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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 33 (1992), No. 2, 85--90

Persistent URL: http://dml.cz/dmlcz/701979

The Brouwer-Jordan Theorem

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Received 10 May 1992

Some stronger version of the Leray product formula is proved. This enables us to find a new homotopy invariant for continuous maps $f: K \to K$ from compact subsets of Euclidean space $\mathbb{R}^n$.

1. The Brouwer-Jordan theorem. As an application of the new product formula we shall show that there exists a function c assigning to any continuous map $f : K \to K$ from a compact set $K \subset \mathbb{R}^n$, an integer or the infinity $c(f, K) \in \mathbb{Z} \cup \{\infty\}$, having the following properties:

(c1) $c(\mathrm{id}_K, K)$ is equal to the number of bounded connected components of $\mathbb{R}^n \setminus K$, where $\mathrm{id}_K : K \to K$ means the identity map.

(c2) If maps $f, g : K \to K$ are homotopic, $f \simeq g : K \to K$, (i.e. there exists a continuous map $h : K \times [0, 1] \to K$ such that for each $x \in K$, $h(x, 0) = f(x)$ & $h(x, 1) = g(x)$), then $c(f, K) = c(g, K)$.

(c3) If maps $f : K \to M$ and $g : M \to K$ are continuous then $c(gf, K) = c(fg, M)$.

Recall, that compact subsets $K, M \subset \mathbb{R}^n$ have the same homotopy type iff there exist continuous maps $f : K \to M$ and $g : M \to K$ such that $gf \simeq \mathrm{id}_K$ and $fg \simeq \mathrm{id}_M$.

Let us observe that the properties (c1)–(c3) imply the celebrated Brouwer-Jordan theorem:

If compact sets $K, M \subset \mathbb{R}^n$ have the same homotopy type then $\mathbb{R}^n \setminus K$ and $\mathbb{R}^n \setminus M$ have the same number of connected components.

Indeed, let $f : K \to M$ and $g : M \to K$ be continuous maps such that $gf \simeq \mathrm{id}_K$ and $fg \simeq \mathrm{id}_M$. Then we have: the number of bounded connected components of $\mathbb{R}^n \setminus K \overset{(c1)}{=} c(\mathrm{id}_K, K) \overset{(c2)}{=} c(gf, K) \overset{(c3)}{=} c(fg, M) \overset{(c2)}{=} c(\mathrm{id}_M, M) \overset{(c1)}{=}$ the number of bounded connected components of $\mathbb{R}^n \setminus M$.

2. Product formula. Let us accept the following notations mostly taken from [D].

$$\overline{a, b} := \{x \in \mathbb{R}^n : x = ta + (1 - t)\, b\,, \quad t \in [0, 1]\}\,, \quad a, b \in \mathbb{R}^n$$

$$d(A, B) := \inf\{|x - y| : x \in A, y \in B\} \quad A, B \subset \mathbb{R}^n$$

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$$B(A, \varepsilon) := \{y \in \mathbb{R}^n : \exists\, x \in A\,; \quad |x - y| < \varepsilon\} \quad A \subset \mathbb{R}^n$$
$$B(a, \varepsilon) := B(\{a\}, \varepsilon) \quad a \in \mathbb{R}^n$$

$C(\mathbb{R}^n)$ — the set of all continuous maps $f : \mathbb{R}^n \to \mathbb{R}^n$, $C^\infty(\mathbb{R}^n)$ — the set of all maps $f : \mathbb{R}^n \to \mathbb{R}^n$ of class C^∞. Symbols $\bar{A}$ and ∂A mean the closure and the boundary of a set $A \subset \mathbb{R}^n$.

It will be convenient to introduce for any triple (f, A, a), where $f \in C(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$ is a bounded set such that $a \notin f(\partial \bar{A})$, the following definition of degree

$$\deg(f, A, a) := d(f|\operatorname{int}\bar{A}, \operatorname{int}\bar{A}, a)$$

where d is a degree function described in [D].

To prove results of this note we shall need to know some properties of the function deg.

(p1) If $f \in C^\infty(\mathbb{R}^n)$ and $a \in \mathbb{R}^n \setminus f(\partial \bar{A})$ is a regular value of the map $f \mid \operatorname{int}\bar{A}$ then $\deg(f, A, a) = \sum\{\operatorname{sgn}\det f'(x) : x \in \bar{A} \cap f^{-1}(a)\}$, (agreement: $\sum_{\emptyset} = \emptyset$).

(p2) For each two open disjoint sets $A_1, A_2 \subset A$, $A_1 \cap A_2 = \emptyset$, if $a \notin f(\bar{A} \setminus (A_1 \cup A_2))$ then $\deg(f, A, a) = \deg(f, A_1, a) + \deg(f, A_2, a)$.

