P. Kůrka

One dimensional dynamics and factors of finite automata

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 34 (1993), No. 2, 83--95

Persistent URL: http://dml.cz/dmlcz/701998

One Dimensional Dynamics and Factors of Finite Automata

P. KŮRKA

Prague*)

Received 14 April 1993

We argue that simple dynamical systems are factors of finite automata, regarded as dynamical systems on discontinuum. We show that any homeomorphism of the real interval is of this class. An orientation preserving homeomorphism of the circle is a factor of a finite automaton iff its rotation number is rational. Any S-unimodal system on the real interval, whose kneading sequence is either periodic odd or preperiodic, is also a factor of a finite automaton, while S-unimodal systems at limits of period doubling bifurcations are not.

1 Introduction

The simplest dynamical systems are characterized by the presence of single fixed points, which attract all other points, the typical example being the halving of the interval $H(x) = x/2$. Chaotic systems, on the other hand, keep visiting every region of the state space in apparently chaotic manner, the simplest example being the doubling of the circle $D(x) = 2x \bmod 1$. Nevertheless, the latter system does not seem to be more complex, and can be regarded as a dual, of the former. Indeed if $x = 0 . x_0x_1x_2 \ldots$ is a binary expansion of $x \in [0, 1]$, then $H(x) = 0 . 0x_0x_1 \ldots$, while $D(x) = 0 . x_1x_2 \ldots$. It turns out that both systems are factors of finite automata regarded as dynamical systems on discontinuum. The halving of the interval writes zero to the output tape, while the doubling of the circle reads a digit from the input tape.

These considerations lead to a complexity classification of dynamical systems. According to a generalized Alexandrov Theorem, every dynamical system on a compact metric space is a factor of some dynamical system on discontinuum (see Balcar and Simon [1] for a proof). Every type of automata studied in computer science can be regarded as a class of dynamical systems on discontinuum. The complexity hierarchy ranging from finite automata, stack automata, Turing

*) Faculty of Mathematics and Physics, Charles University, Malostranské náměstí 25, 11800 Praha 1, Czech Republic

automata, and cellular automata to neural networks can be transferred to dynamical systems on compact metric spaces via factorization. The factors of finite automata are at the bottom of this hierarchy. We have shown in [7] that they include systems with finite attractors and hyperbolic systems.

In the present paper we extend and generalize these results. We use a more general concept of multitape finite automata with variable advance of their tapes. We show that an orientation preserving homeomorphism of the circle is a factor of a finite automaton iff its rotation number is rational. We show that any S-unimodal system on real interval with preperiodic or odd periodic kneading sequence is a factor of a finite automaton. To obtain these results we modify the kneading theory using the two closed intervals on either side of the critical point instead of the open ones. There are multiple itineraries in some cases, and we choose as a standard the least one. This modification substantially simplifies the theory, and we obtain immediately a homomorphism from itineraries to points. Our approach is dual to that of Cruthfield and Young [4], who investigate the regularity of languages generated by quadratic systems.

Finally we prove that the S-unimodal systems with aperiodic kneading sequences, which occur at common limits of period doubling and band merging bifurcations, are not factors of finite automata. Thus we leave open the case of even periodic and arbitrary aperiodic kneading sequences. The negative results are obtained with the use of a topological property of having chaotic limits. This is another simplicity criterion for dynamic systems. The standard Devaney's [5] definition reads that a chaotic system is one, which is topologically transitive, whose periodic points are dense, and which is sensitive to initial conditions. However, Brooks et al. [2] have recently proved, that on infinite metric spaces, the first two conditions imply the third. In light of this result it is natural to relax the definition of chaotic system to topological transitivity and density of periodic points. According to this relaxed definition, finite dynamical systems, which are orbits of periodic points, are chaotic too. These considerations provide some support for the feeling that chaotic systems are simple too. We define a system with chaotic limits as one, whose each point is included in a set, whose ω-limit set is chaotic. We show that systems which do not have chaotic limits are not factors of finite automata.

2 System with chaotic limits

Definition 1. *A dynamical system (X, f) is a continuous map $f\colon X \to X$ of a compact metric space X to itself. A homomorphism $\varphi\colon (X, f) \to (Y, g)$ of dynamical systems is a continuous mapping $\varphi\colon X \to Y$ such that $g\varphi = \varphi f$. We say that (Y, g) is a factor of (X, f), if φ is a factorization (i.e. a surjective map).*

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \varphi \downarrow & & \downarrow \varphi \\ Y & \xrightarrow{g} & Y \end{array}$$

The n-th iteration of f defined by $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$. A point $x \in X$ is periodic with period $n > 0$, if $f^n(x) = x$ and $f^i(x) \neq x$ for $0 < i < n$. A point x is eventually periodic, if there exists $i \geqq 0$, such that $f^i(x)$ is periodic. It is preperiodic, if it is eventually periodic but not periodic. It is aperiodic, if it is not eventually periodic.

A subset $A \subseteq X$ is invariant, if $f(A) \subseteq A$. The ω-limit set of a set $A \subseteq X$ is $\omega(A) = \bigcap_n \overline{\bigcup_{m>n} f^m(A)}$. It is easy to see that if A is nonempty, then $\omega(A)$ is a nonempty, closed, invariant set, so $(\omega(A), f)$ is a dynamical system. If $\omega\colon (X, f) \to (Y, g)$ is a homomorphism, then $\varphi(\omega_f(A)) = \omega_g(\varphi(A))$.

Definition 2. *A dynamical system (X, f) is chaotic, if the set of its periodic points is dense in X and if for any open nonempty sets $U, V \subseteq X$ there exists $k > 0$ with $f^k(U) \cap V \neq 0$ (topological transitivity). The system (X, f) has chaotic limits, if for each $x \in X$ there exists $A \subseteq X$, such that $x \in A$ and $\omega(A)$ is chaotic.*

In particular a finite system is chaotic iff it is an orbit of a periodic point, and any finite system has chaotic limits. If $\omega(x)$ is a periodic orbit for each $x \in X$, then (X, f) is a system with chaotic limits. Moreover if X or $\omega(X)$ is chaotic, then (X, f) has chaotic limits. It is easy to see that a factor of a chaotic dynamical system is chaotic, and a factor of a system with chaotic limits has chaotic limits.

