David Yost
Asplund spaces for beginners

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 34 (1993), No. 2, 159--177

Persistent URL: http://dml.cz/dmlcz/702006

Asplund Spaces for Beginners

D. Yost

Milano*)

Received 14 April 1993

It is now apparent, to Banach spacemen and women in particular, and perhaps to analysts in general, that Asplund spaces are an interesting and useful class of Banach spaces. Put simply, an Asplund space is a Banach space, on which every continuous convex real-valued function is automatically Fréchet differentiable on a dense set. This is not quite the standard definition, but we will soon see that it is equivalent. A problem for newcomers to this area is that there is a plethora of properties equivalent to Asplundness; finding a reasonable proof of the particular implication that one needs may involve a wild goose chase through the literature. Even though books have now been written about this topic ([Bo, Chapter 5], [G2], [P3]), they quite rightly embed Asplund spaces into the general theory of differentiation of convex functions on infinite-dimensional spaces. One encounters Hadamard differentiability, bump functions, variational principles, optimization problems, dentability and usco maps in this now vast area. One also finds the basic facts about Asplund spaces, but not necessarily all in one place. (Admittedly, anyone with access to [P1] will have an easier time).

The first section of this note gives a quick account of the differentiability properties of convex functions. The proofs are by now standard, but not entirely trivial. Our main aim is to give a succint introduction to Asplund spaces and their basic properties; this is done in the second section. Experts will find nothing remotely surprising here, but we hope that others will find the exposition useful. After a few basic definitions, we give just one Theorem, stating that half a dozen other conditions are equivalent to Asplundity.

The third section briefly studies some properties enjoyed by Asplund spaces. Some of these are also characterizations of Asplundty, but including them all in § 2 would have cluttered the exposition.

*) Università di Milano, Dipartimento di Matematica, Via C. Saldini 50, 20133 Milano, Italy

In § 4 we show that this theory is not vacuous, by briefly discussing all the examples of Asplund spaces which we have been able to find. We claim no novelty for the results in this section either.

We adopt the widespread and harmless convention that topological concepts refer to the norm topology, unless explicitly stated otherwise.

1. Convex functions and differentiability

We assume that the reader is familiar with the basic properties of convex functions from $\mathbf{R}$ to itself, or from an open interval into $\mathbf{R}$. Namely, every such function is continuous, has left and right derivatives at every point, these one-sided derivatives are monotonic and agree except perhaps on a countable set. Thus every convex function on $\mathbf{R}$ is differentiable almost everywhere.

It will be convenient to work with the subdifferential of a convex mapping. Given a continuous convex $\varphi: X \to \mathbf{R}$, its subdifferential is the set-valued mapping $\partial\varphi: X \to 2^{X^*}$ defined by $\partial\varphi(x) = \{f \in X^*: f(y) - f(x) \leqq \leqq \varphi(y) - \varphi(x)$ for all $y \in X\}$. (In case φ is only defined on a convex open subset D of X, the condition $\forall y \in X$ must be replaced by $\forall y \in D$). As a motivation for this concept, we note that a convex $\varphi: \mathbf{R} \to \mathbf{R}$ is differentiable at the point x iff there is a unique $f \in \mathbf{R}$ satisfying $fy - fx \leqq \varphi(y) - \varphi(x)$ for all $y \in \mathbf{R}$, iff there is a unique affine $a: \mathbf{R} \to \mathbf{R}$ with $a \leqq \varphi$ and $a(x) = \varphi(x)$. In case X is one-dimensional, we may identify X^* with $\mathbf{R}$; then $\partial\varphi(x)$ reduces to the interval between the left derivative of φ at x and the right derivative.

It is routine to show that $\partial\varphi(x)$ is always weak* closed, convex and non-empty; the following Lemma implies that it is bounded, hence weak* compact.

Lemma 1. *Let $\varphi: X \to \mathbf{R}$ be locally bounded (in particular continuous) and convex. Then φ is locally Lipschitz, and $\partial\varphi$ is locally bounded.*

Proof. Fix $a \in X$. Local boundedness gives us $M, \delta > 0$ so that $|\varphi(x)| \leqq M$ whenever $x \in B(a, 2\delta)$. For any $x, y \in B(a, \delta)$, set $\alpha = \|x - y\|$ and $z = \|x - y\|\delta/\alpha$. Then

$$\|z - a\| \leqq 2\delta \Rightarrow y = \frac{\alpha}{\alpha + \delta} z + \frac{\delta}{\alpha + \delta} x \in B(a, 2\delta)$$

$$\Rightarrow \varphi(y) \leqq \frac{\alpha}{\alpha + \delta} \varphi(z) + \frac{\delta}{\alpha + \delta} \varphi(x)$$

$$\Rightarrow \varphi(y) - \varphi(x) \leqq \frac{\alpha}{\alpha + \delta} (\varphi(z) - \varphi(x)) \leqq \frac{2M}{\delta} \|x - y\| .$$

Interchanging x and y, we see that $|\varphi(x) - \varphi(y)| \leqq \frac{2M}{\delta} \|x - y\|$ for all $x, y \in$ $\in B(a, \delta)$. Thus φ is locally Lipschitz, which easily implies the local boundedness of $\partial\varphi$.

Let's say that a function $\varphi: X \to Y$ between two normed spaces is *Gâteaux differentiable* at a point x iff the partial derivative $\frac{\partial \varphi}{\partial v} = \lim_{t \to 0} \frac{1}{t}(\varphi(x + tv) - \varphi(x))$ exists for every direction v. Many authors insist also that $\frac{\partial \varphi}{\partial v}$ depends linearly and continuously on v, although Gâteaux himself did not [Ga]. In case $Y = \mathbf{R}$ and φ is convex, it is routine to show $\frac{\partial \varphi}{\partial v}(x)$ is a linear function of v whenever it exists; so this distinction is irrelevant for our purposes. One calls this continuous linear functions $\varphi'(x): X \to Y$ the Gâteaux derivative of φ at x; thus $\varphi'(x)(v) = \frac{\partial \varphi}{\partial v}(x)$. Another easy exercise is to show that $\partial\varphi(x)$ contains precisely one point (namely the Gâteaux derivative of φ at x) iff φ is Gâteaux differentiable at x.

Fréchet differentiability is the most commonly studied version of differentiability, for functions defined on spaces of more than one dimension. One says that $\varphi: X \to Y$ is Fréchet differentiable at the point x iff there exists a continuous linear $T: X \to Y$ such that

$$\lim_{h \to 0} \frac{\varphi(x + h) - \varphi(x) - Th}{\|h\|} = 0 .$$

It is easy to see that Fréchet differentiability implies Gâteaux differentiability, and that $T = \varphi'(x)$ in this case.

Given two topological spaces T and U, one says that a set-valued function $\Psi: T \to 2^U$ is upper semicontinuous at the point $x \in T$ iff for every open set V containing $\Psi(x)$, there exists a neighborhood N of x such $\Psi(y) \subseteqq V$ for every $y \in N$. Clearly Ψ is upper semicontinuous at every point if and only if $\{x: F \cap$ $\cap \Psi(x) \neq \emptyset\}$ is closed, for every closed $F \subset U$. In case U is a locally convex space and $\Psi(x)$ is compact, we have a more manageable formulation: $\Psi: T \to 2^U$ is upper semicontinuous at the point $x \in T$ if for every neighborhood N of 0 in U, $\{y: \Psi(y) \subset \Psi(x) + N\}$ is a neighborhood of x.

