Acta Universitatis Carolinae. Mathematica et Physica

Eva Matoušková

How small are the sets where the metric projection fails to be continuous

Acta Universitatis Carolinae. Mathematica et Physica, Vol. 33 (1992), No. 2, 99--108

Persistent URL: http://dml.cz/dmlcz/701981

1992 ACTA UNIVERSITATIS CAROLINAE—MATHEMATICA ET PHYSICA VOL. 33. NO. 2

How Small Are the Sets Where the Metric Projection Fails to Be Continuous

EVA MATOUŠKOVÁ

Czechoslovakia*)

Received 10 May 1992

Let M be a nonempty closed subset of a Banach space X with an uniformly Fréchet differentiable norm. It is shown that the set of points whre the nearest point mapping fails to have uniqueness is σ-cone supported. It is also shown, that there exists a σ-cone supported set $A \subset X - M$ and a cone-small set $B \subset X - M$ such that the distance function d_M is Fréchet differentiable on $C = (X - M) - (A \cup B)$. As a corollary in spaces which have moreover Fréchet differentiable dual norm, the nearest point problem is well posed on C.

1. Introduction

For a nonempty closed subset M of a real Banach space X, let

$$d_M(x) = \inf\{\|x - y\|; y \in M\}$$

be the *distance function* associated to M and let

$$P_M(x) = \{y \in M; \|x - y\| = d_M(x)\}$$

be the *metric projection* (or the *nearest point mapping*) of x onto M, for each $x \in X$.

In [2] De Blasi and Myjak use the following termnology: a point $x_0 \in P_M(x)$ is called a *solution* of the nearest point problem $\min(x, M)$, a sequence $\{x_n\} \subset M$ such that $\lim_{n\to\infty} \|x_n - x\| = d_M(x)$ is called a *minimizing sequence*. A nearest point problem is said *well posed* if it has a unique solution, say x_0, and every minimizing sequence for this problem converges to x_0. For $M \subset X$ they define

$$A_u(M) = \{x \in X; P_M(x) \text{ contains at least 2 points}\}$$

$$A_{wp}(M) = \{x \in X; \text{the nearest point problem } \min(x, M) \text{ is not well posed}\}$$

The set $A_u(M)$ (resp. $A_{wp}(M)$) is called the *ambiguous locus of uniqueness* (resp. *well posedness*) of M for the nearest point mapping.

De Blasi and Myjak show that if X uniformly convex than A_{wp} is σ-porous in X.

*) Department of Mathematical Analysis, Charles University, Sokolovská 83, 186 00 Praha 8, Czechoslovakia.

In this note we show that under some conditions on smoothness of the space X (resp. X^*) are the sets A_u and A_{wp} small in an even more restrictive sense (Theorem 2.9, Corollary 2.8). We use the fact that the properties of P_M (uniqueness, well posedness) have very close relations to the differentiability of d_M (see [3]).

Now let us mention few definitions, which enable „to measure" the smallness of sets.

In the following we denote by $B(x, r)$ (resp. $\bar{B}(x, r)$) an open (resp. closed) ball with a center x and a radius r.

Definition 1.1 *Let X be a metric space. A subset M of X is called porous at $x \in X$ if there exist $\gamma > 0$ and $R > 0$ such that for every $r \in (0, R]$ there exists $y \in X$ such that $B(y, \gamma r) \subset B(x, r) \cap (X - M)$. A set is said to be porous if it is is porous at all its points. A set is called σ-porous if it is a countable union of porous sets.*

Let us remind that this definition of σ-porosity differs slightly from the one introduced by Dolženko in real analysis. He requires for $x \in X$ only existence of such $\gamma > 0$ and $R > 0$ that for every $r_0 \in (0, R]$ there exist $0 < r < r_0$ and $y \in X$ such that $B(y, \gamma r) \subset B(x, r) \cap (X - M)$. For more details see [5].