3 Finite automata

Let A be a finite alphabet and $N = \{0, 1, \ldots\}$ the set on non-negative integers. We frequently use alphabet $2 = \{0, 1\}$. Denote $A^N = \{u = u_0u_1\ldots \mid u_i \in A\}$ the set of infinite sequences of letters of A, A^* the set of finite sequences, and $\overline{A^*} = A^* \cup A^N$. Denote $|u|$ the length of a sequence $u \in \overline{A^*}$, $(0 \leqq |u| \leqq \infty)$, and $\wedge$ the word of zero length so, $u_i = \wedge$ for $i > |u|$. Denote $u_{|i} = u_0 \ldots u_{i-1}$ the initial substring of u of length i, and write $u \sqsubseteq v$, if u is an initial substring of v. If $u \in A^*$, denote $\bar{u} \in A^N$ the infinite repetition of u, defined by $\bar{u}_{k|u|+i} = u_i$. Define a metric ϱ_A on $\overline{A^*}$ by $\varrho_A(u, v) = 2^{-n}$, where $n = \inf\{i \geqq 0 \mid u_i \neq v_i\}$.

We define the shift $\sigma\colon \overline{A^*} \to \overline{A^*}$ by $\sigma(u)_i = u_{i+1}$, thus $\sigma(\wedge) = \wedge$. We use also inverse shifts. To unify the formalism, suppose that $\bullet$, $\#$ are symbols not occurring in A, and define $\sigma_a\colon A^N \to A^N$ for $a \in A \cup \{\bullet, \#\}$ by

$$\sigma_\#(u_0u_1u_2\ldots) = u_1u_2u_3\ldots$$
$$\sigma_\bullet(u_0u_1u_2\ldots) = u_0u_1u_2\ldots$$
$$\sigma_a(u_0u_1u_2\ldots) = au_0u_1\ldots \quad \text{for } a \in A$$

An automaton (finite, stack or Turing) is a finite control attached to one or more tapes with some rules of access to them. For formal reasons it is simpler to conceive an automaton as a finite system of infinite tapes, which interact at their ends. In general case these tapes are stacks, so a letter can be either read or written to them. Two stacks already can be used to simulate any Turing machine. Finite automata possess only input or output tapes, which move in one direction only.

Definition 3. *Let T be a finite set (of tapes) and for each $t \in T$ let A_t be a finite alphabet. Denote $Q = \prod A_t$, $X = \prod A_t^N$ the state space with a product metric γ, and $\pi\colon X \to Q$ the projection defined by $\pi(u)_t = u_{t0}$. An automaton with tapes T, alphabets A_t and transition functions $f_t\colon Q \to A_t \cup \{\bullet, \#\}$ is a dynamical system (X, f) defined by*

$$f(u)_t = \sigma_{f_t\pi(u)}(u_t)\,, \qquad u \in X,\ t \in T$$

We say that $t \in T$ is input tape, if $(\forall q \in Q)\,(f_t(q) \in \{\bullet, \#\})$. We say that $t \in T$ is an output tape, if $(\forall q \in Q)\,(f_t(q) \in A_t \cup \{\bullet\})$. An automaton is a finite automaton, if every tape is either input tape or output tape.

Definition 4. *For an automaton (X, f) over T, $u \in X$, $n \geqq 0$ define the advance $d_t(u, n)$ of t in n steps by $d_t(u, 0) = 0$, $d_t(u, 1) = 1$ if $f_t\pi(u) = \#$, $d_t(u, 1) = 0$ if $f_t\pi(u) = \bullet$, $d_t(u, 1) = -1$ if $f_t\pi(u) \in A_t$, and $d_t(u, n+1) = d_t(u, 1) + d_t(f(u), n)$ for $n > 0$. Clearly $|d_t(u, n)| \leqq n$.*

Theorem 1. *Any finite automoton has chaotic limits.*

Proof. Let (X, f) be a finite automaton and $u \in X$. Denote $Q_0 = \{q \in Q \mid (\forall m)\,(\exists n \geqq m)\,(\pi f^n(w) = q)\}$, $T_{in} = \{t \in T \mid (\exists q \in Q_0)\,(f_t(q) = \#)\}$, $T_{out} = \{t \in T \mid (\exists q \in Q_0)\,(f_t(q) \in A_t)\}$, $T_\bullet = T - T_{in} - T_{out}$. Since (X, f) is a finite automaton, $T_{in} \cap T_{out} = 0$. Let j be the first integer, such that $(\forall n \geqq j)\,(\pi f^n(w) \in Q_0)$ and denote

$$Y = \{u \in X \mid (\forall i)\,(\pi f^i(u) \in Q_0)\ \&\ (\forall t \in T_\bullet)\,(\forall i)\,(u_{ti} = f^j(w)_{ti})\}\,.$$

Then Y is a closed invariant set. Let p be the least integer, for which there exists $z \in Y$ with $\{\pi f^i(z) \mid i < p\} = Q_0$ and $\pi f^p(z) = \pi(z)$. Denote

$$W = \{u \in Y \mid (\forall q \in Q_0)\,(\forall k > 0)\,(|\{i < kp \mid \pi f^i(u) = q\}| \geqq k/2)\}$$

(here $|A|$ means the number of elements of A). We shall prove in the following lemmas that $\omega(W \cup \{w\})$ is chaotic.

Lemma 1. *Let $u \in Y$ and let for all k there exist n, m, and $v \in Y$, such that $\varrho(f^n(v), u) \leqq 2^{-k}$, $(\forall t \in T_{out})\,(d_t(v, n) \leqq -k)$, and $(\forall t \in T_{in})\,(d_t(f^n(v), m) \geqq k)$. Then $u \in \omega(W)$.*

Proof. Let $z \in Y$ be such that $\{\pi f^i(z) \mid i < p\} = Q_0$, $\pi(z) = \pi(v)$, and $(\forall i)(\pi^{p+i}(z) = \pi f^i(z))$. Let $r > n + m + p$ be an integer divisible by p. There exists $s \in Y$ satisfying $\pi(s) = \pi(v)$, and for all $t \in T_{in}$

$$\begin{aligned} s_{t,i} &= z_{t,i} && \text{for } 1 \leq i \leq a = d_t(z, r) \\ s_{t,a+1} &= v_{t,i} && \text{for } 1 \leq i \leq b = d_t(v, n+m) \\ s_{t,a+b+i} &= z_{t,\ell+i} && \text{for } 1 \leq i, \text{ where } \pi f^{n+m}(v) = \pi f^{\ell}(z) \end{aligned}$$

Then $s \in W$, $f^r(s)_{ti} = v_{ti}$ for all $i \leq d_t(v, n+m)$, $t \in T_{in}$, and $\varrho(f^{r+n}(s), f^n(v)) \leq 2^{-k}$, so $\varrho(f^{r+n}(s), u) \leq 2^{-k}$, and therefore $u \in \omega(W)$.