In the case of the subdifferential of a convex function $\varphi: X \to \mathbf{R}$, we consider only the norm topology on X; thus τ-upper semicontinuity refers always to the τ topology on X^*. Weak* upper semicontinuity of the subdifferential $\partial\varphi$ is routine to prove.

Lemma 2. *Let* $\varphi: X \to \mathbf{R}$ *be continuous and convex. Then* φ *is Fréchet differentiable at* x, *with* $\varphi'(x) = f$, *if and only if* $\partial\varphi(x) = \{f\}$ *is a singleton, and* $\partial\varphi$ *is norm upper semicontinuous at* x.

Proof. ($\Leftarrow$) Upper semicontinuity of $\partial\varphi$ ensures that for every $\varepsilon > 0$, we can find a positive δ so that, for every y with $\|y\| < \delta$, we have $\partial\varphi(x + y) \subseteqq$ $\subseteqq B(f, \varepsilon)$. For all sufficiently small y, we then have

$$\begin{aligned} 0 &\leqq \varphi(x+y) - \varphi(x) - f(y) && \text{since } f \in \partial\varphi(x) \\ &\leqq \varphi(x+y) - \varphi(x) - g(y) + \|f-g\|\,\|y\| && \text{for any } g \\ &\leqq 0 + \varepsilon\|y\| && \text{choosing } g \in \partial\varphi(x+y). \end{aligned}$$

($\Rightarrow$) Gâteaux differentiability guarantees that $\partial\varphi(x) = \{f\}$ is a singleton. Now suppose that $x_n \to x$ and $f_n \in \partial\varphi(x_n)$; we must show that $f_n \to f$. Since $\partial\varphi$ is locally bounded, we may assume that the sequence (f_n) is bounded. For all y, we then have

$$\begin{aligned} (f_n - f)(y) &= f_n(x+y) - f_n(x_n) - f(y) + f_n(x_n - x) \\ &\leqq \varphi(x-y) - \varphi(x_n) - f(y) + f_n(x_n - x) \\ &\leqq \varphi(x+y) - \varphi(x) - \varphi'(x)(y) + (\varphi(x) - \varphi(x_n)) + f_n(x_n - x)\,. \end{aligned}$$

The last two terms converge to 0, as $n \to \infty$, uniformly with respect to y. Thus, for any $\varepsilon > 0$, there is a $\delta > 0$, so that for all y with $\|y\| < \delta$ and for all sufficiently large n, we have $(f_n - f)(y) \leqq \varepsilon\|y\| + \varepsilon\delta$.

As usual in functional analysis, it will often be convenient to work in the dual, rather than the space we had to start with. Our first dual concept is the following. Given any bounded set $A \subset X$, we define its support function $\sigma^A\colon X^* \to \mathbf{R}$ by $\sigma^A(f) = \sup_{x \in A} f(x)$. A point $x \in A$ is said to be *exposed* by a functional $f \in X^*$ if f attains its maximum on A at x and at no other point. Obviously every exposed point is an extreme point; the converse is false, even in two dimensions. One says that x is *strongly exposed* by f if, in addition, every sequence (x_n) in A which satisfies $f(x_n) \to f(x)$ is actually convergent to x.

Very often, we will be working with a bounded set $A \subset X^*$, and it will be convenient to work with a support function σ_A defined only on X. Thus we set $\sigma_A(x) = \sup_{f \in A} f(x)$. If the point $f \in A$ is exposed by a functional which lies in X, not merely in X^{**}, we say that it is *weak* exposed*. Similarly we say f is *strongly weak* exposed* by $x \in X$ if $\|f_n - f\| \to 0$ for every sequence (f_n) in A for which $f_n(x) \to f(x)$. Note that the support function of a set in X^* coincides with the support function of its weak* closed convex hull; thus the assumption which begins the next result is not at all restrictive.

Lemma 3. *For any weak* compact convex set A in X^*, we have*

(i) σ_A is continuous and convex,

(ii) $f \in \partial\sigma_A(x)$ if and only if $f \in A$ and $f(x) = \sigma_A(x)$,

(iii) σ_A is Gâteaux differentiable at x, with $\sigma_A'(x) = f$ if and only if $f \in A$ is weak exposed by x,*

(iv) σ_A is Fréchet differentiable at x, with $\sigma_A'(x) = F$ if and only if $f \in A$ is strongly weak exposed by x.*

Proof. (i) and (ii⇐) are very easy.

(ii⇒) For all $y \in X$, $\lambda > 0$, we have $f(\lambda y) - f(x) \leqq \sigma_A(\lambda y) - \sigma_A(x)$. Dividing by λ and letting $\lambda \to \infty$ yields $f(y) \leqq \sigma_A(y)$. The separation theorem then forces $f \in A$. Finally $-f(x) = f(0) - f(x) \leqq \sigma_A(0) - \sigma_A(x) = -\sigma_A(x)$.

(iii) follows immediately from (ii).

(iv⇒) Part (iii) shows that $f = \sigma'_A(x)$ belongs to A. Differentiability in the Fréchet sense means that for every $\varepsilon > 0$, we can find $\delta > 0$ so that every y with $\|y\| < \delta$ satisfies $\sigma_A(x + y) - \sigma_A(x) - f(y) < \varepsilon\|y\|$. Now suppose that $f_n \in A$ and that $f_n(x) \to f(x)$. Then

$$\begin{aligned}(f_n - f)(y) &= f_n(x + y) - f(x) - f(y) - (f_n - f)(x) \\ &\leqq \sigma_A(x + y) - \sigma_A(x) - \sigma'_A(x)(y) - (f_n - f)(x) \\ &\to 0 \qquad\qquad \text{uniformly with respect to } y.\end{aligned}$$

This implies that $\|f_n - f\| \to 0$.

(iv⇐) is easy now. From (iii) we know that σ_A is Gâteaux differentiable at x, and that $f = \sigma'_A(x)$ is the unique member of $\partial\sigma_A(x)$. By Lemma 2, it suffices to show that $\partial\sigma_A$ is norm upper semicontinuous at x. So let $x_n \to x$, and $f_n \in \partial\sigma_A(x_n)$. Using (iii) again, $f_n \in A$ and $f_n(x_n) = \sigma_A(x_n) \to \sigma_A(x) = f(x)$. By hypothesis $f_n \to f$.

Lemma 3(iv) is essentially due to Šmulian [Sm, Theorem 1]. He established this in the case when A is the dual unit ball (equivalently, when σ_A is the norm), but the same argument works in the general case. This seems to have been observed first by Asplund [As, Propositon 1]. It indicates the fundamental connection between differentiabilty and geometric properties of the dual space, and could be said to mark the beginning of this area of research. Lemma 3(iv⇒) can be proved more simply, in the same manner as its converse, by using the Bishop-Phelps Theorem. Since the latter was not available to Šmulian, we thought it more interesting to present a direct argument.

For completeness, we state the result dual to Lemma 3. It is not needed until § 3.

Lemma 4. *For any closed bounded conves set A in X, we have*

(i) σ^A is convex and weak lower semicontinuous,,*

(ii) if A is compact, then σ^A is weak continuous,*

(iii) σ^A is Fréchet differentiable at f, with $(\sigma^A)'(f) = x$ if and only if $x \in A$ is strongly exposed by f.

Proof. (i) and (ii) should be straightforward by now.

