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On Borel Properties of Semi-Continuous Multifunctions

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We study Borel properties of lower and upper semi-continuous multifunctions, their intersections, and multifunctions with dense families of selectors.

1. Introduction

Let X and Y be two arbitrary topological spaces. Denote by $N(Y)$, $C(Y)$ and $K(Y)$ the families of all non-empty subsets of Y, of all non-empty and closed subsets of Y, and of all non-empty, closed and compact subsets of Y, respectively. By a multifunction F from X and Y we mean any function F from X to $N(Y)$. We say that a multifunction $F : X \to N(Y)$ is lower semi-continuous (upper semi-continuous) if for every open subset A of Y the lower (upper) inverse image of A

$$F^{-}(A) = \{x \in X : F(x) \cap A \neq \emptyset\} \qquad (F^{+}(A) = \{x \in X : F(x) \subset A\}$$

is an open subset of X. *If* O is a subset of $N(Y)$ then by an inverse image $F^{-1}(O)$ we mean the usual set of all $x \in X$ such that $F(x) \in O$. Let v be the Vietoris topology on $N(Y)$ (for basic facts see [Mi1], and e.g. [K + T]). It is known that a function $F : X \to (N(Y), v)$ is continuous if and only if the multifunction $F : X \to N(Y)$ is both lower and upper-semicontinuous.

We say that a multifunction $F : X \to N(Y)$ is of Borel class 1 whenever the inverse images $F^{-1}(O)$ of open sets $O \subset C(Y)$ are F_σ-sets in X. The classical result of Kuratowski [K2, p. 70] states that if X and Y are metric spaces, with Y compact and $F : X \to C(Y)$ is an upper or lower semi-continuous multifunction from X to Y then F is of Borel class 1.

In [K3] Kuratowski posed the following question: Can metrizability and compactness of the spaces X and Y, respectively be replaced by weaker assumptions?

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Maritz [Ma] examined this question and formulated some results. In this note we formulate other results on Borel properties of semi-continuous multifunctions.

We will write $X \in (G\delta)$, whenever the space X has the property

$(G\delta)$ every closed subset of X is a G_δ-set in X,

equivalently: every open subset of X is an F_σ-set in X. Metric spaces have the property $(G\delta)$, and also many so called generalized metric spaces have this property too (see [Mi2], [Gr] and [Le + Sp + Ur]). If F is both lower and upper semicontinuous multifunction from X to $N(Y)$ and $X \in (G\delta)$ then F is of Borel class 1. This is trivial result because F is continuous with respect to the Vietoris topology.

More interesting are results where given multifunctions are only lower or upper semi-contunuous. The following are Maritz's results:

(1) Let X, $Y \in (G\delta)$ and Y be a T_1 and a second countable space.
If $F : X \to N(Y)$ is an upper semi-continuous multifunction then F is of Borel class 1.

(2) Let $X \in (G\delta)$ and Y be a separable metric space.
If $F \to K(Y)$ is a lower semi-continuous multifunction then F is of Borel class 1.

Remark that the above results were obtained under a rather strong assumption, namely, that the space Y has a countable base. Maritz formulate [Ma, Corollary 2.2] also a result concerning the lower Baire class 1 property. This is the following:

(3) Let X be a topological and Y a metric space.
If $F : X \to K(Y)$ is lower semi-continuous then $F^+(A)$ is an F_σ-set for every open subset A of Y.

G. Beer [Be, Theorem 5.1] considered closed valued multifunctions from a metric space X to a metric and separable space Y. In particular, he proved that if F is lower semicontinuous then it is of Borel class 1 with respect to the Wijsman topology on $C(Y)$, which is a topology weaker than the Vietoris one.

In this note we will consider the Borel properties expressible by the lower or upper inverse images (see [K], [K + RN], [K4], [K5], [K6], [Ga], [Be]).

We say that $F : X \to N(Y)$ is of lower (upper) Borel class 1 whenever the lower (upper) inverse image $F^-(A)(F^+(A))$ of every open set A of Y is an F_σ-set in X.