Definition 1.2 *Let X be a Banach space. If $x^* \in X^*$, $x^* \neq 0$, and $0 < \alpha < 1$, define*

$$C(x^*, \alpha) = \{x \in X; \alpha\|x\| \cdot \|x^*\| < \langle x, x^* \rangle\} .$$

*We say that a set $M \subset X$ is α-*cone porous *at $x \in X$ if there exists $R > 0$ such that for each $r > 0$ there exist $z \in B(x, r)$ and $0 \neq x^* \in X^*$ such that*

$$M \cap B(x, R) \cap (z + C(x^*, \alpha)) = \emptyset .$$

*A subset of X is said to be α-cone porous if it is α-cone porous at all its points. A set is called $\sigma - \alpha$-*cone porous *if it can be written as a union of countably many α-cone porous sets. A set is said to be* cone-small *if it is $\sigma - \alpha$-cone porous for each $0 < \alpha < 1$.*

In the following we will use also an another definition of a cone.

Definition 1.3 *Let X be a Banach space. If $v \in X$, $\|v\| = 1$, and $0 < c < 1$, define*

$$A(v, c) = \{x; x = \lambda v + w, \lambda > 0, \|w\| < c\lambda\} = \bigcup_{\lambda > 0} \lambda B(v, c) .$$

We say that a set $M \subset X$ is cone supported *at $x \in X$ if there exists $R > 0$, $v \in X$, $\|v\| = 1$ and $0 < c < 1$ such that*

$$M \cap B(x, R) \cap (x + A(v, c)) = \emptyset .$$

*A subset of X is said to be cone supported if it is cone supported at all its points. A set is called σ-*cone supported *if it can be written as a union of countably many cone supported sets.*

Let us review some properties of these notions (see [6]).

Each „cone" $C(x^*, \alpha)$ contains a „cone" $A(v, c)$. Every σ-cone supported set and also every cone small set is σ-porous. In finite dimensional spaces it is easy to prove that each cone-small set is σ-cone supported. In R^2 there exists a σ-cone supported set which is not cone-small. In a separable Hilbert space there exists a set which is cone-small, but is not σ-cone supported.

Lemma 1.4 [6] *Let X be a Banach space and let $0 < \alpha < 1$. Suppose that $M \subset X$ is not $\sigma - \alpha$-cone porous (resp. σ-cone supported). Then there exists $\emptyset \neq N \subset M$ such that N is α-cone porous (resp. cone supported) at no point of N.*

The proof of the Theorem 2.6 presented in this note depends on the fact that similarly as the subdifferential of a continuous convex function is a maximal monotone mapping, the almost superdifferential of the distance function in spaces of our interest proves to be locally almost nonincreasing mapping. And locally almost nonincreasing mappings prove to have some nice properties of maximal monotone mappings. So let us now define these notions.

Definition 1.5 *Let X be a Banach space and F a real function defined on an open set $G \subset X$. We say that the mapping $T: G \to 2^{X^*}$ is a uniform almost superdifferential of F on G if for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$(F(x + h) - F(x) - \langle h, x^* \rangle) \|h\|^{-1} \leqq \varepsilon$$

whenever $0 < \|h\| \leqq \delta$, $x, x + h \in G$ and $x^* \in T(x)$.

Definition 1.6 *Let X be a Banach space and let $G \subset X$ be open. We say that the mapping $T: G \to 2^{X^*}$ is locally almost nonincreasing on G if for any $x \in G$ and $\varepsilon > 0$ there exists a neighborhood U of x such that for any $y, z \in U$ and $y^* \in T(y)$, $z^* \in T(z)$*

$$\langle y - z, y^* - z^* \rangle \leqq \varepsilon \|y - z\| .$$

L. Zajíček defines in [4] these notions for singlevalued mapping T; for our purposes is the multivaluedness more suitable. The following lemma is proved in [4] for the singlevalued case, but in our case the proof works similarly.

Lemma 1.7 [4] *Let X be a Banach space, F a real function defined on an open set $G \subset X$ and $T: G \to 2^{X^*}$ an uniform almost superdifferential of F on G. Then*

(i) T is locally almost nonincreasing on G, and

(ii) if there exists a selection of T which is continuous at $x \in G$, then is Fréchet differentiable at x.