(iii) Denote by B the weak* closure of A in X^{**}. Lemma 3 tells us that $\sigma_B: X^{**} \to \mathbf{R}$ is Fréchet differentiable at f, with $\sigma'_B(f) = F$, say, iff B is strongly weak* exposed at F by f. To establish (⇒), it suffices to show that $F \in A$. Let (x_n) be a sequence in A with $f(x_n) \to \sigma_A(f)$. Since $\sigma^A(f) = \sigma_B(f) = F(f)$, we have $\|x_n - F\| \to 0$, and so $F \in X$.

Since ($\Leftarrow$) is not needed until much later, and not for our main theorem, we feel free to use the language of *slices*, which is introduced in the next section. Suppose then that A is strongly exposed at x by f. This means that $\operatorname{diam} S(A, f, \varepsilon) \to 0$ as $\varepsilon \to 0$. Note that if Y is any subset of a topological space, and G is an open subset, then $G \cap Y$ is dense in $G \cap \bar{Y}$. Thus $S(B, f, \varepsilon)$ is weak* dense in $S(A, f, \varepsilon)$, and the two slices have the same diameter. It follows that B is strongly weak* exposed at x by f.

Lemma 4(iii) was also proved by Šmulian. In fact, his proof covered Lemmas 3 and 4 simultaneously. We stress that the differential of σ^A actualy lies in X, not merely in X^{**}. Combining Lemma 3(iv) and Lemma 4(iii), and taking A to be the unit ball of X^*, we obtain the result, implicit in [Sm], that if the norm on X is Fréchet smooth at the point x, then the norm on X^{**} is also Fréchet smooth at x.

2. Characterizations of Asplund spaces

An Asplund space is usually defined as a Banach space X with the property that if D is any open convex subset of X and $\varphi: D \to \mathbf{R}$ is any continuous convex function, then φ is Fréchet differentiable on a dense G_δ subset of D. We will soon see that this is equivalent to the definition given in the introduction. In fact, the set of points of Fréchet differentiability of φ is necessarily a G_δ set [P3, Proposition 1.25], so this requirement could be omitted from the definition. Asplund spaces were first studied seriously by Asplund [As] under a different name, and were renamed in his honor in [NP].Let us mention two easy examples now. It is clear that $\mathbf{R}$ is an Asplund space, whereas ℓ_1 is not, as its norm is not Fréchet differentiable at any point.

Of all the properties equivalent to Asplundity, the following seems to be one of the most useful: X is Asplund if and only if every separable subspace of X has separable dual. Indeed, this now seems to be the most popular way of proving that a given space is Asplund. Therefore our first task in presenting Theorem 6 below was to find a reasonably simple proof of the sufficiency of this condition. For better or worse, we still need a lot of definitions; let's get them over and done with.

Given a subset A of a Banach space X, *a slice* of A is the nonempty intersection of A with an open half-space. In case A is bounded, one often writes

$$S(A, f, \varepsilon) = \{x \in A : f(x) > \sup f(A) - \varepsilon\},$$

where $f \in X^*$ and $\varepsilon > 0$. A set is called *sliceable* if it contains slices of arbitrarily small diameter. Here, one needs to exclude the empty set from being a slice. In case X is a dual space, one defines a *weak* slice* as the nonempty intersection of A with a weak* open half-space; thus the determining functional comes from the predual of X, not merely from its dual. A subset of a dual space is then said to be *weak* sliceable* if it contains weak* slices of arbitrarily small diameter. A Banach space is said to have the *Radon-Nikodým Property* if every bounded subset is sliceable. (The restriction to bounded sets is essential; obviously the whole space is never sliceable). Although it is not standard terminology, we will say that a dual space has the *weak* Radon-Nikodým Property* if every bounded subset is weak* sliceable. The reason that this term is not commonly used is (as we will soon see) that the weak* Radon-Nikodým Property is equivalent to the Radon-Nikodým Property in dual spaces.

The Radon-Nikodým Property for X is equivalent to the validity of the Radon--Nikodým Theorem for X-valued vector measures — hence the name. For a proof of this, see [DU] and [GU]. This is a much studied property, which we cannot do justice to in this short note, so we simply point to [Bo] and [DU] for further enlightenment. The duality between Asplund spaces and spaces with the Radon--Nikodým Property ((5) $\Leftrightarrow$ (7) in Theorem 6) indicates that the two properties cannot successfully be studied in isolation from one another; nevertheless we limit ourselves here to introducing Asplund spaces.

Obviously any set with a strongly (weak*) exposed point is (weak*) sliceable; a sort of converse is given by [NP, Lemma 1]. Thus Šmulian's result (Lemma 3) implies that the dual of an Asplund space has the weak* Radon-Nikodým Property. This seems to have been first observed in [NP]. That the converse is true was first proved much later. This dual property is central to the study of Asplund spaces; indeed it is involved in the proofs of the majority of the implications in Theorem 6.

More generally, given a topological space T and a metric d on T, not necessarily related to the topology, one says that T is *fragmented* by d if every subset of T contains non-empty relatively open subset of arbitrarily small d-diameter. Thus every sliceable set is fragmented by (the metric induced by) the norm. Since the following argument is used more than once, we state it separately.

Lemma 5. *Suppose that C is a subset of X^* which is not weak* fragmented by the norm. Then there exists an $\varepsilon > 0$, a sequence of non-empty relatively weak* open sets $V_n \subset C$, and a sequence of norm one vectors x_n in X, so that $V_{2n} \cup V_{2n+1} \subset V_n$ for all n, and with $f(x_n) - g(x_n) \geqq \varepsilon$ whenever f is in the weak* closure of V_{2n} and g is in the weak* closure of V_{2n+1}.*

Proof. Let ε be such that $\operatorname{diam} V > 3\varepsilon$ for every relatively weak* open $V \subset C$.

Set $V_1 = C$. Then there exist $f_0, g_0 \in V$ and $x_1 \in X$ with $\|x_1\| = 1$ so that $f_0(x_1) - g_0(x_1) > 3\varepsilon$. Put $V_2 = \{h \in V_1 : h(x_1) > f_0(x_1) - \varepsilon\}$ and $V_3 = \{h \in V_1 : h(x_1) < f_0(x_1) + \varepsilon\}$. It is clear that $f(x_1) - g(x_1) \geqq \varepsilon$ whenever f is in the weak* closure of V_2 and g is in the weak* closure of V_3. Since V_2 and V_3 are weak* open (in C), they must have diameter greater than 3ε.

Repeating this construction, we obtain the desired sequences.

The *density character* of a Banach space X is the minimum possible cardinality of a dense subset of X. Thus a Banach space is separable if and only if its density character is $\aleph_0$. A routine argument shows that $\operatorname{dens} X \leqq \operatorname{dens} X^*$ for every Banach space X^*. The inequality may be strict.

Theorem 6. *For any Banach space X, the following are equivalent.*

(1) Every separable subspace X has separable dual.

(2) Every subspace of X has the same density character as its dual.

(3) The unit ball of X^*, considered in the weak topology, is fragmented by the norm.*

(4) X^* has the weak Radon-Nikodým Property.*

(5) X is an Asplund space.

(6) Every weak compact convex subset of X^* is the weak* closed convex hull of its strongly weak* exposed points.*

(7) X^* has the Radon-Nikodým Property.