Lemma 2. $\omega(W \cup \{w\}) = \omega(W)$.

Proof. It suffices to show $\omega(w) \subseteq \omega(W)$. Let $u \in \omega(w)$ and $k > 0$. We have $f^j(w) \in Y$ and there exist n, m, such that $d_t(f^j(w), n) \leq -k$ for all $t \in T_{out}$, $\varrho(f^{j+n}(w), u) \leq 2^{-k}$, and $d_t(f^{j+n}(w), m) \geq k$. By Lemma 1, $u \in \omega(W)$.

Lemma 3. *Periodic points are dense in* $\omega(W)$.

Proof. Let $U \subseteq X$ be an open set, and $U \cap \omega(W) \neq 0$. There exists $u \in U$ and k such that $u \in C_k(u) \cap \omega(W) \subseteq U \cap \omega(W)$, where $C_k(u) = \{v \in X \mid u_{|k} = v_{|k}\}$ is a cylinder around u. There exist m and $v \in W$ such that $\varrho(f^m(v), u) \leq 2^{-k}$, and $|d_t(v, m)| \geq k$ for all $t \in T_{in} \cup T_{out}$. There exists $r > m + k$ and $s \in Y$ such that $\pi(s) = \pi(v)$, and

$$\begin{aligned} & s_{t,i} = v_{t,i} && \text{for } t \in T_{in},\ 1 \leq i \leq d_t(v, m) + k \\ & \{\pi f^i(s) \mid i < r\} = Q_0 \\ & \pi f^r(s) = \pi(v), \\ & s_{t,a+i} = s_{t,i} && \text{for } t \in T_{in},\ 1 \leq i,\ a = d_t(s, r) \\ & s_{t,i} = f^r(s)_{t,i} && \text{for } t \in T_{out} \end{aligned}$$

Then $s \in Y$, $f^r(s) = s$, $|d_t(s, r)| \geq k$ for $t \in T_{in} \cup T_{out}$, and $\varrho(f^m(s), u) \leq 2^{-k}$. By Lemma 1, $s \in \omega(W)$.

Lemma 4. $\omega(W)$ *is topologically transitive.*

Proof. Let $U, U' \subseteq X$ be open sets, $u \in C_k(u) \cap \omega(W) \subseteq U \cap \omega(W)$, $u' \in C_k(u') \cap \omega(W) \subseteq U' \cap \omega(W)$. There exist m, m' and $v, v' \in W$ with $\varrho(f^m(v), u) \leq 2^{-k}$, $\varrho(f^{m'}(v'), u') \leq 2^{-k}$, $|d_t(v, m)| \geq k$, $|d_t(v', m')| \geq k$ for all $t \in T_{in} \cup T_{out}$. There exist $r > m + k$, $r' > r + m' + k$ and $s \in Y$ such that

$$\begin{aligned} & s_{t,i} = v_{t,i} && \text{for } t \in T_{in},\ 1 \leq i \leq d_t(v, m) + k \\ & \pi f^r(s) = \pi(v'), \\ & s_{t,a+i} = v'_{t,i} && \text{for } t \in T_{in},\ 1 \leq i \leq d_t(v', m') + k,\ a = d_t(s, r) \\ & \{\pi f^i(s) \mid i < r'\} = Q_0 \\ & \pi f^{r'}(s) = \pi(v), \end{aligned}$$

$$s_{t,a+i} = s_{t,i} \qquad \text{for} \quad t \in T_{in},\ 1 \leqq i,\ b = d_t(s, r')$$
$$s_{t,i} = f^{r'}(s)_{t,i} \qquad \text{for} \quad t \in T_{out}$$

Then $f^m(s) \in U$, $f^{r+m'}(s) \in V$, and by Lemma 1, $s \in \omega(W)$.

4 Homeomorphisms

Lemma 5. *Let $g: I \to I$ be an increasing continuous function on a compact real interval I, whose only fixed point is one of the endpoints of I. Then (I, g) is a factor of $(2^N, \sigma_0)$.*

Proof. Let $I = [a, b]$, $g(a) = a$, $g(b) < b$ (the dual case $g(b) = b$ is similar). For $k \geqq 0$ denote $I_k = [g^{k+1}(b), g^k(b)]$, $X_k = \{u \in 2^N \mid u_{|k+1} = 0^k 1\}$. For $u \in X_0$ define $\varphi(u) = g(b) + (b - g(b)) \sum_{i=1}^{\infty} u_i 2^{-i}$. For $u \in X_k$, $k > 0$ define $\varphi(u) = g^k \varphi \sigma^k(u)$, $\varphi(\bar{0}) = a$. Then $\varphi: (2^N, \sigma_0) \to (I, g)$ is a factorization.

Lemma 6. *Let (X, f) be a two tape finite automaton defined by $f(u, v) = (\sigma_{\#}(u), \sigma_{\max(u_0, v_0)}(v))$. Let $g: I \to I$ be an increasing continuous function on a compact real interval I, whose only fixed points are the two endpoints of I. Then (I, g) is a factor of (X, f).*

Proof. Let $I = [a, b]$ and pick some $c \in (a, b)$. Suppose $f(c) > c$ (the dual case $f(c) < c$ is similar). For $k \in Z$ denote $I_k = [g^k(c), g^{k+1}(c)]$. There exist closed sets

$$\begin{aligned}
X_{-\infty} &= \{(u, v) \in X \mid v_0 = 0 \ \&\ (\forall i \geqq 0)(u_i = 0)\} \\
X_k &= \{(u, v) \in X \mid v_0 = 0 \ \&\ \min\{i \geqq 0 \mid u_i = 1\} = -k\}, \quad k \leqq 0 \\
X_k &= \{(u, v) \in X \mid \min\{i \geqq 0 \mid v_i = 0\} = k\}, \quad k > 0 \\
X_{\infty} &= \{(u, v) \in X \mid (\forall i \geqq 0)(v_i = 1)\}
\end{aligned}$$

For $k \in Z$ define a projection $\pi_k: X_k \to 2^N$ by $\pi_k(u, v) = \sigma_{\#}^k(v)$ if $k \geqq 0$, and $\pi_k(u, v) = \sigma_0^{-k}(v)$ if $k \leqq 0$. Then $\pi_{k+1} f(u, v) = \pi_k(u, v)$ for $(u, v) \in X_k$. Let $\psi: 2^N \to I_0$ be a factorization. Define $\varphi(u, v) = g^k \psi \pi_k(u, v)$ for $(u, v) \in X_k$, $\varphi(u, v) = a$ for $(u, v) \in X_{-\infty}$, and $\varphi(u, v) = b$ for $(u, v) \in X_{\infty}$. Then $\varphi: (X, f) \to (I, g)$ is a factorization.

