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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 35 (1994), No. 2, 5--12

Persistent URL: http://dml.cz/dmlcz/702008

On the Dunford-Pettis Property in Banach Spaces

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Received 15. March 1994

We give a brief survey of recent results and examples related with the Dunford-Pettis property, and collect some open question.

0. Introduction

A Banach space X has the *Dunford-Pettis property* (*DPP*, in short) if, for any Banach space Y, every weakly compact operator $T: X \to Y$ transforms weakly convergent sequences into convergent sequences.

This property was introduced by Grothendieck in [G]. A nice survey up to 1979 is J. Diestel's paper [D1]; however, many of the open problems stated in [D1] have been solved by now.

Even though it has been known since the fifties that $L_1(\mu)$ spaces and $C(K)$ spaces have the *DPP*, it has been exceedingly difficult to find other spaces with that property. In fact, for a long time it was unknown if X has the *DPP* and X^* has the *DPP* were equivalent.

The are very few general criteria to recognize Banach spaces with the *DPP*, and many different methods have been designed to prove this property for the concrete spaces that are known to have it.

Here we give a brief up-to-date supplement to the survey [D1], and record some open questions.

Throughout the paper, X will denote a Banach space.

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Supported in part by DGICYT Grant PB91-0307 (Spain)

1. Basic examples of spaces with the DPP (a) $C(K)$-spaces and $L_1(\mu)$-spaces [G]. In particular, c_0, ℓ_1 and ℓ_∞ have the *DPP*.

(b) Complemented subspaces of spaces with the *DPP*. Hence, $\mathcal{L}_\infty$-spaces and $\mathcal{L}_1$-spaces have the *DPP* (see [B2]).

(c) Spaces having the *Schur property*; i.e., spaces in which weakly convergent sequences are norm convergent.

(d) Infinite dimensional reflexive spaces (like ℓ_2) fail the *DPP*. Hence, the *DPP* is not stable under subspaces or quotients.

The following characterization is quite useful.

2. Proposition *The space X has the DPP if and only of given weakly null sequences* (x_n) *in* X *and* (f_n) *in* X^* *we have* $f_n(x_n) \to 0$.

3. Corollary *If* X^* *has the DPP, then X has the DPP.*

An example of C. Stegall shows that the converse implication fails:

4. Example [S] The space $\ell_1(\ell_2^n)$ has the Schur property, hence the *DPP*. Moreover, its dual space $\ell_\infty(\ell_2^n)$ contains a complemented copy of ℓ_2. The latter fact can be seen as follows:

Let $\mathcal{U}$ be a non-trivial ultrafilter in the set of all integers. Every $x \in \ell_\infty(\ell_2^n)$ can be represented by a lower triangular matrix in which the nth row is a vector $(a_{n1}, \dots a_{nn}) \in \ell_2^n$. Since the columns of the matrix are bounded sequences of numbers, the limit over $\mathcal{U}$ exists.

Let us denote $a_{\infty j} := \lim_{\mathcal{U}(i)} a_{ij}$. One has

$$x = \begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \\ \dots & \dots & \dots \end{pmatrix}. \quad \begin{matrix} \downarrow & \downarrow & \downarrow \\ a_{\infty 1} & a_{\infty 2} & a_{\infty 3} \end{matrix} \quad \text{along } \mathcal{U}.$$

Let us define $P\colon \ell_\infty(\ell_2^n) \to \ell_\infty(\ell_2^n)$ by

$$Px = \begin{pmatrix} a_{\infty 1} & & \\ a_{\infty 1} & a_{\infty 2} & \\ a_{\infty 1} & a_{\infty 2} & a_{\infty 3} \\ \dots & \dots & \dots \end{pmatrix}.$$

Since for any n we have $a_{\infty 1}^2 + \dots + a_{\infty n}^2 \leqq \sup_{i \geqq n} a_{i1}^2 + \dots + a_{in}^2 \leqq \|x\|^2$, the map P is a continuous projection. Moreover,

$$U : a = (a_k) \in \ell_2 \to (Ua)_{ij} := a_j\,, \qquad 1 \leqq j \leqq i = 1, 2, 3, \dots$$

defines an isometry from ℓ_2 onto the range of P.

