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A Note to Decompositions of Real Line

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Pro žádnou spojitou bijekci h reálné přímky na sebe neexistuje homogenní podgrupa $A \subset R$, která by tvořila spolu s $h(A)$ rozklad R na dvě disjunktní části.

For an arbitrary continuous bijection h of the real line onto itself there exists no homogeneous subgroup A of R such that $A \cup h(A) = R$ is a decomposition into two disjoint parts.

In [2], J. Menu proved that the real line R can be decomposed in two homogeneous homeomorphic subsets. In [3], J. van Mill showed that there exists a homeomorphism $h: R \to R$ and a subset A of R such that $R = A \cup h(A)$ is a decomposition of R into two disjoint homogeneous parts. Moreover, he proved that such a decomposition is not topologically unique and that h can be chosen as a shift of real line. In [4], J. van Mill found an example of a decomposition $R = A \cup (R \setminus A)$ such that $R \setminus A$ homeomorphic to A, A is homogeneous and it does not admit the structure of a topological group. There were also other types of decompositions of R into two homogeneous homeomorphic parts constructed – e.g. a decompositon of R into two homeomorphic rigid parts in [1].

These results led to the question whether there exist a homeomorphism $h: R \to R$ and a decomposition of $R = A \cup h(A)$ such that $A \cap h(A) = \emptyset$ and A is a homogeneous subset of R satisfying $A = A + A$, $A = -A$ (i.e. A is a subgroup of R). The goal of this note is to shgow that one cannot find not even a continuous bijection with this property.

Theorem. *If $R = A \cup B$ with $A \cap B = \emptyset$, A, B homogeneous, A a subgroup of R (with the usual additive structure), and $h: R \to R$ is a bijection satisfying $h(A) = B$, then h is not continuous.*

We shall prove this theorem by three lemmas:

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Lemma A. *Let* $R = A \cup B$ $(A \cap B = \emptyset)$ *be a decomposition of* R *into two homogeneous parts such that* A *is a subgroup of* R. *Suppose that* $h: R \to R$ *is a bijection satisfying* $h(A) = B$. *Then* A, B *are dense in* R.

Proof. Since A is a subgroup of R, there is $0 \in A$. Suppose that $\inf\{x \in A; x > 0\} > 0$. Then A is a discrete set, hence countable which contradicts the assumption $A \cup h(A) = R$. Thus, there exists a sequence $\{a_n \in A \setminus \{0\}; n = 1, 2, \dots\}$ such that $\lim a_n = 0$. For any $k = 1, 2, \dots$ there exists $n(k)$ such that $|a_{n(k)}| < k^{-1}$. If $b \in B$ and k is a positive integer then there exist an integer p such that $b - k^{-1} < pa_{n(k)} < b + k^{-1}$. Hence, A is dense in R.

If $b_0 \in B$ then $b_n = b_0/n \in B$ for $n = 1, 2, \dots$ For any $a \in A$ there is $a + b_n \in B$, $a = \lim(a + b_n)$. Hence, B is dense in R.

Lemma B. *Let the assumptions of Lemma A be satisfied. If* $a \leqq b$, $h(x) - x = h(a) - a$ *for any* $x \in \langle a, b \rangle$, *then* $a = b$. *(In other words,* h *is not a shift on any non-degenerated interval.)*

Proof. Suppose that $a < b$. Let there exist $d \in R$ such that $h(x) = x + d$ for any $x \in \langle a, b \rangle$. According to Lemma A, there exists $c \in (a, b) \cap A$, $h(c) = c + d \in B$. Hence, $d \in B$. Moreover, there exists a positive integer n such that $a < c - d/(2n) < c + d/(2n) < b$. Clearly, $c - d/(2n) \in B$, $c + d/(2n) \in B$, $h(c - d/(2n)) = c + d(2n - 1)/(2n) \in A$, $h(c + d/(2n)) = c + d(2n + 1)/(2n) \in A$. Therefore, $d/n \in A$ and $d \in A$ which is a contradiction.

Lemma C. *Let the assumptions of Lemma A be satisfied,* $a < b$. *Then* h *is not continuous on* $\langle a, b \rangle$.

Proof. According to Lemma B, there exist $u, v \in \langle a, b \rangle$ such that $u < v$, $h(u) - u \neq h(v) - v$. Without loss of generality, assume that $h(u) - u > h(v) - v$. By Lemma A, there is $p \in A$ such that $h(u) > u + p$, $h(v) < v + p$. Denote $s = \sup\{x \in \langle u, v \rangle; h(x) > x + p\}$. If h is continuous at s, then $h(s) = s + p$. Since $s \in A \Leftrightarrow h(s) \in B$, there is $p \in B$ which is a contradiction.

Lemma C finishes the proof of theorem.

Of course, there remains a following

Open problem (J. van Mill). Does there exist a homogeneous subgroup of R which is homeomorphic to its complement?

According to the result presented above, in the case of a positive solution the homeomorphism $f: A \to R \setminus A$ cannnot be extended to a continous function on any interval.

References

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