Proposition 1. *Let $g: I \to I$ be a non-decreasing continuous map on a compact real interval I. Then (I, g) is a factor of a finite automaton.*

Proof. Define a finite automaton (X, h), where $X = 2^N \times 2^N \times 2^N \times 2^N$, and $h(y, u, v, w) = (\sigma_0(y), f(u, v), w)$, where f is the finite automaton from Lemma 6. We shall prove that (I, g) is a factor of (X, h). Denote $F = \{x \in I \mid g(x) = x\}$ the set of fixed points. F is closed, therefore $I - F = \bigcup_{k \in K} A_k$ is a finite or countable union of open intervals $A_k = (a_k, b_k)$. Define $\vartheta\colon 2^N \to I$ by $\vartheta(u) = a + (b - a) \sum_{i=0}^{\infty} u_i 2^{-i+1}$, where $I = [a, b]$. Each $\vartheta^{-1}(A_k)$ contains a cylinder $C_k \subsetneqq 2^N$. Let $<$ be a lexicographic ordering on 2^N. Denote $K_0 = \{k \in K \mid g(a_k) = a_k, g(b_k) = b_k\}$. If $k \notin K_0$, let $\psi_k\colon (2^N, \sigma_0) \to (\overline{A_k}, g)$ be a factorization from Lemma 5. If $k \in K_0$, let $\psi_k\colon (2^N \times 2^N, f) \to (\overline{A_k}, g)$ be a factorization from Lemma 6. Define $\varphi\colon (X, h) \to (I, g)$ by

$$
\begin{aligned}
\varphi(y, u, v, w) &= \vartheta(w) && \text{if } \vartheta(w) \in F \\
\varphi(y, u, v, w) &= a_k && \text{if } \vartheta(w) \in A_k, \text{ and } (\forall x \in C_k)(w < x) \\
\varphi(y, u, v, w) &= b_k && \text{if } \vartheta(w) \in A_k, \text{ and } (\forall x \in C_k)(x < w) \\
\varphi(y, u, v, w) &= \psi_k(y) && \text{if } w \in C_k, \quad \text{and } k \notin K_0 \\
\varphi(y, u, v, w) &= \psi_k(u, v) && \text{if } w \in C_k, \quad \text{and } k \in K_0
\end{aligned}
$$

Theorem 2. *Any homeomorphism of a real interval into itself is a factor of a finite automaton.*

Proof. If $g\colon I \to I$ is increasing, then it is a factor of finite automaton by Proposition 1. If it is decreasing, then $g^2\colon I \to I$ is increasing and therefore there is a factorization $\psi\colon (X, f) \to (I, g^2)$, and (X, f) is a finite automaton. We add to this automaton an output tape with alphabet 2 and define $h(u, 0v) = (u, 10v)$, $h(u, 1v) = (f(u), 01v)$ $(u \in X, v \in 2^*)$. Then $h\colon X \times 2 \to X \times 2$ is a finite automaton. Define $\varphi(u, 0v) = \psi(u)$, and $\varphi(u, 1v) = g\psi(u)$. Then $\varphi\colon (X \times 2, h) \to (I, g)$ is a factorization. Indeed $\varphi h(u, 0v) = \varphi(u, 10v) = g\psi(u) = g\varphi(u, 0v)$, and $\varphi h(u, 1v) = \varphi(f(u), 01v) = \psi f(u) = g^2\psi(u) = g\varphi(u, 1v)$.

We turn now to the orientation preserving homeomorphisms of the circle, which we regard as the unit circle in the complex plane. For $x \in R_1$ denote $\vartheta(x) = \exp(2\pi i x)$. An increasing map $G\colon R_1 \to R_1$ is a lift of an orientation preserving homeomorphism (S_1, g), if $\vartheta\colon (R_1, G) \to (S_1, g)$ is a factorization. The rotation number $\varrho(g)$ of (S_1, g) is defined as the fractional part of $\varrho_0(G) = \lim_{n \to \infty} |G^n(y)|/n$, where G is any lift of g, and $y \in R_1$ ($\varrho(g)$ does not depend on the choice of either G or y). Then an orientation preserving homeomorphism of the circle (S_1, g) has a periodic point iff $\varrho(g)$ is rational (see Devaney [5] for a proof).

Theorem 3. *An orientation preserving homeomorphism (S_1, g) of the circle is a factor of a finite automaton iff $\varrho(g)$ is rational.*

Proof. If (S_1, g) is a factor of a finite automaton, then it has chaotic limits by Theorem 1, and therefore also periodic points. Thus its rotation number cannot be irrational. If $\varrho(g)$ is rational, then $\varrho(g^n) = 0$ for some n, and g^n has a fixed point. We can suppose that 1 is a fixed point of g, and choose a lift G, with fixed points 0 and 1. Then (S_1, g^n) is a factor of $([0, 1], G)$, and this is a factor of a finite automaton by Theorem 2. We modify the finite automaton obtained by letting in act only every n-th step similarly as in the proof of Theorem 2. Then (S_1, g) is a factor of this modified automaton.