(p3) If $\deg(f, A, a) \neq 0$ then $a \notin f(\bar{A})$.

(p4) If there exists a continuous map $h : \partial\bar{A} \times [0, 1] \to \mathbb{R}^n$ such that $\forall_{x \in \partial \bar{A}} \ \forall_{t \in [0,1]}$ $h(x,t) \neq a$ & $\forall_{x \in \partial\bar{A}}$ $h(x,0) = f(x)$, $h(x,1) = g(x)$, then $\deg(f, A, a) = \deg(g, A, a)$.

(p5) If $f \mid \partial\bar{A} = g \mid \partial\bar{A}$ then $\deg(f, A, a) = \deg(g, A, a)$.

(p6) If $a \notin \overline{f(x), g(x)}$ for each $x \in \partial\bar{A}$, then $\deg(f, A, a) = \deg(g, A, a)$.

(p7) If $M \subset \mathbb{R}^n \setminus f(\partial\bar{A})$ is a connected set then for each two points $x, y \in M$, $\deg(f, A, x) = \deg(g, A, y)$.

The symbol $\deg(f, A, M)$ means that the degree $\deg(f, A, x)$ is constant for each $x \in M$.

Lemma. *Assume that $g : X \to \mathbb{R}^n$ is a continuous map from a compact space X and let $c \in \mathbb{R}^n$ be a point. Then for each open set U; $g^{-1}(c) \subset U \subset X$, there exists an $\varepsilon > 0$ such that for each continuous map $g_1 : X \to \mathbb{R}^n$, $\|g - g_1\| < \varepsilon$ implies $g^{-1}[B(c, \varepsilon)] \subset U$.*

Proof. Since the set $X \setminus U$ is compact hence the set $V := \mathbb{R}^n \setminus g(X \setminus U)$ is an open neighbourhood of the point c and $g^{-1}(c) \subset g^{-1}(V) \subset U$. Choose an $\varepsilon > 0$ such that $B(c, 3\varepsilon) \subset V$. Let $g_1 : X \to \mathbb{R}^n$, $\|g - g_1\| < \varepsilon$, be a continuous map. Observe that

$$W := g_1^{-1}[B(c, \varepsilon)] \subset U\,.$$

Indeed, suppose that there exists a point $x \in W \setminus U$. Then $g(x) \notin B(c, 3\varepsilon)$ and $g_1(x) \in B(c, \varepsilon)$. This implies that $|g(x) - g_1(x)| \geqq 2\varepsilon$, a contradiction with $\|g - g_1\| < \varepsilon$.

Now we establish the main result of this paper.

Theorem (the product formula). *Suppose that we are given two compact sets $A, M \subset \mathbb{R}^n$ and a map $f \in C(\mathbb{R}^n)$ such that $f(A) \subset M$. Let $\{M_s : s \in S\}$ be the family of all bounded connected components of $\mathbb{R}^n \setminus M$. Then for any compact set $K \subset \mathbb{R}^n$, any map $g \in C(\mathbb{R}^n)$ and any point c, such that $\partial K \subset A$ and $c \in \mathbb{R}^n \setminus g(M)$ the following formula holds*

$$\deg(gf, K, c) = \sum_{s \in S} \deg(f, K, M_s) \,.\, \deg(g, M_s, c)$$

where only finitely many terms are different from zero and $\sum_{s\in S} = 0$ *when* $S = \emptyset$.

Proof. Let us fix a map $g \in C(\mathbb{R}^n)$, a compact set $K \subset \mathbb{R}^n$ and a point $c \in \mathbb{R}^n$ such that $\partial K \subset A$ and $c \notin g(M)$.

(I) First, let us note that for the sets K, M_s, $s \in S^* := S \cup \{\infty\}$, where M means the unbounded component of $\mathbb{R}^n \setminus M$, the following conditions hold

(1) $$M_s \cap f(\partial K) = \emptyset \quad \text{and} \quad c \notin (gf)(\partial K) \cup g(\partial M_s)$$

Define for each $s \in S^*$

(2) $$K_s := K \cap \overline{f^{-1}(M_s)}$$

The sets K_s have the following properties

(3) $$f(\partial K_s) \subset \partial M_s \subset M \quad \text{and} \quad c \notin (gf)(\partial K_s)$$

(4) $$\operatorname{int} K_s \cap \operatorname{int} K_t = \emptyset \quad \text{for each} \quad s \neq t\,,$$

(5) $$(gf)^{-1}(c) \cap K \subset \sum_{s \in S^*} \operatorname{int} K_s\,.$$

From (5) and by the compactness of the set K it follows that there exists a finite set $T \subset S^*$ such that

