regularly open $U \subset X$. Then we have $\emptyset, X \in B$. For $U, V \in B$ we easily see $U \cap V \in B$, and for any $U \in B$ we obtain $U^* := \operatorname{int}(X \backslash U) = X \backslash \operatorname{cl} U \in B$. It is well-known and not too difficult to prove that B is a Boolean algebra under these operations. Moreover, B is even complete (as a Boolean algebra), in particular σ-complete.

Indeed, for $Z \subset B$, $\operatorname{int} \operatorname{cl} \bigcup_{u \in Z} u$ is a join (supremum) of Z in B. But note that even finite joins in B are in general not set-theoretic unions; for $x \in X \backslash \{\overline{0}, \overline{1}\}$ we have $[\overline{0}, x[,]\overline{x}, 1] \in B$, but $\operatorname{int} \operatorname{cl}([\overline{0}, x[\cup]x, 1]) = \operatorname{int} X = X \neq [\overline{0}, x[\cup]x, \overline{1}]$.

Now endow B with the sequential topology from given by limes superior and limes inferior. As we saw there, B is a monotonely complete weakly Hausdorff sequential Boolean algebra.

Now finite meets in B are set-theoretic intersections, every $U \in B \backslash \{\emptyset\}$ contains a non-empty open interval and every sequence of pairwise disjoint non-empty open intervals is countable; thus B satisfies (ccc). Moreover, for $U \in B \backslash \{\emptyset\}$ there exist $u, w \in X$ with $u < w$, $\emptyset \neq]u, w[\subset U$, and for $v \in]u, w[$ we have $]u, v[\in B$ and $\emptyset \neq]u, v[\subsetneqq U$. Thus B is atomless.

It remains to be shown that B is not path-connected. The main idea is due to A. Bella [1], who used it in a different context. Assume that $f: [0, 1] \to B$ is continuous with $f(0) = \emptyset$ and $f(1) = X$. For every $t \in [0, 1] \cap \mathbb{Q}$, $f(t) \subset X$ is regularly open and therefore a union of open intervals, which can even be chosen pairwise disjoint. By our hypothesis about X, this set of open intervals is countable; hence its set Z_t of endpoints is also countable. If the countable set $Z := \bigcup_{t \in [0,1] \cap \mathbb{Q}} Z_t$ were order-dense in X (i.e. $Z \cap]u, v[\neq \emptyset$ for all $u, v \in X$ with $u < v$), then it would follow that there exists an order-isomorphism $X \cong [0, 1]$, contradicting our assumption on X. Thus Z is not order-dense, i.e. there exist $u, v \in X$ with $u < v$ and $Z \cap K = \emptyset$ for $K :=]u, v[$. By definition of Z, this implies $f(t) \bigcap K = \emptyset$ or $K \subset f(t)$ for all $t \in [0, 1] \cap \mathbb{Q}$.

Now $f: [0, 1] \to B$ and "$\cap$": $B \Pi B \to B$ are continuous; thus the map $[0, 1] \to B$, $t \mapsto f(t) \cap K$ is also continuous, and for all $t \in [0, 1] \cap \mathbb{Q}$ we have $f(t) \cap K \in \{\emptyset, K\}$. Since B is weakly Hausdorff, $\{\emptyset, K\} \subset B$ is closed, we obtain $f(t) \cap K \in \{\emptyset, K\}$ for all $t \in [0, 1]$. But since $[0, 1]$ is connected and $\{\emptyset, K\}$ is discrete, it follows that the continuous map $[0, 1] \to B$, $t \mapsto f(t) \cap K$ is constant. But this is not true because $f(0) \cap K = \emptyset \neq K = X \cap K = f(1) \cap K$.

3 Necessity of (ccc) and Monotone Completeness

Up to now we only proved connectedness for monotonely complete T_0 sequential Boolean alogebras with (ccc); note that in 2.1 (M1) implies (ccc) and (M2) implies monotone completeness. It will turn out that connectedness of all maximal totally ordered sets implies (ccc) and "almost" implies monotone completeness. On the other hand, there exist T_0 sequential Boolean algebras without (ccc) or monotone completeness, in which some maximal totally ordered sets are connected; consequently these sequential Boolean algebras are connected.