5 Unimodal systems

Definition 5. *A dynamical system (I, g) defined on real interval $I = [a, b]$ is unimodal, if $g(a) = g(b) = a$, and if there exists a critical point $c \in (a, b)$, such that g is increasing in $[a, c]$, and decreasing in $[c, b]$.*

Proposition 2. *Let g be unimodal on $[a, b]$ with critical point c. Denote $I_0 = [a, c]$, $I_1 = [c, b]$, and $I_u = \{x \in I \mid 0 \leqq i < |u| \Rightarrow f^i(x) \in I_{u_i}\}$ for $u \in \overline{2^*}$. Then I_u are closed intervals (possibly empty or singletons).*

Proof. $I_{au} = I_a \cap f^{-1}(I_u)$ for $a \in 2$ and $u \in \overline{2^*}$.

Definition 6. *For $u \in \overline{2^*}$, $n < |u|$ denote $\tau_n(u) = \sum_{i=0}^{n} u_i \bmod 2$. We say that $u \in 2^*$ is even (odd), if $\tau_{|u|-1}(u)$ is zero (one). An infinite periodic sequence $u \in 2^\infty$ is even (odd), if the shortest sequence $v \in 2^*$ with $u = \bar{v}$ is even (odd). For $u, v \in \overline{2^*}$ define*

$$u \prec v \quad \textit{iff} \quad (\exists k < \min(|u|, |v|))\,(u_{|k} = v_{|k} \;\&\; \tau_k(u) < \tau_k(v))$$

$$u \preceq v \quad \textit{iff} \quad u \prec v \textit{ or } u \sqsubseteq v$$

Proposition 3. *Let $u, v \in \overline{2^*}$. Then*

$$u \prec v \Rightarrow (\forall x \in I_u)(\forall y \in I_v)(x \leqq y) \overset{def}{\Leftrightarrow} I_u < I_v$$

$$I_u \neq 0 \;\&\; u \preceq v \Rightarrow (\forall y \in I_v)(\exists x \in I_u)(x \leqq y) \overset{def}{\Leftrightarrow} I_u \leqq I_v$$

Proof. Let $u \prec v$, $k < |u|$, $k < |v|$, $u_{|k} = v_{|k}$, $\tau_k(u) < \tau_k(v)$, $x \in I_u$, $y \in I_v$. We prove $x \leqq y$ by induction on k. If $k = 0$, then $u_0 = 0$, $v_0 = 1$, so $I_u \subseteq I_0 < I_1 \supseteq I_v$, and $I_u < I_v$. If $k > 0$, we apply the Proposition to $\sigma(u)$ and $\sigma(v)$. If $u_0 = v_0 = 0$, then $\sigma(u) \prec \sigma(v)$, so $f(x) \leqq f(y)$, and $x \leqq y$ since $x, y \in I_0$. If $u_0 = v_0 = 1$, then $\sigma(v) \prec \sigma(u)$, so $f(y) \leqq f(x)$, and $x \leqq y$ since $x, y \in I_1$. Suppose $u \preceq v$, and $y \in I_v$. Since $I_u \neq 0$, we have some $x \in I_u$. If $u \prec v$, then $x \leqq y$ by the above proof. If $u \sqsubseteq v$, then $y \in I_u$, and $y \leqq y$. □

Definition 7. *The itinerary $\mathfrak{I}(x) \in 2^N$ of $x \in I$ is defined by*

$$g^i(x) < c \Rightarrow \mathfrak{I}(x)_i = 0$$

$$g^i(x) > c \Rightarrow \mathfrak{I}(x)_i = 1$$

$$g^i(x) = c \Rightarrow \tau_i(\mathfrak{I}(x)) = 0$$

The kneading sequence of g is $\mathfrak{K}(g) = \mathfrak{I}(g(c))$.

Proposition 4

$$x \in I_u \;\Rightarrow\; \mathfrak{I}(x) \leqq u$$

$$\mathfrak{I}(x) \prec \mathfrak{I}(y) \;\Rightarrow\; x \leqq y$$

$$x < y \;\Rightarrow\; \mathfrak{I}(x) \leqq \mathfrak{I}(y)$$

Proof. The first formula follows from the definition. If $\mathfrak{I}(x) \prec \mathfrak{I}(y)$, then $I_{\mathfrak{I}(x)} < I_{\mathfrak{I}(y)}$. Since $x \in I_{\mathfrak{I}(x)}$, $y \in I_{\mathfrak{I}(y)}$, we get $x \leqq y$ by Proposition 3. If $x < y$, then $\mathfrak{I}(y) \not\prec \mathfrak{I}(x)$, so $\mathfrak{I}(x) \precsim \mathfrak{I}(y)$. □

Definition 8. *For* $u, v \in \overline{2^*}$ *define* $u \preccurlyeq v$ *iff* $(\forall i > 0)\,(\sigma^i(u) \leqq v)$, *and* $u \preccurlyeq^* v$ *if* $u \preccurlyeq v$ *and* $u \sqsubseteq v$. *A sequence* $w \in \overline{2^*}$ *is maximal, if* $w \preccurlyeq w$.

Theorem 4. *If* (I,g) *is unimodal, then* $(\forall x \in I)\,(\mathfrak{I}(x) \preccurlyeq \mathfrak{K}(g))$, *and* $\mathfrak{K}(g)$ *is maximal.*

Proof. Let $x \in I$ and $k > 0$. Denote $u = \mathfrak{I}(x)$ and suppose $\sigma^k(u) > \mathfrak{K}(g)$. We have $g^k(x) \leqq g(c)$. If $g^k(x) = g(c)$, then $g^{k-1}(x) = c$, and $\tau_{k-1}(\mathfrak{I}(x)) = 0$, so $\sigma^k(u) = \mathfrak{I}(g^k(x)) = \mathfrak{K}(g)$, which is a contradiction. Thus $g^k(x) < g(c)$, so $g^{k-1}(x) \neq c$, and by Proposition 4, $\mathfrak{I}(g^k(x)) \leqq \mathfrak{K}(g)$. Let $i \geqq k$ be the least number with $g^i(x) = c$ (such a number exists, since otherwise $\mathfrak{I}(g^k(x)) = \sigma^k(u)$). There exists $y \in I$ with $\mathfrak{I}(y) = 0u_k \dots u_i \dots$ and $f^j(y) \neq c$ for $0 \leqq j \leqq i - k + 1$. Then $f(y) < f(c)$ and $\mathfrak{I}(f(y)) = u_k \dots u_i \dots \precsim \mathfrak{K}(g)$, which is a contradiction. □

Theorem 5. *If* (I, g) *is unimodal,* $u \in \overline{2^*}$, *and* $u \preccurlyeq \mathfrak{K}(g)$, *then* $I_u \neq 0$.