In [3] Fitzpatrick proved the following relations between properties of the metric projection and the distance function:

Theorem 1.8 [3] *Suppose that M is a closed subset of a Banach space X such that the norm of X is both Fréchet differentiable and uniformly Gâteaux differentiable and the norm of X^* is Fréchet differentiable. Then the following are equivalent for $x \in X - M$:*

(i) d_M is Fréchet differentiable at x;

(ii) P_M *is continuous at* x *(so especially it is singlevalued at* x*);*
(ii) *every minimizing sequence in* M *for* x *converges.*

2. Results

The following two lemmas generalize lemmas from [6] replacing monotone mappings by locally almost nonincreasing.

Lemma 2.1 *Let* X *be a Banach space and let* $T: X \to 2^{X^*}$ *be a locally almost nonincreasing mapping with an arbitrary domain* $D(T) = \{x; T(x) \neq \emptyset\}$, *Let* $0 < 3a < A$, $x \in X$ *and* $N \subset D(T)$ *be given such that*

$$\lim_{\delta \to 0+} diam T(B(x, \delta) \cap N) < a \tag{1}$$

and

$$\lim_{\delta \to 0+} diam T(B(x, \delta)) > A \tag{2}$$

Then N *is* $3a/A$*-cone porous at* x.

Proof: Because the mapping T is locally almost nonincreasing and (1) holds, we can choose $R > 0$ such that

$$diam T(B(x, R) \cap N) < a \quad \text{and} \quad \langle y - z, y^* - z^* \rangle \leqq (a/2) \|y - z\| \tag{3}$$

whenever $y, z \in B(x, R)$, $y^* \in T(y)$, $z^* \in T(z)$.

If $T(B(x, R) \cap N) = \emptyset$, then the assertion of the lemma is obviously satisfied. Otherwise choose $f \in T(B(x, R) \cap N)$ and consider an arbitrary $r > 0$. By (2) we can find $z \in B(x, r)$ and $z^* \in T(z)$ such that $\|z^* - f\| > A/2$. To show that N is $3a/A$-cone porous at x it is sufficient to prove that

$$B(x, R) \cap N \cap \{y \in X; \langle y - z, f - z^* \rangle > (3a/A) \| z^* - f\| \cdot \|y - z\|\} = \emptyset .$$

Suppose on the contrary that there exists $y \in B(x, R) \cap N$ for which

$$\langle y - z, f - z^* \rangle > (3a/A) \|z^* - f\| \cdot \|y - z\|$$

and choose $y^* \in T(y)$. Since (3) implies $\|y^* - f\| < a$ we obtain

$$\begin{aligned} a\|y - z\| &\geqq \langle y - z, f - y^* \rangle = \langle y - z, z^* - y^* \rangle + \\ &+ \langle y - z, f - z^* \rangle \geqq \langle y - z, f - z^* \rangle - (a/2) \|y - z\| > \\ &> (3a/A) \|z^* - f\| \cdot \|y - z\| - (a/2) \|y - z\| \geqq \\ &\geqq (3a/A)(A/2) \|y - u\| - (a/2) \|y - z\| = a\|y - z\| . \end{aligned}$$

This is a contradiction which completes the proof.

Lemma 2.2 *Let* X *be a Banach space and* $T: X \to 2^{X^*}$ *be a locally almost nonincreasing mapping with an arbitrary domain* $D(T) = \{x; T(x) \neq \emptyset\}$. *Let* $H \subset D(T)$, $x \in H$, $v \in X$, $\|v\| = 1$, $c \in R$, $\varepsilon > 0$, $x^* \in T(x)$, *(i)* $\langle v, x^* \rangle > c + \varepsilon$ *and*

(ii) $\lim_{\delta \to 0+} \operatorname{diam} T(B(x, \delta) \cap H) < K$.