Proof. We'll show (1) ⇒ (3) ⇒ (4) ⇒ (7) ⇒ (2) ⇒ (1) and that (4) ⇒ (5) ⇒ ⇒ (6) ⇒ (4).

(1) ⇒ (3) *Suppose* that C is a bounded subset of X^*, equipped with the weak* topology, which is not fragmented by the norm. Let (x_n), (V_n) and (f_n) be the sequences given by Lemma 5.

Then Y, the closed linear span of (x_n), is a separable subspace of X. For every sequence $\alpha = (\alpha_n)$ of natural numbers with $\alpha_{n+1} - 2\alpha_n \in \{0, 1\}$ for every n, weak* compactness guarantees the existence of some $f_\alpha \in \bigcap_{n=1}^{\infty} \overline{V_{\alpha_n}}$. If α and β are distinct, then, for a suitable value of n, $\|f_\alpha|Y - f_\beta|Y\| \geqq \|f_\alpha(x_n) - f_\beta(x_n)\| \geqq \varepsilon$. Since there are uncountably many α with the given property, we see that Y^* is not separable.

(3) ⇒ (4) Let C be a bounded subset of X^*; without loss of generality, we may suppose that it is weak* compact and convex.

Let A be the weak* closure of ext C, the set of extreme points of C. By hypothesis, we can find, for any $\varepsilon > 0$, a weak* open set V in X^* so that $V \cap A$ is non-empty and has diameter less than ε. Now let C_1 be the weak* closure of $A \setminus V$, and C_2 the weak* closure of $V \cap \operatorname{ext} C$. Then C_1 might be empty, but C_2 certainly is not. The Krein-Milman Theorem ensures that $C = \operatorname{co}(C_1 \cup C_2)$. In fact,

$$\begin{aligned} V \cap A \neq \emptyset &\Rightarrow V \cap \operatorname{ext} C \neq \emptyset \\ &\Rightarrow \exists p \in \operatorname{ext} C \text{ with } p \notin A \setminus V \\ &\Rightarrow C_1 \neq C. \end{aligned}$$

Clearly $\operatorname{diam} C_2 \leqq \varepsilon$. For $0 \leqq r \leqq 1$, set $C_1 = \{\lambda c_1 + (1 - \lambda)c_2 \colon c_i \in C_i$ and $\lambda_i \in [r, 1]\}$. This is consistent with the original definition of C_1! We also have $C_0 = C$, and $\operatorname{diam}(C \setminus C_r) < 2\varepsilon$ for r sufficiently small.

Choose $f \in \text{ext } C \setminus C_1$; then $f \notin C_r$ whence f and C_r can be weak* separated. The resulting weak* slice of C lies in $C \setminus C_r$ and thus has diameter less than 2ε.

(4) $\Rightarrow$ (7) is trivial.

(7) $\Rightarrow$ (2) Suppose that Z is a subpsace of X, with $\text{dens } Z^* > \text{dens } Z$. Then [SY] X has another subspace $Y \supseteq Z$, with the same density character as Z, which admits a linear extension operator $T: Y^* \to X^*$. Obviously $\text{dens } Y^* > \text{dens } Y$. It will be shown that Y^* contains a bounded subset A which is not sliceable. Then $T(A)$ will be a bounded but unsliceable set in X^*.

Let us call a set *small* if its cardinality is less than or equal to dens Y, and *large* if its cardinality is strictly greater than dens Y. Note that a bounded net is weak* convergent in Y^* iff it converges pointwise on a dense subset of Y. Choose a small dense subset A of Y and a countable base $\mathfrak{C}$ for the topology of $[-1, 1]$. Let $\mathfrak{B}$ be the collection of all sets of the form $\{f \in Y^*: \|f\| \leqq 1, f(F) \subset U\}$, where F is a finite subset of A and $U \in \mathfrak{C}$. Then $\mathfrak{B}$ is a base for the weak* topology on the unit ball of Y^*, and it is small.

One can find an $\varepsilon > 0$ and a large set C in the unit ball of Y^*, so that each two points in C are at distance at least ε apart. (Smallness of all such sets, for all positive rational ε, would imply than $\text{dens } Y^* \leqq \text{dens } Y$). Let C_0 be the collection of all $f \in C$ which have a small relative weak* neighborhood N_f. The existence of a small base for this topology implies that the covering $\{N_f: f \in C_0\}$ has a small subcovering. This means that C_0 itself is small; we may assume that it is empty.

Thus every relatively weak* open set in C is large, and so has diameter at least ε. Lemma 5 gives us (possibly for a different ε) relatively weak* open sets $V_n \subset C$ and norm one vectors $y_n \in Y$ so that $V_{2n} \cup V_{2n+1} \subset V_n$ for all n, and with $(f - g)(y_n) \geqq \varepsilon$ whenever $f \in V_{2n}$ and $g \in V_{2n+1}$. Let K_n be the weak* closed convex hull of V_n, and set $K = \prod_{n=1}^{\infty} K_n$ and $A_n = \{(f_k)_{k=1}^{\infty} \in K: f_n = \frac{1}{2}(f_{2n} + f_{2n+1})\}$. It is easily seen that the sets A_n are closed and have the finite intersection property. (To see that $\bigcap_{n=1}^{k} A_n \neq \emptyset$, choose f_n arbitrary for $n > k$, and then define $f_k, f_{k-1}, \ldots, f_2, f_1$ by $f_n = \frac{1}{2}(f_{2n} + f_{2n+1})$.) Since K is compact, we have $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. So there exist $f_n \in K_n$ with $f_n = \frac{1}{2}(f_{2n} + f_{2n+1})$ for all n. Now

$$f_{2n} - f_{2n+1} \in K_{2n} - K_{2n+1} \subseteq \overline{\text{co}}\,(V_{2n} - V_{2n+1}) \Rightarrow$$
$$\Rightarrow (f_{2n} - f_{2n+1})(y_n) \geqq \varepsilon \Rightarrow \|f_{2n} - f_{2n+1}\| \geqq \varepsilon\,.$$

It follows that $\|f_n - f_{2n}\| \geqq \varepsilon/2$ and $\|f_n - f_{2n+1} \geqq \varepsilon/2$ for every n. Since any half-space containing f_n must contain either f_{2n} or f_{2n+1}, we see that $\{f_n: n \in \mathbf{N}\}$ is bounded but not sliceable.

(2) $\Rightarrow$ (1) is trivial.

(4) ⇒ (5) Let $\varphi: D \to \mathbf{R}$ be continuous and convex, where D is an open convex subset of X. For each natural number n, let us write $x \in G_n$ iff x has a neighborhood N with $\operatorname{diam}(\bigcup_{y \in N} \partial\varphi(y)) < \frac{1}{n}$. Obviously each G_n in open, and Lemma 2 shows that φ is Fréchet differentiable at each point in $\bigcap_{n=1}^{\infty} G_n$. It remains only to show that each G_n is dense. So fix n, and let U be an open subset of D. We must show that $G_n \cap U \neq \emptyset$.

By Lemma 1, we may suppose that $A = \bigcup_{x \in U} \partial\varphi(x)$ is bounded. By hypothesis, it has a slice $S = S(A, y, \alpha)$ with diameter less than $1/n$. Choose $f_1 \in S$ and $x_1 \in U$ so that $f_1 \in \partial\varphi(x_1)$. Then choose $\delta > 0$ so that $x_0 = x_1 + \delta y \in U$; we claim that $x_0 \in G_n$.