(6) $$(gf)^{-1}(c) \cap K \subset \bigcup_{t \in T} \operatorname{int} K_t$$

Consequently according to the property (p2)

(7) $$\deg(gf, K, c) = \sum_{t \in T} \deg(gf, K_t, c)\,.$$

For $s \in S^* \setminus T$ we have; $\deg(gf, K_s, c) = 0$. Thus the equality (7) can be written as

(8) $$\deg(gf, K, c) = \sum_{s \in S^*} \deg(gf, K_s, c)\,.$$

To conclude the proof it suffices to show that for each $s \in S$

(i) $$\deg(gf, K_s, c) = \deg(f, K, M_s) \,.\, \deg(g, M_s, c)$$

and

(ii) $$\deg(gf, K_\infty, c) = 0\,.$$

(II) Let us fix an index $s \in S^*$ and choose an $\varepsilon_1 > 0$ such that

(9) $$3\varepsilon_1 < \min\{d[(gf)^{-1}(c), \partial K_s], d(g^{-1}(c), M), d(c, g(M))\}$$

and let $Q \subset \mathbb{R}^n$ be an arbitrary closed ball with

(10) $$f(K) \subset \operatorname{int} Q\,.$$

According to the Lemma and the Weierstrass approximation theorem there exist maps f_s, $g_s \in C^\infty(\mathbb{R}^n)$ and an $\varepsilon \in (0, \varepsilon_1)$ such that

$$Q \cap \overline{M_s} \cap g_s^{-1}[B(c, \varepsilon)] \subset M_s \cap B[g^{-1}(c), \varepsilon] \tag{11}$$

$$K_s \cap (g_s f_s)^{-1}\, [B(c, \varepsilon)] \subset \operatorname{int} K_s \cap B[(gf)^{-1}\,(c), \varepsilon_1] \tag{12}$$

$$f_s[K_s \cap (g_s f_s)^{-1}\, [B(c, \varepsilon)] \subset M_s\,, \quad f_s(K) \subset \operatorname{int} Q\,, \tag{13}$$

$$\|f - f_s\|_K\,, \quad \|g - g_s\|_Q\,, \quad \|gf - g_s f_s\|_K < \varepsilon_1\,. \tag{14}$$

From the Sard Lemma it follows that there exists a point $c_s \in B(c, \varepsilon)$ being a regular value of the map $g_s f_s$. From (14), (13) and (9) we have

$$\underset{y \in B(c,\varepsilon_1)}{\forall} \ \underset{x \in \partial K_s}{\forall} \ y \notin \overline{(gf)\,(x), (g_s f_s)\,(x)}$$

Since $c_s \in B(c, \varepsilon)$, according to the property (p6) we get

$$\deg(gf, K_s, c) = \deg(g_s f_s, K_s, c_s)\,. \tag{15}$$

From (14), (9) and (1) it follows that

$$\underset{y \in B(c,\varepsilon)}{\forall} \ \underset{x \in \partial M_s}{\forall} \ y \notin \overline{g(x), g_s(x)}$$

and the above fact in the case when M_s is a bounded component yields

$$\deg(g, M_s, c) = \deg(g_s, M_s, c_s)\,. \tag{16}$$

The condition (13) implies

$$f_s[K_s \cap (g_s f_s)^{-1}\,(c_s)] \subset M_s$$

and consequently

$$K_s \cap (g_s f_s)^{-1}\,(c_s) = \bigcup\{K_s \cap f_s^{-1}(y) : y \in M_s \cap g_s^{-1}(c_s)\} \tag{17}$$

Observe that from (11) we get

$$Q \cap \overline{M_s} \cap g_s^{-1}(c_s) \subset M_s \cap B[g^{-1}(c), \varepsilon_1]$$

and in view of (14), (3) and (9) this implies that

$$\underset{y \in Q \cap M_s \cap g_s^{-1}(c_s)}{\forall} \ \underset{x \in \partial K_s}{\forall} \ y \notin \overline{f(x), f_s(x)}$$

Moreover, according to (p6) we get

$$\underset{y \in Q \cap M_s \cap g^{-1}(c_s)}{\forall} \deg(f, K_s, y) = \deg(f_s, K_s, y) \tag{18}$$

Definition 2 of the set K_s along with the property (p2) yields $\deg(f, K, M_s) \overset{(p2)}{=} \deg(f, K_s, M_s) = \deg(f, K_s, Q \cap M_s \cap g_s^{-1}(c_s)) \overset{(18)}{=} \deg(f_s, K_s, Q \cap M_s \cap g_s^{-1}(c_s))$.