2. Generalized metric spaces

Let us recall a few notions from the theory of generalized metric spaces (see [Ar], [Mi2], [Gr] and [Le + Sp + Ur]) and describe some classes of topological spaces which are also useful to study Borel properties of multifunctions. Almost all of these spaces have some countable separability properties, for example such spaces are: $\aleph_0$-spaces (spaces with countable pseudobases), cosmic spaces, spaces with countable networks, and separably submetrizable spaces.

Let X be a topological space. We say that X is countably R-separated (countably C-separated) whenever there exists a sequence (W_n) of closed subsets of X such that for every $x \in X$ (for every nonempty compact subset $B \subset X$) and every nonempty closed $A \subset X$ not containing x (not intersecting B) there exists $n \in N$ such that $x \in W_n$ $(B \subset W_n)$ and $A \subset X \backslash W_n$ (see [Sp + Ur] and [Le + Sp + Ur]). It is clear that if X is a T_1-space then countable C-separability implies countable R-separability. Moreover it is known that separable metric spaces are countably C-separated. If X is a regular space then countably R-separated spaces become spaces with countable networks in Archangelskii's terminology or spaces with countable point-pseudobases in Michael's terminology, and countably C-separated spaces become $\aleph_0$-spaces, i.e. spaces with countable pseudobases in Michael's terminology. Every countably R-separated space, and thus every countably C-separated space, is a separable space with $(G\delta)$ property.

Other useful spaces to study measurable multifunctions are separably submetrizable spaces, i.e. spaces with a weaker separable and metrizable topology (see [Le + Sp + Ur] and references therein).

3. Borel properties

In [SP + Ur] and [Le + Sp + Ur] methods of countable separability has been recently examined to study measurable multifunctions in non necessarily metrizable topological spaces. Here we use these methods to study Borel properties of lower and upper semi-continuous multifunctions. It is obvious that every lower semi-continuous multifunction is of lower Borel class 1, and every upper semi-continuous multifunction is of upper Borel class 1, provided $X \in (G\delta)$. Now we have the following results.

Theorem 1. Let X be a topological space and Y a countably C-separated space. If $F: X \to K(Y)$ is lower semi-continuous then F is of upper Borel class 1.

Proof. We show that the upper inverse image of each open set A in Y is a F_σ-set in C. It is sufficient to prove that the lower inverse image of each closed set B in Y is a G_δ-set in X. Let $B \subset Y$ be closed and (W_n) be a sequence of closed subsets of Y that C-separates compact and closed sets in Y. Then

$$F^-(B) = X \backslash \cup \{F^+(W_n) : B \subset X \backslash W_n\}.$$

The sets $F^+(W_n)$ are closed because F is lower semi-continuous. Therefore $F^-(B)$ is a G_δ-set in X.

Theorem 2. Let X be a topological space and Y be a countably R-separated space. If $F: X \to N(Y)$ is upper semi-continuous then F is of lower Borel class 1.

Proof. Let $A \subset Y$ be an open set and (W_n) a sequence of closed subsets of Y that

R-separates points and closed sets in Y. Observe first that A is a sum of countably many of W_n's, say $A = \cup W_n$. It follows that $F^-(A) = \cup F^-(W_n)$. The upper semicontinuity of F ends the proof.

4. Intersection of multifunctions

It is well known (see [K1, p. 179]) that if Y is a normal space then the intersection of two upper semi-continuous multifunctions from a topological space X to Y is upper semi-continuous too. Moreover if Y is a T_1-space then the closed valued multifunction: $(A, B) \mapsto A \cap B$ from $C(Y) \times C(Y)$ to $C(Y)$ is upper semi-continuous if and only if Y is normal.

We say that a multifunction $F : X \to N(Y)$ is of (upper, lower) Borel class 2 if the (upper, lower) inverse images of the open sets are $G_{\delta\sigma}$-sets. Without normality assumption on Y we can obtain the following result.

Theorem 3. Let X be a topological space and Y be a Hausdorff countably C-separated space. If F_1 and F_2 are two upper semi-continuous multifunctions from X to $K(Y)$ then their intersection $F_1 \cap F_2$ is of upper Borel class 2.