Proof. We proceed by induction on $n = |u|$. If $n \leqq 1$, then $I_u \neq 0$. Suppose that the Theorem holds for $n - 1$. Then the assumption holds for $\sigma(u)$, and therefore $I_{\sigma(u)} \neq 0$. Since $\sigma(u) \leqq \mathfrak{K}(g)$, and $g(c) \in I_{\mathfrak{K}(g)}$, there exists by Proposition 3 some $y \in I_{\sigma(u)}$ with $y \leqq g(c)$. There exists $x \in I_{u_0}$ with $g(x) = y$, and therefore $x \in I_u \neq 0$. Thus we have proved the Theorem for finite $|u|$. If $|u| = \infty$, then $I_u = \bigcap\{I_v \mid v \sqsubseteq u, v \in 2^*\}$. Since $v \sqsubseteq u$ implies $v \preccurlyeq \mathfrak{K}(g)$, and therefore $I_v \neq 0$, we get $I_u \neq 0$ by compactness.

Definition 9. *A unimodal system* (I, g) *is S-unimodal, if it has continuous third derivation,* $g''(c) < 0$, *and* $Sg(x) < 0$ *for all* $x \neq c$, *where*

$$Sg(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2$$

is the Schwarzian derivative.

Theorem 6. *Suppose that g is S-unimodal. If* $\mathfrak{K}(g)$ *is not periodic, then* I_u *has empty interior for any* $u \in 2^N$. *If* $\mathfrak{K}(g)$ *is periodic with period n, and* $u \in 2^N$, *then* I_u *has nonempty interior iff* $u \preccurlyeq \mathfrak{K}(g)$ *and there exists* $k \geqq 0$ *with* $\sigma^k(u) = \mathfrak{K}(g)$.

See Guckenheimer [6] or Collet and Eckmann [3] for a proof.

Definition 10. *Let* $w \in 2^N$ *be maximal and define* $\Sigma_w = \{u \in 2^N \mid u \ll w\}$, $\Sigma_w^* = \{u \in 2^* \mid u \ll w\}$. *Then* Σ_w *is a subshift, i.e. a closed* σ*-invariant subset.*

Theorem 7. *Let* (I, g) *be S-unimodal system such that* $\mathfrak{K}(g)$ *is not periodic. Then* (I, g) *is a factor of* $(\Sigma_{\mathfrak{K}(g)}, \sigma)$.

Proof. By Theorems 5 and 6, I_u is a singleton for each $u \in \Sigma_{\mathfrak{K}(g)}$, so we define $\varphi: \Sigma_{\mathfrak{K}(g)} \to I$ by $\varphi(u) \in I_u$. By Theorem 4, φ is surjective. Let us prove that it is continuous. If $\varphi(u) \in U$, and U is open, then there is $v \sqsubseteq u$ with $v \in 2^*$ and $I_v \subseteq U$. It follows that for all $w \in 2^N$ with $v \sqsubseteq w$ we have $\varphi(w) \in U$.

6 Eventually periodic kneading sequences

Theorem 8. *If* $w \in 2^N$ *is preperiodic and maximal, then* (Σ_w, σ) *is a factor of a finite automaton.*

Proof. We can write $w = w_0 \dots w_{m-1}\overline{w_m \dots w_{m+n-1}}$, where $w_{m-1} \neq w_{m+n-1}$, and $w_m \dots w_{m+n-1}$ is even. Then $\sigma^i(w) < w$ for all $i > 0$, and since the set $\{\sigma^i(w) \mid i > 0\}$ if finite, there exists p such that $\sigma^i(w)_{|p} < w_{|p}$ for all $i > 0$. Let k be an inateger with $nk > p$ and put $r = m + nk$, $s = m + (n+1)k$. Assume $w_{|r}u \ll^* w$ and let us prove $\sigma^i(w_{|s}u) \leqq w$ for all $i \geqq 0$. For $i = 0$ we distinguish two cases. If $w_{|m}$ is even then $w_{|r}$ is even, and $u \leqq \sigma^r(w) = \sigma^s(w)$ by the assumption, so $w_{|s}u \leqq w$. If $w_{|m}$ is odd then $u \geqq \sigma^r(w) = \sigma^s(w)$, and again $w_{|s}u \leqq w$. For $0 < i < s - p$ we have $\sigma^i(w_{|s}u) < w$ by $\sigma^i(w)_{|p} < w_{|p}$ For $i \geqq s - p$ we have $i > n$ and $\sigma^i(u_{|s}u) = \sigma^{i-n}(w_{|r}u) \leqq w$. Conversely, if $w_{|s}u \ll^* w$, then $w_{|r}u \ll^* w$. By induction, $(\forall u \in 2^N)\,(w_{|r}u \ll^* w \Leftrightarrow w_{|r+jn}u \ll^* w)$ for all $j > 0$. We construct a finite automaton with input alphabet $A_0 = 2$, and output alphabets $A_1 = 2$, $A_2 = \{0, \dots, s-1\}$. Put $f_0(q) = \#$, $f_1(q) = q_0$ if $w_{|q_2}q_0 \leqq w$, $f_1(q) = 1 - q_0$ otherwise. Then $w_{|q_2}f_1(q) \leqq w$. If $w_{|q_2}f_1(q) = w_{|q_2+1}$, then put $f_2(q) = q_2 + 1$ if $q_2 + 1 < s$, and $f_2(q) = r$ if $q_2 + 1 = s$. If $w_{|q_2}f_1(q) < w_{|q_2+1}$, then put $f_2(q) = \max\{k < q_2 + 1 \mid \sigma^{q_2+1-k}(w_{|q_2}f_1(q)) = w_{|k}\}$. Define $\varphi: X \to \Sigma_w$ by $\varphi(u)_i = (\pi f^i(u))_1$. Then φ is continuous and $\varphi f = \sigma\varphi$. Let $v \in \Sigma_w$, and put $q_1 = v_0$, $u_{0i} = v_{i+1}$, $q_2 = 0$. Then $\varphi(u) = v$, so φ is surjective.

Corollary 1. *If* (I, g) *is S-unimodal and* $\mathfrak{K}(g)$ *is preperiodic, then* (I, g) *is a factor of a finite automaton.*

Proof. Theorems 7 and 8.