Then there exists $\varrho > 0$ *such that for every*

$$y \in B(x, \varrho) \cap H \cap (x + A(-v, \varepsilon/2K)) \quad \textit{and} \quad y^* \in T(y)$$

the inequality $\langle v, y^* \rangle > c$ *holds.*

Proof: Because the mapping T is locally almost nonincreasing and (ii) holds, we can choose $\varrho > 0$ such that for every $y \in B(x, \varrho) \cap H$ and $y^* \in T(y)$ holds that $\|x^* - y^*\| \leqq K$ and

$$\langle x - y, x^* - y^* \rangle \leqq (\varepsilon K/(2K + \varepsilon)) \|x - y\| . \tag{4}$$

Suppose that $y \in B(x, \varrho) \cap H \cap (x + A(-v, \varepsilon/2K))$ *and* $y^* \in T(y)$ are given. By Definition 1.3 we can find $\lambda > 0$ and $w \in X$, $\|w\| < \lambda\varepsilon/2K$ such that $y = x - \lambda v + w$.

By (4) it holds

$$(\varepsilon K/(2K + \varepsilon)) \|x - y\| = (\varepsilon K/(2K + \varepsilon)) \|w - \lambda v\| \geqq \langle y^* - x^*, w - \lambda v \rangle ,$$

hence

$$\begin{aligned} \langle \lambda v, y^* \rangle &\geqq \langle \lambda v, x^* \rangle + \langle w, y^* - x^* \rangle - (\varepsilon K/(2K + \varepsilon)) \|w - \lambda v\| > \\ &> \lambda(c + \varepsilon) - \|w\| \, \|y^* - x^*\| - (\varepsilon K/(2K + \varepsilon)) (\|w\| + \lambda \|v\|) \geqq \\ &\geqq \lambda(c + \varepsilon) - \lambda\varepsilon/2 - (\varepsilon K/(2K + \varepsilon)) (\lambda + \lambda\varepsilon/2K) = \lambda c . \end{aligned}$$

Consequently $\langle v, y^* \rangle > c$.

Proposition 2.3 *Let X be an Asplund space and let $T: X \to 2^{X^*}$ be a locally bounded and locally almost nonincreasing mapping with an arbitrary domain $D(T) = \{x; T(x) \neq \emptyset\}$. Then there exists a σ-cone supported set $A \subset D(T)$ such that T is single-valued at each point of $D(T) - A$.*

Proof: Suppose on the contrary that

$$A := \{x \in D(T);\ T \text{ is not single-valued at } x\}$$

is not σ-cone supported. Obviously $A = \bigcup_{n=1}^{\infty} A_n$ where

$$A_n = \{x \in D(T);\ \operatorname{diam} T(x) > 1/n\} .$$

Consequently we can choose a positive integer n such that A_n is not σ-cone supported. By Lemma 1.4 there exists a set $\emptyset \neq N \subset A_n$ which is cone supported at no its point. Choose $x \in N$. Since T is locally bounded, there exists $r > 0$ such that $T(B(x, r))$ is bounded. Putting $H := N \cap B(x, r)$, we easily see that $\emptyset \neq H$ is cone supported at no point of H and $T(H)$ is a bounded (let us say by a number K) subset of X^*. Since X is an Asplund space, every nonempty bounded subset of X^* admits weak* slices of arbitrarily small diameters. Consequently there exist $v \in X$, $\|v\| = 1$ and $c > 0$ such that the weak* slice of $T(H)$

$$S := \{x^* \in T(H);\ \langle v, x^* \rangle > c\}$$

is nonempty and has diameter less that $1/n$. Since $S \neq \emptyset$, we can choose $x \in H$ and

$x^* \in T(x) \cap S$. Choose $\varepsilon > 0$ such that $\langle v, x^* \rangle > c + \varepsilon$. By Lemma 2.2 there exists $\varrho > 0$ such that for each $y \in B(x, \varrho) \cap H \cap (x + A(-v, \varepsilon/(2K)))$ and $y^* \in T(y)$ the inequality $\langle v, y^* \rangle > c$ holds. Since H is not cone supported at x we can choose $y \in B(x, \varrho) \cap H \cap (x + A(-v, \varepsilon/(2K)))$. Since $H \subset A_n$ we have $diam T(y) > 1/n$. But $T(y) \subset S$ and $diam S < 1/n$, and that is a contradiction.