Given $f_0 \in \partial\varphi(x_0)$, we have $f_0(x_1 - x_0) \leqq \varphi(x_1) - \varphi(x_0)$ and $f_1(x_0 - x_1) \leqq \leqq \varphi(x_0) - \varphi(x_1)$. Then $\delta(f_1 - f_0)(y) = (f_1 - f_0)(x_0 - x_1) \leqq 0$, whence $f_0(y) \geqq f_1(y) \geqq \sup_{a \in A} a(y) - \alpha$, i.e. $f_0 \in S$. This shows that $\partial\varphi(x_0) \subset S$.

Since $\partial\varphi$ is weak* upper semicontinuous, x_0 has a neighborhood N with $\partial\varphi(z)$ contained in the half-space which determines S, for all $z \in N$. Without loss of generality, we suppose that $N \subseteqq U$. Then $\bigcup_{z \in N} \partial\varphi(z)$ is contained in S, which has diameter less than $1/n$. Thus $x_0 \in G_n$.

(5) ⇒ (6) Let A be a weak* compact convex subset of X^*, and set B equal to the weak* closed convex hull of the strongly weak* exposed points of A. Lemma 3(iv) guarantees that B is not empty. If B were a proper subset of A, then $\{x \in X: \sigma_B(x) < \sigma_A(x)\}$ would be open and non-empty. By hypothesis, we could find an x in this set at which σ_A would be differentiable. But this would imply $f = \sigma'_A(x) \in B$ and $f(x) = \sigma_B(x) < \sigma_A(x) = f(x)$.

(6) ⇒ (4) As we remarked earlier, any set which contains a strongly weak* exposed point is weak* sliceable.

Some references for the non-trivial implications above seem to be in order. (1) ⇒ (4), including Lemma 5, was first proved by Stegall [S2]. The proof (1) ⇒ ⇒ (3) ⇒ (4) given above is a simplification of Stegall's proof, due to Namioka. It appears in [P1], [NP] and [DU, p. 213]. The intermediate property (3) was originally introduced simply to streamline the proof of (1) ⇒ (4); the underlying idea can be found as early as [NA]. Fragmentability was formally defined somewhat later [JR, p55] and has since turned out to be of interest in its own right.

The proof that (7) ⇒ (2) is based on a simplification due to van Dulst and Namioka [DN] of the proof of (7) ⇒ (1) due to Stegall [S1]. The idea of using a small base for the topology to strengthen the conclusion from (1) to (2) comes from [Bo, § 4.2]. Our use of linear extension operators simplifies the proof further. Instead, the original proof established the existence of a minimal weak* compact set in X^* whose image under the restriction map is equal to C; this set then leads to an unsliceable set in X^*, by the same argument. In view of [AS, Theorem 4], Asplund spaces are a natural place in which to look for linear extension operators; see also some results from a previous Winter School in [FG].

The direct implication (4) ⇒ (5) was established by Kenderov [Ke]. (5) ⇒ (6) was proved (in the separable case) but not published by E. Bishop; this proof is due to Asplund [As, Proposition 5].

Another popular proof of (1) ⇒ (5) is the "separable reduction method". This means showing that Asplundness is *separably determined*, i.e. that a Banach space is Asplund iff every separable subspace is Asplund. (Then it remains only to show that X is Asplund whenever X^* is separable.) The idea which follows is due to D. A. Gregory and appeared first in [G1, Theorem (ii) (a)]. It is not hard to show that a continuous convex $\varphi : X \to \mathbf{R}$ is Fréchet differentiable at the point x iff $x \in G_\varepsilon$ for all $\varepsilon > 0$, where $x \in G_\varepsilon$ means that there is a $\delta > 0$ so that $\varphi(x + y) + \varphi(x - y) - 2\varphi(x) < \varepsilon\|y\|$ for all $y \in B(0, \delta)\backslash\{0\}$. With a bit of work, one can then construct, in any non-Asplund space, a separable subspace which is not Asplund. (We note also that each G_ε is open [G1]; thus this characterization also shows that the set of points of Fréchet differentiability is always a G_δ set, since it is just $\bigcap_{n=1}^{\infty} G_{1/n}$.) There are numerous proofs of the Asplundity of spaces with separable duals. One such proof, which recovers differentiability except on a countable set in the one-dimensional case, is in [PZ]. Perhaps the shortest proof is in [G1]. This method of proof for (1) ⇒ (5) is probably about the same length as the one we have given. Since we needed to use Lemma 5 anyway, we thought the proof above would minimize total effort.

To show as quickly as possible that (5) ⇒ (1), first recall that (5) ⇒ (4) is easy. To show (4) ⇒ (1), note that „X is a Banach space whose dual has the weak* Radon-Nikodým property" is a property which passes to subspaces [G1]. The third paragraph of the proof of (7) ⇒ (2) shows that the dual of a separable space with this property must be separable.

Not all readers will be interested in all of these equivalences. For the benefit of those needing, for example, just one implication from this theorem, we indicate some more alternative proofs.

The first proof that (1) ⇒ (2) is in [LW]. The construction given there yields a ***rough norm*** on any space satisfying densX^* > dens$X$; see §3. Note that the condition densX^* = dens$X$ is not sufficient for X to be an Asplund space. If H is a suitably large Hilbert space, then $\ell_1 \oplus H$ has this property, but cannot be Asplund, since it contains ℓ_1.

Note that (5) ⇒ (4) ⇒ (3) is trivial, so shorter proofs of (6) ⇒ (3) and (5) ⇒ (7) are available. Separable determination provides an alternative proof of (5) ⇒ (1). A simple direct proof of (7) ⇒ (4) does not seem to be available. For the Radon-Nikodým purists, a direct proof of (4) ⇒ (6) appears in [P2, p.86], and of (6) ⇒ (5) in [NP]. In [Su] it is shown that (4) is a sepaably determined property; this makes various rearrangements of the proof possible. Those familiar with the many different characterizations of the Radon-Nikodým Property will know that there are several proofs that (1) ⇒ (7); see [DU, Chapter 3].

3. More properties of Asplund spaces

We begin with stability of the Asplund property. It is clear from Theorem 6 that every subspace of an Asplund space is Asplund (and conversely, that a Banach space is Asplund if every separable subspace is Asplund). The following result is a generalization of this. It is probably well known, but we have no reference.

Proposition 7. *Let Y be an Asplund space and $T: X \to Y$ a bounded linear operator whose second adjoint is injective. Then X is also an Asplund space.*

Proof. Assume without loss of generality that X is separable. Then T has separable range, so we may suppose that Y is separable. Then Y^* is separable also. Since $T^*: Y^* \to X^*$ has dense range, we see that X^* is separable.

It is fairly clear that every quotient of an Asplund space is again an Asplund space. This can be proved, as in [As, Proposition 4], directly from the definition, by composing the given convex function with the quotient map. It is necessary to use the fact that the set of points of Fréchet differentiability is always a G_δ set. One can also use the separable subspace criterion. It is too much to expect a space to be Asplund, every time it contains a dense subspace which is the continuous linear image of an Asplund space; for example there is a linear map with dense range from c_0 to ℓ_1. It is trivial that the product of two Asplund spaces is an Asplund space. More generally, being Asplund is a three-space property; the original proof of this [NP, Theorem 14] is a bit longer than the following, which appears in [S1, Corollary 6].