Since $f(K) \cup f_s(K_s) \subset Q$, we see, applying (p3) that

$$\underset{y \notin Q}{\forall} \deg(f, K, y) = 0 = \deg(f_s, K_s, y)$$

and in consequence

$$\deg(f, K, M_s) = \deg(f_s, K_s, M_s \cap g_s^{-1}(c_s)) \tag{19}$$

(III) Now let us prove the conditions (i), (ii).

$$\deg(gf, K_s, c) \overset{(15)}{=} \deg(g_s f_s, K_s, c_s) \overset{(p1)}{=}$$
$$= \sum\{\operatorname{sgn}\det(g_s f_s)'(x) : x \in K_s \cap (g_s f_s)^{-1}(c_s)\} \overset{(17)}{=}$$
$$= \sum\{\operatorname{sgn}\det g_s'(y) \cdot \det f_s'(x) : y \in M_s \cap g_s^{-1}(c_s), x \in K_s \cap f_s^{-1}(y)\} =$$
$$= \sum\{\operatorname{sgn}\det g_s'(y) : y \in M_s \cap g_s^{-1}(c_s)\} \cdot \sum\{\operatorname{sgn}\det f_s'(x) : x \in K_s \cap f_s^{-1}(y)\} \overset{(p1)}{=}$$
$$= \sum\{\operatorname{sgn}\det g_s'y) \cdot \deg(f_s, K_s, y) : y \in M_s \cap g_s^{-1}(y)\} \overset{(19)}{=}$$
$$= \deg(f_s, K_s, M_s) \cdot \sum\{\operatorname{sgn}\det g_s'(y) : y \in M_s \cap g_s^{-1}(c_s)\}\,.$$

It was shown that

$$\deg(gf, K_s, c) = \deg(f, K, M_s) \cdot \sum\{\operatorname{sgn}\det g_s'(y) : y \in M_s \cap g_s^{-1}(y)\}\,. \tag{20}$$

From (p3) and (p7) we infer that $\deg(f, K, M_\infty) = 0$, and this together with (20) completes the proof of the equality (ii). In the case when $s \in S$ we have

$$\sum\{\operatorname{sgn}\det g_s'(y) : y \in M_s \cap g_s^{-1}(c_s)\} \overset{(p1)}{=} \deg(g_s, M_s, c_s) \overset{(16)}{=} \deg(g, M, c)$$

Comparing the above with (20) we obtain (i). The proof of the theorem is completed.

Remark. The well known in literature the Leray product formula (cf. [D] or [S]) is a special case of presented in this note the product formula, when $A = \partial K$ and $M = f(\partial K)$.

3. Definition of the function c. Let $K \subset \mathbb{R}^n$ be a compact set and let $\{K_i : i \in I\}$ be the family of all bounded connected components of $\mathbb{R}^n \setminus K$. For any map $f \in C(\mathbb{R}^n)$ such that $f(K) \subset K$ let us define

$$c(f, K) := \sum_{i \in I} \deg(f, K_i, K_i)$$

when only finitely many terms are different from zero, and

$$c(f, K) := \infty$$

in the other case.

It is clear that the properties (p1), (p5) and (p4) imply the properties (c1) and (c2). We shall prove that the property (c3) holds, too.

Assume that one of the numbers $c(gf, K)$ or $c(fg, M)$ is finite, for example let $c(gf, K) < \infty$. Let $\{K_i : i \in I\}$, $\{M_j : j \in J\}$ mean the families of all bounded connected components of $\mathbb{R}^n \setminus K$, $\mathbb{R}^n \setminus M$, respectively. From the product formula we get

$$\deg(gf, K_i, K_i) = \sum_{j \in J} \deg(f, K_i, M_j) \cdot \deg(g, M_j, K_i) \tag{21}$$

$$\deg(fg, M_j, M_j) = \sum_{i \in I} \deg(g, M_j, K_i) \cdot \deg(f, K_i, M_j) \tag{22}$$

Then

$$c(gf, K) := \sum_{i\in I} \deg(gf, K_i, K_i) \overset{(21)}{=} \sum_{i\in I} \sum_{j\in J} \deg(f, K_i, M_j) \cdot \deg(g, M_j, K_i) =$$
$$= \sum_{j\in J} \sum_{i\in I} \deg(g, M_j, K_i) \cdot \deg(f, K_i, M_j) \overset{(22)}{=}$$
$$= \sum_{j\in J} \deg(fg, M_j, M_j) =: c(fg, M)\,.$$

From the above part it follows that

$$c(gf, K) = \infty \Leftrightarrow c(fg, M) = \infty$$

The proof of the property (c3) is completed.

References

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