We like to introduce two more notions. We call a sequential Boolean algebra B *exhaustable* if every disjoint sequence in B converges to 0. Every monotonely complete sequential Boolean algebra B is exhaustable, because for a disjoint sequence $(x_n)_{n \in \mathbb{N}}$ the increasing sequence $y_n := (\bigvee_{k=1}^{n} x_k)_{n \in \mathbb{N}}$ converges to some $y \in B$, thus $(x_n)_{n \in \mathbb{N}} = (y_n \wedge y_{n-1}^*)$ converges to $y \wedge y^* = 0$ (where $y_0 = 0$). On the other hand, an infinite Boolean algebra endowed with the discrete topology is not exhaustable.

Furthermore, we call a sequential Boolean algebra B *solid* if it satisfies the following condition: If $(x_n)_{n\in\mathbb{N}}$, and $(y_n)_{n\in\mathbb{N}}$ are sequences in B such that $(y_n)_{n\in\mathbb{N}}$ converges to 0 and such that $x_n \le y_n$ for all n, then $(x_n)_{n\in\mathbb{N}}$ converges to 0. If B is σ-complete, then B is obviously solid in the sequential topology given by limes superior and limes inferior. Moreover, if the topology of B is given by a (not necessarily continuous) submeasure, then B is also solid because the discrete topology is given by the submeasure μ with $\mu(0) := 1$ for $x \neq 0$.

3.1 Example

Consider the *interval algebra* $B :=$ Interval($[0, 1[$) of the interval $[0, 1[\subset \mathbb{R}$. Elements of B are all finite unions of intervals $[\alpha, \beta[$, where $\alpha, \beta \in \mathbb{R}$, $0 \le \alpha < \beta \le 1$; we have $\emptyset \in B$ since 0 is the nullary union. The Boolean operations are set-theoretical. Let $\lambda : B \to \mathbb{R}$ be the restriction of the Lebesgue measure, i.e. λ is the unique measure with $\lambda([\alpha, \beta[) = \beta - \alpha$ whenever $0 \le \alpha < \beta \le 1$. Moreover, for $a \in B$ let $\nu(x)$ be the least non-negative integer such that a is a union of n intervals; equivalently, $\nu(x)$ is the number of connected components of x. Then for all $x, y \in B$ we have $\nu([0, 1]\backslash x) \le \nu(x) + 1$ and $\nu(x \vee y) \le \nu(x) + \nu(y)$. We define a sequential topology on B with unique sequential limits in the following way: a sequence $(x_n)_{n\in\mathbb{N}}$ converges to $\emptyset$ if $(\lambda(x_n))_{n\in\mathbb{N}}$ converges to 0 and $\sup\{\nu(x_n) | \, n \in \mathbb{N}\} < \infty$. Moreover, $(x_n)_{n\in\mathbb{N}}$ converges to $x \in B$ if and only if $(x_n \Delta x)_{n\in\mathbb{N}}$ converges to 0. It is readily checked that this notion of convergence satisfies (CS1$-$4); thus it is induced by a sequential topology with unique sequential limits. It is easy to see that complementation and binary meets are sequentially continuous. Therefore B is a T_0 sequential Boolean algebra.

We claim that B is not solid. For $n \in \mathbb{N}$ define $x_n := \bigcup_{k=n}^{2n} [1/(2k+1), 1/2k[\in B$, $y_n := [0, 1/2n[\in B$. Then for all $n \in \mathbb{N}$ we have $x_n \subset y_n$, $\lambda(y_n) = 1/2n$, $\nu(y_n) = 1$; hence $(y_n)_{n\in\mathbb{N}}$ converges to 0 in B. But $(x_n)_{n\in\mathbb{N}}$ does not converge to 0 because $\nu(x_n) = n + 1$ for all $n \in \mathbb{N}$, hence $\sup\{\nu(x_n) | \, n \in \mathbb{N}\} = \infty$. This shows that B is not solid.

3.2 Theorem

Let B be a T_0 sequential Boolean algebra such that all maximal totally ordered subsets of B are connected. Then the following assertions hold:

(i) *B is σ-complete.*

(ii) *If $(a_n)_{n\in\mathbb{N}}$ is an increasing sequence in B, then there are a strictly order-preserving map $\varphi : \mathbb{N} \to \mathbb{N}$ and a sequence $(b_n)_{n\in\mathbb{N}}$ such that $a_{\varphi(n)} \le b_n \le a_{\varphi(n+1)}$ for all $n \in \mathbb{N}$ and such that $(b_n)_n$ converges to $\bigvee_{n\in\mathbb{N}} b_n = \bigvee_{n\in\mathbb{N}} a_{\varphi(n)} = \bigvee_{n\in\mathbb{N}} a_n$.*

(iii) *If B is solid or exhaustable, then B is monotonely complete.*

(iv) *B satisfies* (ccc).