Proposition 8. *Let X be any Banach space, M a closed subspace for which both M and X/M are Asplund. Then X is an Asplund space.*

Proof. Let S be any separable subspace of X. First we note that $\overline{M+S}/M$ is separable; let $\varphi: \overline{M+S} \to \overline{M+S}/M$ be the quotient map, and choose a countable set C in the open unit ball of $\overline{M+S}$ so that $\varphi(C)$ is dense in the open unit ball of $\overline{M+S}/M$. Let Z be the closed linear span of C, and U the open unit ball of Z. Then U and $\varphi(U)$ are CS-compact sets [Ja, § 22], and $\varphi(U)$ is a dense subset of the unit ball. Since CS-compact sets are semi-closed, $\varphi(U)$ must contain the open unit ball of $\overline{M+S}/M$. It follows that $\varphi(Z) = \overline{M+S}/M$, i.e. that $M + Z = \overline{M+S}$. Of course Z is a separable subpace Z of X. Replacing Z by $\overline{Z+S}$, we may also supoose that Z contains S.

Since it is a subspace of M, we see that $Z \cap M$ is Asplund. Since $M + Z$ is closed, $Z/(Z \cap M) \cong (M + Z)/M \subset X/M$, so $Z/(Z \cap M)$ is also an Asplund space. Put $Y = Z \cap M$. Working in the dual of the separable space Z, we have that $Z^*/Y^0 \cong Y^*$ is separable, and that $Y^0 \cong (Z/Y)^*$ is separable. Thus Z^* is separable.

Since S is a subspace of Z, S^* is also separable.

Now we discuss briefly some more properties equivalent to Asplundness. Proofs vary from complete to sketchy to mere references, and definitions are given only as they are needed.

Proposition 9. *Each of the following is equivalent ot X being an Asplund space.*
(i) Every weak compact subset of X^* has a strongly weak* exposed point.*
(ii) Every equivalent norm for (every subspace of) X is Fréchet differentiable at at least one point.
(iii) No equivalent norm for (any subspace of) X is rough.
(iv) Every maximal monotone operator on X is single-valued and upper semicontinuous on a dense G_δ subset of its domain.
(v) Every bounded separable subset of X is weakly metrizable.
(vi) X^* has the Krein-Milman Property.

Proof (with definitions). (i) This property lies between (6) and (4) of Theorem 6.

(ii) Necessity of this condition is obvious. Suppose that X is not Asplund; then there is a bounded set A in X^* which is not weak* sliceable. Without loss of generality, we assume that A is absolutely convex and weak* compact. The Minkowski sum of the unit ball of X^* and A will also be weak* unsliceable, and so will not have any strongly weak* exposed points. This set is obviously the unit ball for an equivalent dual norm on X^*, for which the corresponding norm on X is nowhere Fréchet differentiable.

(iii) A norm $\|\cdot\|$ is *rough* [LW] iff there exists an $\varepsilon > 0$ so that for all x and all $\delta > 0$ there exist $x_1, x_2 \in B(x, \delta)$ and u with $\|u\| = 1$ so that $n(x_1, u) - n(x_2, u) \geqq \varepsilon$, where $n(a, b) = \lim_{t\downarrow 0} \frac{1}{t}(\|a + tb\| - \|a\|)$. In other words, the norm is at each point uniformly non-differentiable in some sense. It is not hard to show that a norm is rough if, and only if, its dual ball is not weak* sliceable [JZ], so (ii) is applicable.

(iv) It would be remiss of us to say nothing about monotone operators. A map $T: X \to 2^{X^*}$ is called a *monotone operator* iff $(f - g)(x - y) \geqq 0$ whenever $f \in T(x)$ and $g \in T(y)$. For technical convenience, we consider only the restriction of T to $\{x : T(x) \neq \emptyset\}$, and we assume that the latter is an open set. If $\varphi : D \to \mathbf{R}$ is continuous and convex, D an open convex subset of X, it is easy to show that $\partial\varphi$ is a monotone operator. There are many other naturally occurring examples [PS]. A monotone operator is *maximal* if its images cannot be enlarged without destroying the monotonicity; in other words if g must belong to $T(y)$ whenever $(f - g)(x - y) \geqq 0$ for all $f \in T(x)$. It can be shown that the subdifferential of any continuous convex function is a maximal monotone operator; and that any maximal monotone operator (restricted as above) is locally bounded, weak* upper semicontinuous, with weak* compact convex values [P3, pp27–32]. A modification of the proof of (4) $\Rightarrow$ (5) establishes the desired conclusion. In fact, the proof of (4) $\Rightarrow$ (5) given in [Ke] was for monotone operators, and the argument we gave above was a modification of that.

(v) (S. P. Fitzpatrick [Bo, Theorem 5.4.1]) It clearly suffices to show that if X is any separable Banach space, then its unit ball B is weakly metrizable if, and only if, X^* is separable. One direction is easy; if X^* is separable, then the unit ball of

X^{**} is weak* metrizable, so the unit ball of X is weakly metrizable. Conversely, let d be a metric for the weak topology on B. For each n, $\{x \in B : d(x, 0) < 1/n\}$ contains a basic weak neighborhood of 0, i.e. a set of the form $\{x \in B : |f(x)| < 1$ for all $f \in F_n\}$, where F_n is a finite subset of X^*. Denote by A_n the absolutely convex hull of $\bigcup_{k=1}^{n} F_k$, by B^* the unit ball of X^*, and set $A = \bigcup_{n=1}^{\infty} A_n$. For any $g \in X^*$, $\{x \in B : |g(x)| < 1\}$ is a weak neighborhood of the origin, and so must contain $B \cap \{x : |f(x)| < 1$ for all $f \in A_n\}$ for some A_n. The separation theorem shows that $g \in A_n + B^*$. Thus $X^* = A + B^*$, which implies that the separable subspace $\mathbf{R}^+ A$ is dense in X^*.

(vi) There any many results in this area concerning the extreme point structure of convex sets. The property (6) is one such example: here we give just one more. A Banach space has the *Krein-Milman Property* if every closed bounded convex set therein is the closed convex hull of its extreme points. An argument of Lindenstrauss [DU, p190], similar to that used in the proof of the Krein-Milman Theorem, but using also the Bishop-Phelps Theorem, shows that the Radon-Nikodým Property implies the Krein-Milman Property. That the converse is true in dual spaces is a somewhat deeper result due to R. E. Huff and P. D. Morris; we refer to [DU, p 196] for one exposition. It remains unknown whether the two properties are equivalent in every Banach space; Proposition 10 contains a partial answer.

It natural to ask if the duality between the Asplund and Radon-Nikodým Properties is complete. The following result of Collier [Co] answers this question. Note that we cannot simply interchange X and X^* in Theorem 6, since ℓ_1 has the Radon-Nikodým Property, but ℓ_∞ is not an Asplund space.

Proposition 10. *For any Banach space X, the following are equivalent.*
(1) X has the Radon-Nikodým Property,
(2) every continuous, weak lower semicontinuous, convex function* $\varphi : X^* \to \mathbf{R}$ *is Fréchet differentiable on a dense* G_δ *set.*
(3) every closed bounded convex subset of X is the closed convex hull of strongly exposed points.

Proof. This is similar to the proof of the corresponding parts (i.e. (4) ⇒ (5) ⇒ (6) ⇒ (4)) of the proof of Theorem 6. We refer to [G2, Theorem 3.5.8] or [Bo, Chapter 5] for details The implication (1) ⇒ (2) requires the most modification; it is necessary to use a Bishop-Phelps type theorem.