Observation If the reader does not believe in ultrafilters, he could consider the space c of all convergent sequences of scalars as a closed subspace of the space ℓ_∞ of all bounded sequences, and take instead of $\lim\limits_{\mathcal{U}(i)}$ any Hahn-Banach extension of the limit functional $\lim \in c^*$, given by $\lim(a_i) := \lim\limits_{i\to\infty} a_i$, $(a_i) \in c$.

The space $\ell_\infty(\ell_2^n)$ is essentially the only known example of a Banach space X with the *DPP* such that X^* fails to have *DPP*. So the question arises.

QUESTION 1

Find new examples of spaces with the *DPP* whose dual fails it. In particular, it would be interesting to study the following cases:

(a) Let $\{X_n\}$ be a sequence of finite dimensional spaces. Is it possible to obtain conditions implying that $\ell_\infty(X_n)$ has the *DPP*? This is a difficult question. Bourgain [B1] proved that $\ell_\infty(\ell_n^1)$ has the *DPP*, and asked if $\ell_\infty(X_n^*)$ has the *DPP* when $\ell_\infty(X_n)$ has it.

(b) It is known that $(c_0 \hat{\otimes}_\pi c_0)^*$has the the Schur property, hence the *DPP* [R]. Does $(c_0 \hat{\otimes}_\pi c_0)^{**}$ have the *DPP*?

(c) Is it possible to find a quotient map $q\colon \ell_1 \to c_0$ such that the dual space of the kernel of q fails the *DPP*?

The study of the behaviour of the *DPP* under duality seems to be a difficult problem because one needs information about weakly null sequences in the second dual of a Banach space. However, a characterization of the *DPP* for dual spaces can be given as follows.

5. Proposition *[CG2] The following assertions are equivalent:*

(a) The dual space X^* has the DPP.

(b) For every weakly compact operator $T\colon X \to Y$, the second conjugate T^{**} transforms weakly convergent sequences into convergent sequences.

A good characterization can be given in a special case.

6. Theorem *[PT] The following assertions are equivalent:*

(a) X has the DPP and contains no copies of ℓ_1.

(b) X^* has the Schur property.

7. Corollary *If X is an infinite dimensional space with the DPP, then X^* contains a copy of ℓ_1.*

Proof.

If ℓ_1 is contained in X, then $L_1[0, 1]$ is contained in X^* [D2]. Otherwise, apply theorem 6: Schur spaces contains plenty of copies of ℓ_1.

QUESTION 2 [D1] Assume X has the *DPP* and contains no complemented copies of ℓ_1. Does X^* has the *DPP*?

8. Further examples The following classical Banach spaces have the *DPP*:

(a) The disc algebra A and its dual space A^* [Ch].

(b) The Hardy space H^∞ and its dual spaces of any order [B4].

(c) The quotient space L_1/H_1 [Cha], and its ultraproducts [B4].

(d) The hereditarily reflexive, $\mathscr{L}_\infty$-space X with $X^* \cong \ell_1$ [BD].

(e) The ball algebras $A(\mathbb{B}^n)$ and the polydisc algebras $A(\mathbb{D}^n)$ [B3].

(f) The space of smooth function $C^{(k)}(\mathbb{T}^n)$, $\mathbb{T}$ unit circle in $\mathbb{C}$ [B3].

As examples of classical Banach spaces failing the *DPP* we have:

(e) The Hardy space H_1, its predual *VMO* and its dual *BMO*: Note that H_1 is a separable dual which is not Schur.

(f) The Lorentz spaces $\Lambda(\mathrm{w}, 1)$: They are also non-Schur separable duals.

For every subset Λ of the integers, define $C_\Lambda(\mathbb{T})$ as the subspace of $C(\mathbb{T})$ generated by $\{e^{int}\}_{n\in\Lambda}$. We can identify the disc algebra A with the space $C_\Lambda(\mathbb{T})$ obtained taking as Λ the positive integers.

QUESTION 3 [B4, p. 29] Is it possible to find $\Lambda \subset \mathbb{Z}$ so that $C_\Lambda(\mathbb{T})$ fails the *DPP*?

It has also been open for a long time the question whether the *DPP* is stable under injective or projective tensor products; in particular, whether $C(K, X) \cong$ $\cong C(K) \hat{\otimes}_\varepsilon X$ and $L_1(\mu, X) \cong L_1(\mu) \hat{\otimes}_\pi X$ have the *DPP* when X has it (see [D1]). A strongly negative answer was given by Talagrand.