Proposition 2.4 *Let X be an Asplund space and let $T: X \to X^{X^*}$ be a locally bounded and locally almost nonincreasing mapping with a domain $D(T) = \{x; T(x) \neq \emptyset\}$ which has a nonempty interior $G = Int\, D(T)$. Let moreover T be norm-to-weak* upper semicontinous. Then the set A of all points $z \in G$ at which T is single-valued but is not norm-to-norm upper semicontinuous is cone-small.*

Proof: Suppose on the contrary that A is not cone small. Then there exists $0 < \alpha < 1$ such that A is not $\sigma - \alpha$-cone porous. Obviously $A = \bigcup_{n=1}^{\infty} A_n$, where

$$A_n = \{x \in A; \lim_{\delta \to 0+} diam T(B(x, \delta)) > 1/n\} .$$

Consequently we can choose a positive integer n such that A_n is not $\sigma - \alpha$-cone porous. By Lemma 1.4 there exists a set $\emptyset \neq N \subset A_n$ which is α-cone porous at no point of N. Choose $x \in N$. Since T is locally bounded at every point of G we can choose $r > 0$ such that $T(B(x, r))$ is bounded. Since X is an Asplund space, there exist $v \in X$, $\|v\| = 1$ and $c > 0$ such that the weak* slice

$$S := \{x^* \in T(B(x, r) \cap N); \langle v, x^* \rangle > c\}$$

of the set $T(B(x, r) \cap N)$ is nonempty and has diameter less than $\alpha/3n$. Hence there exists $y \in B(x, r) \cap N$ such that $\langle v, T(y) \rangle > c$ (here $T(y) \in X^*$, since T is single--valued on $N \subset A$). Since $\{x^*; \langle v, x^* \rangle > c\}$ is weak* open and since T is norm-to--weak* upper semicontinuous, there exists $d > 0$ such that $B(y, d) \subset B(x, r)$ and $T(B(y, d)) \subset \{x; \langle v, x^* \rangle > c\}$. Consequently $T(B(y, d) \cap N) \subset S$ and therefore

$$\lim_{\delta \to 0+} diam(T(B(y, \delta)) \cap N) \leqq diam S < \alpha/3n .$$

Since $y \in N \subset A_n$, we have $\lim_{\delta \to +0} diam T(B(y, \delta)) > 1/n$. Using Lemma 2.1 with $x = y$, $a = \alpha/3n$, $A = 1/n$ we obtain that N is α-cone porous at y, which is a contradiction.

In the proof of the next theorem we will need the following easy lemma:

Lemma 2.5 *Let X be an Asplund space, $G \subset X$ an open set and $\{f_\alpha; \alpha \in \Lambda\}$ be a system of functions on G such that each f_α is Fréchet differentiable on G, and the limit $\lim_{h \to 0} (f_\alpha(x + hv) - f_\alpha(x))\, h^{-1}$ is uniform with respect to $(\alpha, v, x) \in \Lambda \times \{v; \|v\| = 1\} \times G$. Then the mappings $x \leftrightarrow f'_\alpha(x)$ ($f'_\alpha(x)$ denotes the Fréchet derivative of f_α at x) are equally continuous on G with respect to $\alpha \in \Lambda$.*

Proof: We need to prove that for every $x \in G$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\|f'_\alpha(x) - f'_\alpha(y)\| < \varepsilon$ whenever $y \in G$ and $\|x - y\| < \delta$. So let $x \in G$ and

$\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$|f_\alpha(y + hv) - f_\alpha(y))\, h^{-1} - f'_\alpha(y)(v)| < \varepsilon/2$$

whenever $(\alpha, v, y) \in A \times \{v; \|v\| = 1\} \times G$, $0 < |h| < \delta$, $y + hv \in G$. Consequently for every $y \in G$, $0 < \|x - y\| < \delta$ and $v \in X$, $\|v\| = 1$ we have

$$|f'_\alpha(x)(v) - f'_\alpha(y)(v)| \leqq |f'_\alpha(x)(v) - (f_\alpha(y) - f_\alpha(x))/\|x - y\|| + \\ + |(f_\alpha(x) - f_\alpha(y))/(-\|x - y\|) - f'_\alpha(y)(v)| \leqq \varepsilon\,.$$

Theorem 2.6 *Let X be an Asplund space, $G \subset X$ an open set and $\{f^k; a \in \Lambda\}$ be a system of K-Lipschitz functions on G, for which the following conditions hold:*

(i) Each f_α is Fréchet differentiable on G, and the limit $\lim_{h\to 0}(f_\alpha(x + hv) - f_\alpha(x))\, h^{-1}$ is uniform with respect to $(\alpha, v, x) \in A \times \{v; \|v\| = 1\} \times G$.