We finish this section with a non-characteristic property of Asplund spaces.

Proposition 11 [S3, Theorem 3.5]. *If X is an Asplund space, then the unit ball of* X^* *is weak* sequentially compact.*

Proof. Given a bounded sequence (f_n) in X^*, let A_n be the weak* closed convex hull of $\{f_k : k \geqq n\}$, and set $A = \bigcap_{n=1}^{\infty} A_n$. Note that for $f \in A$ and any $x \in X$, we have $f(x) \in \bigcap_{n=1}^{\infty} \overline{\text{co}}\{f_k(x) : k \geqq n\}$. By hypothesis, A contains a point f which is weak* exposed by some $x \in X$. Thus there is a subsequence (g_n) of (f_n) for which $g_n(x) \to f(x)$. If g is any weak* limit point of (g_n), then $g \in A$ and $g(x) = f(x)$. Since x exposes f, this forces $g = f$, i.e. f is the only weak* limit point of (g_n). Since the sequence (g_n) is relatively weak* compact, it must converge weak* to f.

Note that only Gâteaux differentiability was used in this argument, not Fréchet differentiability. Thus the class of spaces whose dual balls are weak* sequentially compact is much larger than the class of Asplund spaces. For further information about such spaces, see [Di, Chapter 13].

4. Examples of Asplund spaces

All the Asplund spaces with which we are familiar fall into one of the following groups. (Necessary definitions will be given in due course.)

(i) Banach spaces whose duals are separable,
(ii) reflexive spaces,
(iii) subspaces of $C(K)$, where K is a scattered compact Hausdorff space,
(iv) spaces which are weakly Hahn-Banach smooth, in particular
   (iv$\frac{1}{4}$) spaces with Fréchet smooth norms,
   (iv$\frac{1}{2}$) spaces which are M-ideals in their biduals, and
   (iv$\frac{3}{4}$) spaces whose duals have the property (**),
(v) the Long James spaces, their duals and subspaces.

We will deal with these categories separately.

(i) and (ii) It is obvious that all spaces in these categories are Asplund.

(iii) A topological space is said to be *scattered* iff every subset has an isolated point. Standard arguments show that the continuous image of one compact scattered space is another, and that a compact scattered metric space is countable. The following result first appeared in [NP, Theorem 18].

Proposition 12. *Let K be a compact Hausdorff space. Then $C(K)$ is Asplund if and only if K is scattered.*

Proof. ($\Rightarrow$) This can be proved using the separable subspace criterion, but for once it is easier to work directly from the definitions. Identify K with its carrier space in $C(K)^*$; then $\|f - g\| = 2$ for any distinct $f, g \in K$. Thus any subset of K with diameter strictly less than 2 is a singleton. Fragmentability of the unit ball in $C(K)^*$ then says that K is scattered.

($\Leftarrow$) We will establish a slightly more general result, namely that if X is a Banach space and K is a subset of X^* whose closed linear span is the whole space, such that K is compact and scattered in the weak* topology, then X is Asplund. Let

Y be any separable subspace of X. Then Y^* is a quotient of X^*, so is the closed linear span of some weak* compact scattered subset L. Since Y is separable, L must be metrizable, hence countable; thus Y^* is separable.

(iv) A Banach space was defined in [SS] to be *weakly Hahn-Banach smooth* if every norm-attaining functional in its dual has a unique norm-preserving extension to the bidual. In [SS, Theorem 15] it is shown that in such a space, every separable subspace has separable dual. Thus all weakly Hahn-Banach smooth spaces are Asplund; the first explicit statement of this is in [GGS], where the result is generalized further.

Indeed, it is not hard to show that $f \in X^*$ has a unique norm-preserving extension to X^{**} iff every net in $B(0, \|f\|)$ which converges weak* to f is already veakly convergent. Functionals with this property are called *Namioka points* in [Go], where much more information about them can be found. A closed subspace M of X^* is said to be *norming* iff $\|x\| = \sup\{f(x) : f \in M, \|f\| \leqq 1\}$ for all $x \in X$; a standard argument shows that this is equivalent to requiring the unit ball of M to be weak* dense in the unit ball of X^*. Thus every Namioka point belongs to every norming subspace. For every Banach space X, there is a norming subspace M of X^* with dens M = dens X. The Bishop-Phelps Theorem shows that the dual of a weakly Hahn-Banach smooth space has no proper norming subspaces, whence dens$X$ = densX^*. It is easily checked that weak Hahn-Banach smoothness is a property which passes to subspaces; this proves that all weakly Hahn-Banach smooth spaces are Asplund.

More generally, let $\mathscr{P}$ be a property of Banach spaces, which passes to subspaces, and which implies that X^* has no proper norming subspaces. Then every Banach space with $\mathscr{P}$ is Asplund, and every dual space with $\mathscr{P}$ is reflexive. For example, $\mathscr{P}$ could be one of the properties „X has a Fréchet smooth norm", „X^* is locally uniformly convex", „X^{**} has a Gâteaux smooth norm", „X^{***} is strictly convex", or „X is an M-ideal in its bidual". Further examples of such properties appear in **[FP]** and **[HL]**. There is a conjecture that if $\mathscr{P}$ is any property which passes to subspaces, and every dual space with $\mathscr{P}$ is reflexive, then every space with $\mathscr{P}$ is Asplund. We know of no counterexample to this. Some more results of this nature can be found in **[Go]**.

Suppose that the norm on X is Fréchet smooth (except, of course, at the origin). Let $f \in X^*$ be a norm one functional which attains its norm at $x \in X$. Then, by the remarks at the end of § 1, the norm on X^{**} is also smooth at x. Thus there is a unique $F \in X^{***}$ with $\|F\| = F(x) = 1$; we must have $F = f$. This shows that f has only one norm preserving extension to X^{**}, so X is Asplund in this case also. The converse of this is false, even allowig for renorming **[Ha]**. More precisely, there is an Asplund space for which every equivalent norm fails to be even Gâteaux differentiable at some non-zero point.

A Banach space X is said to be an M-ideal in its bidual if the natural decomposition $X^{***} = X^* \oplus X^0$ is an ℓ_1-sum. Obviously such a space is weakly Hahn-

-Banach smooth. Typical examples are $c_0(\Gamma)$ for any set Γ, the space of compact operators on any Hilbert space, and $C(\mathbf{T})/A$, where $\mathbf{T}$ is the unit circle and A is the disc algebra. An impressive collection of examples can be found in [**W**]. These spaces have many remarkable properties; we refer to [**HWW**, Chapter 3] for details.

Finally we recall that a dual space X^* has property (**) [**NP**] if the norm and weak* topologies coincide on the unit sphere. Obviously this implies that the weak and weak* topologies coincide on the unit sphere, so X will be weakly Hahn--Banach smooth in this case.

(v) The Long James spaces $J(\eta)$ are defined in a similar manner to the original quasireflexive space of James, but the index set may be an uncountable ordinal. They were first studied by G. A. Edgar, with a more detailed account appearing in [**Bo**, pp 346–364]. It is known that the duals of all orders of these spaces are nonreflexive Asplund spaces, so they cannot be isomorphic to any $C(K)$ space, or to any space which is an M-ideal in its bidual. Under any equivalent norm, $J(\eta)^*$ is not strictly convex, and its unit sphere contains a subset which is weak* homeomorphic to the ordinal interval $[0, \eta]$. Thus $J(\eta)^*$ does not have the property (**). Whether $J(\eta)$ has an equivalent Fréchet smooth norm sems to be unknown. It is known that $J(\eta)^*$ is not weakly Lindelöf [**Bo**].