Proof. (i): It suffices to show that every increasing sequence $(a_n)_{n\in\mathbb{N}}$ has a supremum, and w.l.o.g. we can assume that $(a_n)_n$ is even strictly increasing, i.e. $a_n < a_{n+1}$ for all $n \in \mathbb{N}$. By Zorn's Lemma, there exists a maximal totally ordered subset $X \subset B$ with $\{a_n | n \in \mathbb{N}\} \subset X$, and X is connected by hypothesis. Moreover, we have $0, 1 \in X$ by maximality, because $X \cup \{0,1\}$ is still totally ordered. For $Z := \{x \in X \mid \exists n\, x < a_n\}$ we see that $X \backslash Z := \{x \in X \mid \forall n\, x_n \geq a\}$ is relatively sequentially closed and hence relatively closed in X (by continuity of Boolean operations and uniqueness of sequential limits). Since $0 \in Z$ and $1 \in X \backslash Z$, connectedness of X yields that Z is not closed in X and thus not sequentially closed in X. Therefore there exists a sequence $(b_n)_{n\in\mathbb{N}}$ in Z that converges to some $s \in X\backslash Z$. Then for every $x \in Z$ there is an $n \in \mathbb{N}$ with $x < a_n \leq s$. For each $x \in X\backslash Z$, we have $b_n \leq x$ for all $n \in \mathbb{N}$, hence $s = \lim_{n\to\infty} b_n \leq x$. Thus $X \cup \{s\}$ is totally ordered, hence $x \in X$ by maximality.

We claim that s is a supremum of $\{a_n \mid n \in \mathbb{N}\}$ in B and we already proved $a_n \leq s$ for all $n \in \mathbb{N}$. Assume $c \in B$ with $a_n \leq c$ for all $n \in \mathbb{N}$. For every $x \in Z$ there is an $n \in \mathbb{N}$ with $x < a_n \leq c$, hence $x \leq s \wedge c$. For $x \in X\backslash Z$ we have $s \wedge c \leq s \leq x$. Thus $X \cup \{s \wedge c\}$ is totally ordered, and by maximality we have $s \wedge c \in X$. Since $a_n \leq s \wedge c$ for all $n \in \mathbb{N}$, we have $s \wedge c \in X\backslash Z$, thus $s \leq s \wedge c \leq c$, proving our claim.

(ii): W.l.o.g. assume that $(a_n)_n$ is strictly increasing, and define $s := \bigvee_{n\in\mathbb{N}} a_n$. As shown above there exists a sequence $(c_n)_{n\in\mathbb{N}}$ converging to s such that $\{a_n \mid n \in \mathbb{N}\} \cup \{c_n \mid n \in \mathbb{N}\}$ is totally ordered and such that for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ with $c_n < a_m \leq s$, hence $c_n \leq a_k$ for all $k \geq m$.

Now we define $\varphi : \mathbb{N} \to \mathbb{N}$ and $(b_n)_{n\in\mathbb{N}}$ by induction. Let $\varphi(1) := 1$ and $b_1 := c_1$. If $\varphi(1), \ldots, \varphi(n)$ have already been defined, define $\varphi(n+1)$ as the smallest $k \in \mathbb{N}$ with $k > \varphi(n)$ and b_n. If we had $c_l \leq a_{\varphi(n)}$ for all $l > \varphi(n)$, we should get the contradiction $s \leq a_{\varphi(n)} < a_{\varphi(n+1)} \leq s$. Thus we can choose $b_{n+1} := c_l$ for the smallest $l \in \mathbb{N}$ with $c_l > \max(a_{\varphi(n)}, b_n)$ and $a_{\varphi(n)} < c_l$. Then has the required properties.