Proposition 5. *If* $w \in 2^N$ *is odd periodic and maximal, then* (Σ_w, σ) *is a factor of a finite automaton. Moreover, if* $u \in \Sigma_w$, *and* $\sigma^i(u) = w$ *for some* $i \geqq 0$, *then* u *is an isolated point of* Σ_w.

Proof. Let n be the period of w. Then $(\forall u \in 2^N)\,(w_{|n}u \ll^* w \Leftrightarrow u = w)$. Indeed if $w_{|n}u \ll^* w$, then $\sigma^n(w_{|n}u) = u \leqq w$ by definition, and $u \geqq \sigma^n(w) = w$, since $w_{|n}$ is odd, thus $u = w$. Thus w is an isolated point of Σ_w, and if $\sigma^j(u) = w$, then u is isolated too. Construct a finite automaton with input alphabet $A_0 = 2$, and output alphabets $A_1 = 2$, $A_2 = \{0, \ldots, 2n-1\}$. Put $f_0(q) = \#$, and suppose $q_2 < n$. If $w_{|q_2}q_0 < w$, then put $f_1(q) = q_0$, $f_2(q) = \max\{k < q_2 + 1 \mid \sigma^{q_2+1-k}(w_{q_2}f_1(q)) = w_{|k}\}$. If $w_{|q_2}q_0 = w_{|q_2+1}$, then put $f_1(q) = q_0$, $f_2(q) = q_2 + 1$. If $w_{q_2}q_0 > w$, then put $f_1(q) = 1 - q_0$, $f_2(q) = q_2 + 1$. If $n \leqq q_2 < 2n - 1$, then put $f_1(q) = w_{q_2+1}$, $f_2(q) = q_2 + 1$. If $q_2 = 2n - 1$, then put $f_1(q) = w_{q_2+1}$, $f_2(q) = n$. Define $\varphi: X \to \Sigma_w$ by $\varphi(u)_i = (\pi f^i(u))_1$. Then $\varphi: (X, f) \to (\Sigma_w, \sigma)$ is a factorization.

Theorem 9. *If (I, g) is S-unimodal and $\mathfrak{K}(g)$ is periodic and odd, then (I, g) is a factor of a finite automaton.*

Proof. Let n be the period of $\mathfrak{K}(g)$. By Proposition 5, there is a factorization $\varphi': (X', f') \to (\Sigma_{\mathfrak{K}(g)}, \sigma)$. By Theorem 2 there is a factorization $\varphi'': (X'', f'') \to (I_{\mathfrak{K}(g)}, g^n)$. We construct a finite automaton (X, f), where $X = X' \times X''$, and $f(u', u'') = (f'(u'), f''(u''))$ if $q'_2 = 0$, or $q'_2 = n$, $f(u', u'') = (f'(u'), u'')$ otherwise. Thus if $\sigma\varphi'(u') = \mathfrak{K}(g)$, then either $q'_2 = 0$, or $q'_2 = n$, so $f(u', u'') = (f'(u'), f''(u''))$. Define $\varphi: X \to I$ by

$$\begin{array}{ll} \varphi(u', u'') \in I_{\varphi'(u')} & \text{if } (\forall i)\,(\sigma^i\varphi'(u') \neq \mathfrak{K}(g)) \\ \varphi(u', u'') = g_{\varphi'(u')_0} g^n \varphi''(u'') & \text{if } \sigma\varphi'(u') = \mathfrak{K}(g) \\ \varphi(u', u'') = g_{\varphi'(u')_0} \varphi(f'(u'), u'') & \text{if } \varphi(f(u'), u'') \text{ has been defined} \end{array}$$

(Here $g_i: [a, g(c)] \to I_i$, are the two inverses of g.) Then $\varphi(u', u'') \in I_{\varphi'(u')}$. Since φ'' is continuous, and since by Proposition 5 $\varphi'(u')$ is an isolated point when $\sigma^i\varphi'(u') = \mathfrak{K}(g)$, φ is continuous. By Theorems 4 and 6, φ is a surjection. If $\sigma\varphi'(u') = \mathfrak{K}(g)$, then $\varphi f(u', u'') = \varphi(f'(u'), f''(u'')) = g_{\varphi' f'(u')_0}\varphi((f')^2(u'), f''(u'')) = g_{\varphi'(u')_1} \cdots g_{\varphi'(u')_{n-1}} \varphi((f')^n(u'), f''(u'')) = g_{\varphi'(u')_1} \cdots g_{\varphi'(u')_n} g^n \varphi'' f''(u'') = \varphi'' f''(u'') = g^n \varphi''(u'') = g\varphi(u', u'')$. If $\varphi'(u')$ is eventually periodic, and if $\sigma\varphi'(u') \neq \mathfrak{K}(g)$, then $\varphi f(u', u'') = \varphi(f'(u'), u'') = g\varphi(u', u'')$. Thus φ is a factorization.

7 Aperiodic kneading sequences

For $s \in 2^*$, denote $\hat{s}$ the sequence obtained from s by changing the last bit. The double of s is $\mathfrak{D}(s) = s\hat{s}$. The doubling operator can be iterated, and since $s \sqsubseteq \mathfrak{D}(s)$, $\mathfrak{D}^\infty(s) = \lim_{i \to \infty} \mathfrak{D}^i(s)$ is well defined. It can be easily proved that if $s \in 2^*$ is an odd, aperiodic maximal sequence, then $\bar{\hat{s}} < \bar{s} < \overline{\mathfrak{D}(s)} < \mathfrak{D}^\infty(s)$, and $\mathfrak{D}^\infty(s) \in 2^N$ is a maximal aperiodic sequence.