We mention the property „X^* is Lindelöf in the weak topology“ because it is also suficient for a space to be Asplund [**Ed**, Proposition 1.18]. However, we know of no concrete example of a Banach space with this property, which does not fall into one of the groups above. This class of Banach spaces includes all spaces whose duals are weakly compactly generated, or more generally spaces whose duals are subspaces of weakly compactly generated spaces and all spaces X for which X^{**}/X is separable. In each of these cases, the separable subspace crierion is easy to apply.

Acknowledgements. Most of this work was done at the University of Granada, while the author held a *Europa Stipendium* from the Alexander von Humbold Foundation. He is indebted to the AvH Foundation for its support, and to the members of the Department of Mathematical Analysis in Granada, for their hospitality.

References

[As] Asplund E., Fréchet differentiability of convex functions, Acta Math. 121 (1968), 31–47.

[Bo] Bourgin R. D., Geometric aspects of convex sets with the Radon-Nikodým property, Lecture Notes in Math. 993, Springer, Berlin (1983).

[Co] Collier J. B., The dual of a space with the Radon-Nikodým Property, Pacific J. Math. 64 (1976), 103–106.

[Di] Diestel J., Sequences and series in Banach spaces, Graduate Texts in Math. 92, Springer, New York, 1983.

[DU] DIESTEL J. and UHL J. J., Vector measures, Math. Surveys 15, Amer. Math. Soc., Providence, 1977.

[DN] VAN DULST D. and NAMIOKA I., A note on trees in conjugate Banach spaces, Indag. Math. 46 (1984), 7−10.

[Ed] EGAR G. A., Measurability in a Banach space, Indiana Univ. Math. J. 26 (1977), 663−677.

[FG] FABIAN M. and GODEFROY G., The dual of every Asplund space admits a projectional resolution of the identity, Studia Math. 91 (1988), 141−151.

[FP] FRANCHETTI C. and PAYÁ R., Banach spaces with strongly subdifferentiable norm, Boll. Un. Mat. Ital. B (7) 7 (1993), 45−70.

[Ga] GÂTEAUX R., Fonctions d'une infinité de variable indépendantes, Bull. Soc. Math. France 47 (1919), 70−96.

[G1] GILES J. R., On the characterisation of Asplund spaces, J. Austral Math. Soc. (Ser. A) 32 (1982), 134−144.

[G2] GILES J. R., Convex analysis with application it the differentiation of convex functions, Research Notes in Math. 58, Pitman, London, 1982.

[GGS] GILES J. R., GREGORY D. A. and SIMS B., Geometrical implications of upper semicontinuity of the duality mapping on a Banach space, Pacific J. Math. 79 (1978), 99−109

[GU] GIRARDI M. and UHL J. J., Slices, the Radon-Nikodým Property, strong regularity and martingales, Bull. Austral. Math. Soc. 41 (1990), 411−415.

[GO] GODEFROY G., Points de Namioka. Espaces, normants. Applications à la théorie isométrique de la dualité, Israel J. Math. 38 (1981), 209−220.

[HWW] HARMAND P., WERNER D. and WERNER W., M-ideals in Banach spaces and Banach algebras, Lecture Notes in Math. 1547, Springer, Berlin, 1993.

[Ha] HAYDON R., A counterexample to several questions about compact scattered spaces, Bull. London Math. Soc. 22 (1990), 261−268.

[HL] HU Z. and LIN B.-L., Smoothness and the asymptotic norming properties of Banach spaces, Bull. Austral. Math. Soc. 45 (1992), 285−296.

[Ja] JAMESON G. J. O., Topology and normed spaces, Chapman and Hall, London, 1974.

[JR] JAYNE J. E. and ROGERS C. A., Borel selectors for upper semicontinuous set-valued maps, Acta Math. 155 (1985), 41−79.

[JZ] JOHN K. and ZIZLER V., On rough norms on Banach spaces, Comment. Math. Univ. Carolinae 19 (1978), 335−349.

[Ke] KENDEROV P. S., Monotone operators in Asplund spaces, C. R. Acad. Bulgar. Sci. 30 (1977), 963−964.

[LW] LEACH E. B. and WHITFIELD J. H. M., Differentiable functions and rough norms on Banach spaces, Proc. Amer. Math. Soc. 33 (1972), 120−126.

[NA] NAMIOKA I. and ASPLUND E., A geometric proof of Ryll-Nardzewski's fixed point theorem, Bull. Amer. Math. Soc. 73 (1967), 443−445.

[NP] NAMIOKA I. and PHELPS R. R., Banach spaces which are Asplund spaces, Duke Math. J. 42 (1975), 735−750.

[PS] PASCALI D. and SBURLAN S., Nonlinear mappings of monotone type, Editura Academiei, Bucharest and Sijthoff Noordhoff, Alphen an den Rijn, 1978,

[P1] PHELPS R. R., Differentiabilty of convex functions on Banach spaces, unpublished lecture notes, University College London, 1977.

[P2] PHELPS R. R, Dentability and extreme points in Banach spaces, J. Funct. Anal. 17 (1974), 78−90.

[P3] PHELPS R. R., Convex functions, monotone operators and differentiability, Lecture Notes in Math. 1364, Springer, Berlin, 1989.

[PZ] PREISS D. and ZAJÍČEK L., Stronger estimates of smallness of sets of Fréchet nondifferentiability of convex functions. Proc. 11th Winter School Abstract Analysis, Bohemia, January 1984, Supp. Rend. Circ. Mat. Palermo 3 (1984), 219−223.

[SY] SIMS B. and YOST D., Linear Hahn-Banach extension operators, Proc. Edinb. Math. Soc. 32 (1989), 53–57.

[SS] SMITH M. A. and SULLIVAN F., Extremely smooth Banach spaces, Proc. Pełczynski conference, Kent, Ohio, July 1976, Springer Lecture Notes in Math. 604 (1977), 125–137.

[Sm] ŠMULIAN V. L., Sur la dérivabilité de la norme dans l'espace de Banach, C. R. (Doklady) Acad. Sci. URSS 27 (1940), 643–648.

[S1] STEGALL, C., The Radon-Nikodým Property in conjugate Banach spaces, Trans. Amer. Math. Soc. 206 (1975), 213–223.

[S2] STEGALL C., The duality between Asplund spaces and spaces with the Radon-Nikodým property, Israel J. Math. 29 (1978), 408–412.

[S3] STEGALL C., The Radon-Nikodým Property in conjugate Banach spaces II, Trans. Amer. Math. Soc. 264 (1984), 507–519.

[Su] SULLIVAN F., On the duality between Asplund spaces and spaces with the Radon-Nikodým Property, Proc. Amer. Math. Soc. 71 (1978), 155–156.

[We] WERNER D., New classes of Banach spaces which are M-ideals in their biduals, Math. Proc. Camb. Phil. Soc. 111 (1992), 337–354.

[YS] YOST D. and SIMS B., Banach spaces with many projections, Miniconference on Operator Theory and Partial Differential Equations, Sydney, September 1986, Proc. Centre Mant. Anal. Austral. Nat. Univ. 14 (1986), 335–342.