9. Example [T] There exists a Banach space $\mathscr{T}$ such that $\mathscr{T}^*$ has the Schur property, but neither $C(K, \mathscr{T})$ nor $L_1(\mu, \mathscr{T}^*)$ have the *DPP*.

10. Observation Nuñez [N] showed that the second dual $\mathscr{T}^{**}$ of Talagrand space contains a complemented copy of ℓ_2, hence it fails the *DPP*. He conjectured that $C(K, X)$ has the *DPP* when X^{**} has it.

QUESTION 4 [N] Assume the second dual X^{**} of the space X has the *DPP*. Do $C(K, X)$ and $L_1(\mu, X^{**})$ have the *DPP*?

There are some partial affirmative answers. If X^* is Schur then $L_1(\mu, \Xi)$ has the *DPP* [A]. For projective tensor products, we also have some positive results.

11. Proposition *[R2] Assume that X and Y have the DPP and contain no copy of ℓ_1. Then $X \hat{\otimes}_\pi Y$ also has those properties.*

12. Corollary The space $c_0 \hat{\otimes}_\pi c_0$ has the *DPP*.

QUESTION 5 For which compact spaces K, does $C(K) \hat{\otimes}_\pi C(K)$ have the *DPP*? In particular, do $C[0, 1] \hat{\otimes}_\pi C[0, 1]$ and $\ell_\infty \hat{\otimes}_\pi \ell_\infty$ have the *DPP*?

Note that $\ell_\infty \hat{\otimes}_\pi \ell_\infty$ is isomorphic to a closed subspace of $(c_0 \hat{\otimes}_\pi c_0)^{**}$, but we do not know if $(c_0 \hat{\otimes}_\pi c_0)^{**}$ has the *DPP*.

Another positive result related with tensor products was obtained by Bourgain.

13. Theorem *The spaces* $C(K, L_1(\mu))$ *and* $L_1(\mu, C(K))$ *and all their dual spaces have the DPP.*

The proof is based in a result that is interesting by itself.

14. Proposition *Let* (X_n) *be a sequence of subspaces of a Banach space* X *and assume that the union* $\bigcup_{n=1}^{\infty} X_n$ *is dense in* X. *If* $\ell_\infty(X_n)$ *has the DPP, then* X *has also the DPP.*

A related result was obtained in [CG1]. This is the only result we know giving conditions for X^{**} to have the *DPP*.

15. Proposition *[CG1] Assume the space* X *has a shrinking basis* $\{e_n\}$; *for example, a space* X *with an unconditional basis containing no copies of* ℓ_1. *If* $\ell\ ([e_1, \ldots, e_n])$ *has the DPP, the* X^{**} *has also the DPP.*

16. Definition A Banach space X has the *hereditary Dunford-Pettis* property (in short, *h-DPP*) if every subspace of X has the *DPP*.

Using a deep result of Elton [E] about weakly null sequences, a useful characterization of these spaces was obtained.

17. Theorem *[Ce] A Banach space* X *has the h-DPP if and only if every normalized weakly null sequence in* X *has a subsequence which is equivalent to the unit vector basis of* c_0.

This characterization was greatly improved by Knaust and Odell.

18. Theorem *[KO] The space* X *has the h-DPP if and only there exists a constant* $C > 0$ *so that every normalized weakly null sequence in* X *has a subsequence which is C-equivalent to the unit vector basis of* c_0.

19 Examples (a) Obviously, Schur spaces have the *h-DPP*.

(b) The space $c_0(I)$ has the *h-DPP*.

(c) A space $C(K)$ has the *h-DPP* if and only if K is dispersed and the ωth derived set of K is empty. In particular, $C(\omega^\omega)$ fails the *h-DPP*.

(d) There exist spaces $X_n \cong c_0$ such $c_0(X_n)$ fails the *h-DPP*. The spaces X_n are renormings of c_0 such that their constants $C_n \to \infty$.

Recall that a class of Banach spaces $\mathscr{C}$ has the three-space property if a space $X \in \mathscr{C}$ when it contains a subspace M such that $M, X/M \in \mathscr{C}$.

20 Theorem *[CG1] The class of spaces with the DPP and the class of spaces with the h-DPP fail the three-space property.*

Proof.