Lemma 7. *Let $s \in 2^*$ be an odd, aperiodic, maximal sequence, $j > 2$, and $U_j = \{u \in \Sigma_{\mathfrak{D}^\infty(s)} \mid (\exists i \leqq 2^{j+1}|s|)\,(\mathfrak{D}^j(s) \sqsubseteq \sigma^i(u)$ or $\widehat{\mathfrak{D}^j(s)}\ \mathfrak{D}^j(s) \sqsubseteq \sigma^i(u)\}$. Then U_j is a σ-invariant set, which is closed and open in $\Sigma_{\mathfrak{D}^\infty(s)}$, and does not contain periodic points with periods less than $2^j|s|$.*

Proof. Clearly U_j is closed and open in $\Sigma_{\mathfrak{D}^\infty(s)}$. To prove it is invariant denote $w = \mathfrak{D}^{j-2}(s)$ and suppose $u \in U_j$. Then we can write $\sigma^j(u) = u_0 u_1 \ldots$, where $|u_i| = 2^{j-2}|s|$. We use repeatedly the fact $u \preccurlyeq \mathfrak{D}^\infty(s)$ and prove that if $u_0 u_1 u_2 u_3 = w\hat{w}ww$, then either $u_4 u_5 u_6 u_7 = w\hat{w}ww$ or $u_4 \ldots u_{11} = w\hat{w}w\hat{w}w\hat{w}ww$. Since $u_0 u_1 u_2 u_3 u_4 \leqq w\hat{w}www$, and $w\hat{w}ww$ is odd, $u_4 \geqq w$. Since $u_4 \leqq w$, we get $u_4 = w$. Since $u_4 u_5 \leqq w\hat{w}$, $u_5 \geqq \hat{w}$. Since $u_0 \ldots u_5 \leqq w\hat{w}www\hat{w}$, and $w\hat{w}www$ is even, $u_5 \leqq \hat{w}$, so $u_5 = \hat{w}$. Since $u_4 u_5 u_6 \leqq w\hat{w}w$, $u_6 = w$. Since $u_6 u_7 \leqq w\hat{w}$, $u_7 \geqq \hat{w}$, so either $u_7 = w$ or $u_7 = \hat{w}$. In the former case we are done. Suppose therefore $u_7 = \hat{w}$. Since $u_0 \ldots u_9 \leqq w\hat{w}www\hat{w}w\hat{w}w\hat{w}$, $u_9 \leqq \hat{w}$, so $u_9 = \hat{w}$. Since $u_8 u_9 u_{10} \leqq w\hat{w}w$, $u_{10} = w$. Since $u_0 \ldots u_{11} \leqq w\hat{w}www\hat{w}www\hat{w}ww$, $u_{11} = w$, so $u_8 u_9 u_{10} u_{11} = w\hat{w}ww$. If there is a periodic point $u \in U_j$ with period $k < n = |s|2^j$, then $u_{2n-1} = u_{2n-1-k} = u_{n-1-k} = u_{n-1}$, and $\bar{w}$ would have period k, which is impossible.

Proposition 6. *If $s \in 2^*$ is an odd, aperiodic, maximal sequence, then the subshift $(\Sigma_{\mathfrak{D}^\infty(s)}, \sigma)$ has not chaotic limits.*

Proof. Let U_j be sets from Lemma 7. Then $\omega(\mathfrak{D}^\infty(s)) \subseteqq U_j$, for $j > 2$, so $\omega(\mathfrak{D}^\infty(s))$ does not contain periodic points. Suppose that $u \in \omega(\mathfrak{D}^\infty(s))$, $u \in A \subseteqq \Sigma_{\mathfrak{D}^\infty(s)}$, and $\omega(A)$ is chaotic. Then it contains a periodic point $v \in \omega(A)$. Choose j so that $v \notin U_j$, and let V be a neighbourhood of v disjoint from U_j. Then $\sigma^k(U_j) \cap V = 0$ for all k, so $\omega(A)$ is not transitive.

Theorem 10. *If (I, g) is S-unimodal system, and $\mathfrak{K}(g) = \mathfrak{D}^\infty(s)$ for some odd, aperiodic, maximal sequence s, then (I, g) has not chaotic limits.*

Proof. By Theorem 7 there is a factorization $\varphi: (\Sigma_{\mathfrak{K}(g)}, \sigma) \to (I, g)$. Let $y \in I$ be an eventually periodic point. Since $\mathfrak{K}(g)$ is aperiodic, $g^i(y) \neq c$ for all i, and therefore there exists only one $u \in \Sigma_{\mathfrak{K}(g)}$ with $y = \varphi(u)$. Since $\omega(\mathfrak{K}(g))$ does not contain periodic points by Proposition 6, neither does $\omega(c) = \varphi(\omega(\mathfrak{K}(g))$. We prove that if $j > 2$, then $\varphi(U_j)$ is a neighbourhood of $\omega(c)$. If $y \in \omega(c)$, then there is unique $u \in \omega(\mathfrak{K}(g))$ with $\varphi(u) = y$. There exists $k \leqq 2^j|s|$ such that $\bar{0}_{|k} \neq u_{|k} \neq 1\bar{0}_{|k}$. Define $u', u'' \in U_j$ by $u'_{|k} = \bar{0}_{|k}$, $u''_{|k} = 1\bar{0}_{|k}$, $u'_i = u''_i = u_i$ for $i \geqq k$. Then $u' < u < u''$ and $y = \varphi(u)$ is an inner point of the interval $[\varphi(u'), \varphi(u'')] \subseteqq \varphi(U_j)$. Since $\varphi(U_j)$ is invariant, we get the result similarly as in Proposition 6.

Corollary 2. *Let (I, g) be S-unimodal system, and $\mathfrak{K}(g) = \mathfrak{D}^\infty(s)$ for some odd, aperiodic, maximal sequence s. Then (I, g) is not a factor of a finite automaton.*

Proof. Theorems 1 and 10.

References

[1] Balcar B., Simon P.: Appendix on general topology. Handbook of Boolean Algebras (J. D. Monk and R. Bonnet, eds.), Elsevier Science Publishers B.VC., 1241 (1989).

[2] BROOKS J., CAIRNS G., DAVIS G., STACEY P.: On Devaney's definition of chaos. The American Mathematical Monthly, 99, 4, 332 (1992).

[3] COLLET P., ECKMANN J. P.: Iterated Maps on the Interval as Dynamical Systems. Birkhauser, Basel (1980).

[4] CRUTCHFIELD J. P., YOUNG K.: Computation at the onset of chaos. In: Complexity, Entropy and the Physics of Information, SFI Studies in the Sciences of Complexity, vol VIII (W. H. Zurek, ed.), Addison Wesley 223 (1990).

[5] DEVANEY R. L.: An Introduction to Chaotic Dynamical Systems. Addison-Wesley, Redwood City (1989).

[6] GUCKENHEIMER J.: Sensitive dependence to initial conditions for one dimensional maps. Commun. Math. Phys. 70, 133 (1979).

[7] KŮRKA P.: Dynamical systems and factors of finite automata, manuscript.