(iii): Let $(a_n)_n$ be a strictly increasing sequence and let $s := \bigvee_{n\in\mathbb{N}} a_n$. By (CS3) it suffices to show that for every strictly increasing $\varphi : \mathbb{N} \to \mathbb{N}$ there is a strictly increasing $\sigma : \mathbb{N} \to \mathbb{N}$ such that $(a_{\varphi\circ\sigma(n)})_n$ converges to s. So let φ be fixed and note that $s = \bigvee_{n\in\mathbb{N}} a_{\varphi(n)}$. By (ii) there are a strictly increasing $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence $(b_n)_n$ converging to s such that $a_{\varphi\circ\sigma(n)} \leq b_n \leq a_{\varphi\circ\sigma(n+1)}$ for all $n \in \mathbb{N}$. Then $(b_n^* \wedge s)_n \in \mathbb{N}$ converges to $s^* \wedge s = 0$, and for every $n \in \mathbb{N}$ we have $a^*_{\varphi\circ\sigma(n+1)} \wedge s \leq b_n^* \wedge s$. If B is solid, it follows that $(a^*_{\varphi\circ\sigma(n+1)} \wedge s)_n$ and hence $(a_{\varphi\circ\sigma(n)} \wedge s)_n$ converges to 0; therefore $(a_{\varphi\circ\sigma(n)})_n = (s \wedge (a^*_{\varphi\circ\sigma(n)} \wedge s)^*)_n$ converges to $s \wedge 0^* = s$, and we are done.

If B is exhaustable, we get $b_n^* \wedge a_{\varphi\circ\sigma(n+1)} \wedge b^*_{n+1} \wedge a_{\varphi\circ\sigma(n+2)} \leq a_{\varphi\circ\sigma(n+1)} \wedge b^*_{n+1} = 0$, because $a_{\varphi\circ\sigma(n+1)} \leq b_{n+1}$. By exhaustability, $(b_n^* \wedge a_{\varphi\circ\sigma(n+1)})_n$ converges to 0, hence

$(a_{\varphi \circ \sigma(n+1)})_n = (b_n \vee (b_n^* \wedge a_{\varphi \circ \sigma(n+1)}))_n$ and thus also $(a_{\varphi \circ \sigma(n)})_n$ converges to $s \vee 0 = s$, because $(b_n)_n$ converges to s, proving our claim.

(iv): It suffices to show that there is no family $(a_\xi)_{\xi < \omega_1}$ with $a_\xi \in B$, $a_\xi \neq 0$ for all $\xi < \omega_1$ and with $a_\xi \wedge a_\eta = 0$ for $\xi < \eta$. Assume the existence of such a family. Then by (i), $b_\xi := \bigvee_{\zeta < \xi} a_\zeta$ exists for all $\xi < \omega_1$, and for $\xi < \eta$ we have $a_\xi \wedge b_\xi = 0$, $a_\xi < b_\eta$, hence $b_\xi < b_\eta$. In particular, $\{b_\xi \mid \xi < \omega_1\}$ is totally ordered (in fact, even well-ordered).

By Zorn's Lemma, there exists a maximal totally ordered set $X \subset B$ with $\{b_\xi \mid \xi < \omega_1\} \subset X$. Then for $Z := \{x \in X \mid \exists \xi < \omega_1,\ x < b_\xi\}$ we see that $0 \in Z$, $1 \notin Z$ and $X \backslash Z = \{x \in X \mid \forall \xi \in W\, b_\xi \leq x\}$ is relatively sequentially closed and therefore closed in X. On the other hand, if $(z_n)_{n \in \mathbb{N}}$ is a sequence in Z, which converges to some $c \in X$, then for every $n \in \mathbb{N}$ there is a $\xi_n \in W$ with $z_n < b_{\xi_n}$. But there is also an $\eta < \omega_1$ with $\xi_n \leq \eta$ for all $n \in \mathbb{N}$ we get $z_n < b_{\xi_n} \leq b_\eta$, and this yields $c \leq b_\eta < b_{\eta+1}$ and hence $c \in Z$. This proves that Z is also relatively sequentially closed and thus closed in X, contradicting our hypothesis that X be connected.

4 Examples

(1) Let the T_0 sequential Boolean algebra B be as in 3.1. Then B is not solid and thus not monotonely complete. B is not even σ-complete because $\bigvee_{n=1}^{\infty}[1/(2n+1),\ 1/2n[$ does not exist in B. Therefore, by 3.2 not all maximal totally ordered sets are connected. It is even quite easy to give a concrete example. For the set $a := \bigcup_{n=1}^{\infty}[1/(2n+1),\ 1/2n[\subset [0,1[$ we easily see $a \notin B$ but $Y := \{a \cup [0,t[\mid t \in]0,1]\} \subset B$ and $Z := \{a \cap]t,1] \mid t \in]0,1]\} \subset B$. Moreover, we see that $X := Y \cup Z$ is maximal totally ordered but not connected because $Y \cap Z = \emptyset$ and both Y, Z are closed in B and therefore relatively closed in X.