A subset $A \subset \mathbb{N}$ is *admissible* if $A = \{n_1 < \ldots < n_k\}$ with $k \leqq n_1$. The Schreier's space S is defined as the completion of the space of all finitely nonzero sequences with respect to the norm

$$\|(a_i)\|_S := \sup \left\{ \sum_{j \in A} |a_j| : A \subset \mathbb{N} \text{ admissible} \right\}.$$

It can be shown that the unit vector basis $\{e_n\}$ in S and its dual sequence $\{e^*\} \subset S^*$ of coefficient functionals are both weakly null; hence S does not have the *DPP*.

Let $i: S \to c_0$ denote the natural inclusion, and let $q: \ell_1 \to c_0$ be a quotient map. Define

$$Q: (x, y) \in S \times \ell_1 \to ix - qy \in c_0 .$$

Clearly Q is surjective, and it can be showed that the kernel $N(Q)$ has the *h-DPP*. So $X \times \ell_1$ fails the *DPP*, although $N(Q)$ and $(S \times \ell_1)/N(Q)$ have the *h-DPP*.

A positive result for the three-space property is the following.

21. Proposition *Let X be a Banach space.*
If X^* and X^{**}/X have the DPP, then X^{**} has the DPP.

Proof.
If we consider the decomposition $X^{***} \cong X^* \oplus (X^{**}/X)^*$, then any weakly null sequence (α_n) in X^{***} can be written as $\alpha_n = f_n + h_n$. So, if (F_n) is a weakly null sequence in X^{**}, then $\alpha_n(F_n) = \alpha_n(f_n) + \alpha_n(h_n)$ converges to 0 because X^* and X^{**}/X have the *DPP*.

22. Corollary *[CG2] Assume X contains no copies of ℓ_1. Then X and X^{**}/X have the DPP $\Leftrightarrow$ X^{**} has the DPP.*

QUESTION 6 Is it true in general the equivalence X and X^{**}/X have the *DPP* $\Leftrightarrow$ X^{**} has the *DPP*?

It is when X is a dual space, because $X^{**} \cong X \oplus X^{**}/X$ in this case.

Let Λ be a Banach space with an unconditional basis, and let (X_n) be a sequence of Banach spaces. We will consider the Banach space

$$\Lambda(X_n) := \{(x_n)/x_n \in X_n \quad \text{and} \quad (\|x_n\|) \in \Lambda\}.$$

23. Theorem *[CG2] If Λ and the spaces X_n have the DPP, then $\Lambda(X_n)$ has the DPP.*

Using standard techniques [LT] this last theorem yields the following result.

24. Theorem *[CG2] Any Banach space with an unconditional basis and the DPP is a complemented subspace of a space with symmetric basis and the DPP.*

In [D1] it was asked if there exists a space with separable dual and the *h-DPP* that is universal for this class of spaces. We finish the paper proving a related result.

25. Proposition *There exists no universal space for the class of dual separable spaces with the DPP.*

Proof.
Szlenk [Sz] proved the non-existence of a universal space for the class of separable reflexive Babach spaces. The method of proof was to introduce an ordinal index, known as Szlenk's index, to show that every separable reflexive Banach has countable index, and to give a procedure to obtain separable reflexive Banach spaces

with arbitrarily large index. Let $\eta(X)$ denote the Szlenk index of X. Proposition 1.4 of [Sz] shows that if X^* is separable then $\eta(X)$ is countable. Our question is then whether it is possible to obtain Banach spaces with the *DPP* having separable dual and large index.

The procedure, following [Sz], is: start with a Banach space X having the *DPP* and dual separable, and put $X_0 = X$. Then, for any countable ordinal α, define $X_\alpha = (X_{\alpha-1} \times c_0)_1$ if α is not a limit ordinal, and $X_\alpha = c_0(X_\beta)_{\beta \leqq \alpha}$ if α is a limit ordinal. It follows from Theorem 23 that X_α has *DPP* and separable dual for any countable α. It is also clear that $\eta(X_\alpha) \geqq \sup\{\eta(X_\beta) : \beta < \alpha\}$. It only remains to verify the following version of lemma 3.1 of [Sz]:

$$\text{if } X^* \text{ is separable, then } \eta((X \times c_0)_1) \geqq \eta(X) + 1\,.$$

(In Szlenk's paper, ℓ_2 plays the role of c_0). This is easy regarding that the properties of ℓ_2 which are needed for the proof are: the canonical basis of the space is weakly null, and the canonical basis of the dual is weakly* null.

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