(ii) $F(x) := \inf\{f_\alpha; \alpha \in \Lambda\} > -\infty$ for each $x \in G$.

Then there exists a σ-cone supported set $A \subset G$ and a cone-small set $B \subset G$ such that F is Fréchet differentiable on $G - (A \cup B)$.

Proof: For an arbitrary $x \in G$ let us denote by $\mathscr{F}_x$ the filter on X^* with the filter basis

$$\{\{f'_\alpha(x); f_\alpha(x) < F(x) + \varepsilon\}; \varepsilon > 0\}\,.$$

Since every function f_α is K-Lipschitz, we have $\|f'_\alpha(x)\| \leqq K$. Since the set $\{g \in X^*; \|g\| \leqq K\}$ is weak*-compact, the set $T(x)$ of all points of accumulation of $\mathscr{F}_x$ in weak* topology is nonempty whenever $x \in G$. Moreover, the mapping $T: x \mapsto T(x)$ is locally bounded (in fact it is bounded by K). Let us show that T is norm-to-weak* upper semicontinuous on G.

We must show that if $x \in G$ and W is any weak* open subset of X^* containing $T(x)$, then for any sequence $\{x_n\} \subset G$ with $\|x_n - x\| \to 0$, we have $T(x_n) \subset W$ for all sufficiently large n. If not, then there exists a subsequence (call it $\{x_n\}$) and $x_n^* \in T(x_n) - W$. Since the mapping T is bounded, there exists weak* cluster point x^* of the sequence $\{x_n^*\}$. Clearly $x^* \in X^* - W$, but we will prove that $x^* \in T(x)$, which is a contradiction. To do so, it is sufficient to show that for arbitrary $\varepsilon > 0$, $\omega > 0$ and nonzero $v_i \in X$, $i = 1, \ldots, k$ there exists $\alpha \in \Lambda$ such that

$$f_\alpha(x) < F(x) + \varepsilon \quad \text{and} \quad |\langle v_i, f'_\alpha(x) - x^*\rangle| < \omega \quad i = 1, \ldots, k\,.$$

Let $\varepsilon > 0$, $\omega > 0$ and nonzero $v_i \in X$, $i = 1, \ldots, k$ be given. Let us choose n large enough, such that $\|x - x_n\| < \varepsilon/2K$, $\|f'_\alpha(x) - f'_\alpha(x_n)\| < \omega/3 \cdot \max \|v_i\|$ whenever $\alpha \in \Lambda$ (this is possible, because f'_α are by Lemma 2.5 equally continuous with respect to α) and $\langle v_\iota, x_n^* - x^*\rangle < \omega/3$. Now let us take $\alpha \in \Lambda$ such that

$$f_\alpha(x_n) < F(x_n) + \varepsilon/2 \quad \text{and} \quad |\langle v_i, f'_\alpha(x_n) - x_n^*\rangle| < \omega/3 \quad i = 1, \ldots, k$$

Then using the fact that every f_α is K-Lipschitz we get

$$f_\alpha(x) \leqq f_\alpha(x_n) + K\|x - x_n\| < F(x_n) + \varepsilon/2 + K \cdot \varepsilon/2K = F(x_n) + \varepsilon\,.$$

Moreover

$$|\langle v_i, f'_\alpha(x) - x^* \rangle| \leqq |\langle v_i, f'_\alpha(x_n) - x_n^* \rangle| + |\langle v_i, x_n^* - x^* \rangle| + $$
$$+ |\langle v_i, f'_\alpha(x) - f'_\alpha(x_n) \rangle| \leqq 2\omega/3 + \|v_i\| \, \omega/3 \max \|v_i\| \leqq \omega$$