Proof. Let $E \subset Y$ be open and F_1 and F_2 as above. For every closed $A \subset Y$ the multifunctions $F_1 \cap A$ and $F_2 \cap A$ are compact-valued and upper semi-continuous. Now observe that

$$(F_1 \cap F_2)^+(E) = \{x \in X : (F_1(x) \setminus E) \cap (F_2(x) \setminus E) = \emptyset\}$$

and that the multifunctions $F_1 \setminus E$ and $F_2 \setminus E$ are compact valued and upper semi-continuous for $F \setminus E = F \cap (Y \setminus E)$. Therefore it is sufficient to consider the set $H = \{x \in X : F_1(x) \cap F_2(x) = \emptyset\}$. Let (W_n) be a sequence of closed subsets of Y that C-separates compact and closed sets in Y. We have

$$\begin{aligned} H &= \cup\{x \in X : F_1(x) \subset W_n \quad \text{and} \quad F_2(x) \cap W_n = \emptyset\} \\ &= \cup\{x \in X : F_1(x) \subset W_n \quad \text{and} \quad F_2(x) \subset W_m\}, \end{aligned}$$

where the first sum is over all $n \in N$ and the second is over all $n, m \in N$ such that $W_n \cap W_m = \emptyset$. Therefore

$$H = \cup F_1^+(W_n) \cap F_2^+(W_m).$$

Since every closed set in Y is a G_δ-set and for every multifunction F the formula $F^+(\cap A_i) = \cap F^+(A_i)$ holds then H is a $G_{\delta\sigma}$-set.

Remark 1. In fact, Theorem 3 holds under a weaker condition on the space Y. Namely, we may assume only that Y has a weaker countably C-separated topology (in particular, this is the case if Y is a separably submetrizable space). Indeed, it is sufficient to remark that if (W_n) is a sequence that C-separates Y under a weaker topology w, then it C-separates Y under its original topology.

Remark 2. It is known (see [K2, p. 73]) that the boundary multifunction $F(A) = A \cap \mathrm{cl}\,(Y \backslash A)$, $A \in C(Y)$, where $\mathrm{cl}\,(D)$ denotes the closure of D, is of Borel class 2 whenever Y is metric and compact, and that the boundary may fail to be of Borel class 1. Note that F is the intersection of two lower semi-continuous multifunctions.

5. Dense families of selectors

Now we will consider closed-valued multifunctions with pointwise dense families of selectors. First, let us see the following example.

Example. Let F be a multifunction from $\mathbb{R}$ to $\mathbb{R}$ defined as follows: $F(0) = \{0\}$ and $F(x) = \mathbb{R}$ for $x \neq 0$. Note that F has a dense sequence of continuous selectors, for instance we may take the sequence of linear functions of the form $f_n(x) = a_n x$, where (a_n) is the sequence of all rationals. However, F is not upper semicontinuous.

Theorem 4. Let X be a topological space, (Y, d) a metric space and F a closed-valued multifunction from X to Y. If F has a pointwise dense family of continuous selectors then F is of upper Borel class 1.

Proof. Let S be a pointwise dense family of continuous selectors of F and take an arbitrary open set $A \subset Y$. We will show that $F^+(A)$ is an F_σ-set in Y. Denote by D the set $Y \backslash A$. D is closed and $F^+(A) = X \backslash \{x \in X : F(x) \cap D \neq \emptyset\}$. Observe that the following three conditions are equivalent:

$$F(x) \cap D \neq \emptyset, \qquad d(F(x), D) = 0, \qquad \inf\{d(f(x), D) : f \in S\} = 0,$$

where $d(B, C) = \inf\{d(b, c) : b \in B,\ c \in C\}$. Moreover for each $f \in S$ the superposition $d(f(\ ), D)$ is a continuous function. Now we may and do assume that the family S is countable. We have the following formula

$$\{x \in X : d(F(x), D) = 0\} = \bigcap_{n=1}^{\infty} \bigcup_{f \in S} \left\{x \in X : d(f(x), D) < \frac{1}{n}\right\}$$

From this we infer that $\{x \in X : F(x) \cap D \neq \emptyset\}$ is a G_δ-set. This ends the proof.

Remark. In the same way as above we may prove that $F^+(A) \in F_{\sigma\delta}$ for every open $A \subset Y$ whenever the family S consists of selectors of Borel class 1.

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