On the other hand, B is even path-connected because the map $[0,1] \to B$, $t \mapsto [0,t[$ is a path from 0 to 1 in B, and $\{[0,t[\mid t \in [0,1]\}$ is a path-connected totally ordered set. Moreover, B satisfies (ccc), because λ as in 3.1 is a strictly positive measure.

(2) Let B be as in (1) as a Boolean algebra, but endow it with the metrizable topology induced by the measure μ. Then B is solid, but by the same arguments as above B has a path-connected maximal totally ordered set, but the maximal totally ordered set X as in (1) is not connected. Moreover, B satisfies (ccc), because λ as in 3.1 is a strictly positive measure.

(3) Let B be a connected T_0 sequential Boolean algebra and let I be a set. Then the sequential modification $B_i := s(B^I)$ of the topological power B^I (with componentwise Boolean operations) is also a T_0 sequential Boolean algebra. If $B_0 \neq \{0\}$ and I is uncountable; B_1 does not satisfy (ccc). If B_0 is monotonely complete

(σ-complete resp.), then so is B_1. The diagonal map $\delta : B_0 \mapsto B_1$, $x \mapsto (x)_{i \in I}$ is sequentially continuous and therefore continuous. If B_0 is (path-)connected, then $\delta[B_0] \subset B_1$ is a (path-)connected subset containing 0 and 1, thus B_1 is (path-)connected by 1.2 or 1.3. If $X \subset B_0$ is a maximal totally ordered set, then $\delta[X] \subset B_1$ is also a maximal totally ordered subset, namely $\delta[\Xi]$ for $X \subset B_0$ maximal totally ordered. We can even choose $B \neq \{0\}, X$ such that X is path-connected. On the other hand, in case $B_0 \neq \{0\}$ and I uncountable we see from 3.2 that not all maximal totally ordered subsets of B_1 are connected because B_1 does not satisfy (ccc). If I is an ordinal $\alpha \geq \omega_1$ we even can give an example. For $X \subset B_0$ as above, define $Z_\xi \subset B$ as the set of all families $(s_\eta)_\eta < \alpha$ such that $x_\xi \in X$ and moreover $x_\eta = 0$ for $\eta < \xi$ and $x_\eta = 1$ for $\eta > \xi$. Then for $\xi + 1 < \alpha$ the set $Z_\xi \cap Z_{\xi+1}$ has a unique element $(z_\xi)_{\xi \in \mathbb{N}}$, where $z_\eta = 0$ for $\eta \leq \xi$ and $z_\eta = 1$ for $\eta > \xi$. For $\xi + 1 < \eta$ we have $Z_\xi \cap Z_\eta = \emptyset$. Now we see that $Z := \bigcup_{\xi < \alpha} Z_\xi \subset B_1$ is a maximal totally ordered subset of B_1, which is not connected because $\bigcup_{\xi < \omega_1} Z_\xi$ is relatively clopen in Z.

References

[1] BELLA, A. A couple of questions concerning nowhere constant continuous functions, Recent Development of General Topology and its Applications, Mathematical Research 67, Akademie-Verlag Berlin (1992), 27 – 32.

[2] BÖRGER, R. Ein kategorieller Zugang zur Integrationstheorie, Habilitationsschrift, Fernuniversität Hagen 1989.

[3] DEVLIN, K. J. and JOHNSBRÅTEN, H. The Souslin problem, Lecture Notes in Mathematics, Springer-Verlag, Berlin 1974.

[4] DOLCHER, M. Topologie e strutture di convergenza, Ann. Scuola Norm. Sup. Pisa (III), 14 (1960), 63 – 92.

[5] HALMOS, P. R. Lectures on Boolean algebras, Springer-Verlag, New York 1974.

[6] KISYŃSKI, J. Convergence du type L, Colloq. Math. 7 (1960), 205 – 211.

[7] KOUTNÍK, V. Closure and topological sequential convergence, Proceedings of the Conference on Convergence in Bechyné 1984, Akademie-Verlag, Berlin (1985), 199 – 204.

[8] WYLER, O. Convenient categories for topology, Gen. Top. Appl. 3 (1973), 225 – 242.