Hence $x^* \in T(x)$ and the mapping T is norm-to-weak* upper semicontinuous. Now we will show that T is a uniform superdifferential of F on G. Let $\varepsilon > 0$ be given. Since f_α are uniformly Fréchet differentiable on G, there exists $\delta > 0$ such that

$$|f_\alpha(x + h) - f_\alpha(x) - \langle h, f'_\alpha(x) \rangle| < \varepsilon/2 \, \|h\| \,, \tag{5}$$

whenever $x \in G$, $x + h \in G$, $0 < \|h\| < \delta$, $\alpha \in \Lambda$.

Let $x \in F$, $h \in X$, $0 < \|h\| < \delta$ and $\eta > 0$ be fixed. Since $T(x)$ is the set of points of accumulation of $\mathscr{F}_x$ in weak* topology, for an arbitrary $g(x) \in T(x)$ there exists $\alpha \in \Lambda$ such that

$$f_\alpha(x) < F(x) + \eta \quad \text{and} \quad |\langle h/\|h\|, f'_\alpha(x) - g(x) \rangle| < \varepsilon/2 \tag{6}$$

By (5) and (6) it holds that

$$F(x + h) \leqq f_\alpha(x + h) \leqq F(x) + \eta + \langle h, g(x) \rangle + 2\varepsilon/2 \, \|h\|$$

and because $\eta > 0$ is an arbitrary number

$$F(x + h) - F(x) - \langle h, g(x) \rangle \leqq \varepsilon \|h\| \,,$$

whenever $0 < \|h\| < \delta$, $x \in G$, $x + h \in G$, $g(x) \in T(x)$. Hence T is a uniform superdifferential of F on G and by Lemma 1.7(i) it is locally almost nonincreasing on G.

By Proposition 2.3 there exists a σ-cone supported set $A \subset D(T) = G$ such that T is single-valued on $G - A$. By Proposition 2.4 the set B of all points where T is single-valued, but is not norm-to-norm upper semicontinuous is cone small. So if $x \in G - (A \cup B)$ then $T(x)$ is single-valued and T is norm-to-norm upper semicontinuous at x, consequently every selection of T is continuous at x and by Lemma 1.7(ii) F is Fréchet differentiable at x.

L. Zajíček proved in [4] that if X is a Banach space with separable dual and uniformly Fréchet differentiable norm then the set of points where the distance function fails to be Fréchet differentiable is cone-small (in fact he proved that it is angle-small, but in separable spaces this notion coincides with our definition of cone-small sets). Using the previous theorem, we can prove without the presumption of a separable dual a result which is only slightly weaker.

Theorem 2.7 *Let X be a Banach space with an uniformly Fréchet differentiable norm (a.e. the limit $\lim_{\delta \to 0} (\|x + \delta v\| - \|x\|) \, \delta^{-1}$ is uniform with respect to $(x, v) \in S_1 \times S_1$, where S_1 denotes the unit sphere of X) and let M be a closed subset of X. Then there exist a σ-cone supported set $A \subset X - M$ and a cone small set $B \subset X - M$ such that the distance function d_M is Fréchet differentiable on $(X - M) - (A \cup B)$.*

Proof: Denote by $F(x) := d_M$, for $\alpha \in M$ denote $f_\alpha(x) := \|x - \alpha\|$, and for a positive integer n denote $G_n := X - (M + B(0, 1/n))$. Since the norm on X is uniformly Fréchet differentiable the limit $\lim_{\delta \to 0} (\|x + \delta v\| - \|x\|) \delta^{-1}$ is uniform also with respect to $(x, v) \in V \times S_1$, where V is an arbitrary set such that $X - V$ is a neighborhood of zero. If we define $V_n := X - B(0, 1/n)$, then obviously V_n satisfies this condition and whenever $x \in G_n$ and $\alpha \in M$ it holds that $x - \alpha \in V_n$. Consequently the functions f_α satisfy the condition (ii) of Theorem 2.6. The norm on X is Fréchet differentiable, so X is an Asplund space and we can apply Theorem 2.6 to obtain that there exist a cone-small set $B_n \subset G_n$ and a σ-cone supported set $A_n \subset G_n$ such that F is Fréchet differentiable on $G_n - (A_n \cup B_n)$. Because $X - M = \bigcup_{n=1}^{\infty} G_n$ we have also that there exist a cone-small set $B \subset G$ and a σ-cone supported set $A \subset G$ such that F is Fréchet differentiable on $G - (A \cup B)$.

As a corollary we obtain using Theorem 1.8 the following:

Corollary 2.8 *Let X be a Banach space with an uniformly Fréchet differentiable norm and Fréchet differentiable norm on X^* and let M be a closed subset of X. Then there exist a σ-cone supported set $A \subset X - M$ and a cone small set $B \subset X - M$ such that $A_{wp}(M) \subset A \cup B$.*

De Blasi and Myjak proved in [2] that if X is uniformly convex than the set A_{wp} is σ-porous. Corollary 2.8 provides an improvement of their result for example for spaces l_p, where $1 < p < \infty$. These are uniformly convex, consequently also their duals are uniformly convex and by [7] the norm on l_p, $1 < p < \infty$ is uniformly Fréchet differentiable.

Now let us deal with the set $A_u(M)$ where the metric projection on M contains at least two points.

Theorem 2.9 *Let X be a strictly convex Banach space with a uniformly Fréchet differentiable norm and let M be a closed subset of X. Then there exists a σ-cone supported set $A \subset X - M$ such that $A_u(M) \subset A$.*

Proof: If we define $F(x) := d_M$, $f_\alpha(x) := \|x - \alpha\|$ for $\alpha \in M$, and $G_n := X - (M + B(0, 1/n))$ for a positive integer n, we obtain in the same way as in the proof of Theorem 2.7 that the conditions of Theorem 2.6 are satisfied. Consequently we can define the mapping $T_n : G_n \to 2^{X^*}$ as in the proof of Theorem 2.6 and obtain that T_n is a uniform almost superdifferential of d_M on G_n. Moreover we obtain that there exists a σ-cone supported set $A_n \subset D(T) = G_n$ such that T_n is single-valued on $G_n - A_n$. If we define $A := \bigcup_{n=1}^{\infty} A_n$, then $A \subset X - M$ is a σ-cone supported subset of $X - M$.

We will prove that $A_u(M) \subset A$.

Let $x \in A_u(M)$ and $y, z \in P_M(x)$ such that $y \neq z$ be fixed. Choose a positive integer n such that $x \in G_n$. Since $f_y(x) = f_z(x) = F(x)$, we have by the definition of T_n that both $f'_y(x)$ and $f'_z(x)$ belong to $T_n(x)$. Since X is strictly convex, $\|x - y\| = \|x - z\|$, and $x - y \neq x - z$ we have that $f'_y(x) \neq f'_z(x)$. Consequently T_n is not singlevalued at x.

References

[1] De Blasi F. S., Myjak J., Ensembles poreux dans la théorie de la meilleure approximation, C.R. Acad. Sci. Paris, t. 308, Série I, (1989), 353—356.

[2] De Blasi F. S., Myjak J., Ambiguous loci in best approximation theory, to appear.

[3] Fitzpatrick S., Metric projections and the differentiability of distance functions, Bull. Austral. Math. Soc. 22 (1980), 291—312.

[4] Zajíček L., On the Fréchet differentiability of distance functions, Proc. 12th Winter School, Suppl. Rend. Circ. Mat. Palermo, Ser. II, No. 5, (1984), 161—165.

[5] Zajíček L., Porosity and σ-porosity, Real Analysis Exchange 13, (1987—88), 314—350.

[6] Zajíček L., Smallness of sets of nondifferentiability of convex functions in non-separable Banach spaces, Czechoslovak Mathematical Journal, 41 (116), (1991), 288—296.

[7] Zizler V., Banach spaces with the differentiable norms, Commentationes Mathematicae Universitatis Carolinae 8 (3), (1968), 415—441.