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# Measures with Idempotent Types and Completely Stable Measures on Nilpotent Groups

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**Introduction.** The original definition of stability of probabilities on $\mathbb{R}$ due to P. Lévy can be formulated as follows: $\mu$ is stable if type$(\mu)*$ type$(\mu)$ = type$(\mu)$ i.e. the type of $\mu$ is idempotent, where the type of $\mu$ is the orbit of $\mu$ under the action of the group of affine maps. The definition makes sense if we replace $\mathbb{R}$ by a vector space or by a group. In the following, since the underlying convolution structure is non abelian, we consider only strict stability. Hence we consider only groups $\Gamma$ of automorphisms – instead of affine transformations – acting on $\mu$. The $\Gamma$-type is the orbit $\Gamma(\mu)$. This definition of stability was used by several authors for finite – dimensional vector spaces (see e.g. [PS], [S], [M1]), in the infinite dimensional situation [MU], [M2]) and for probabilities on groups [Ha1]).

Under fullness conditions, i.e. if a suitable version of the convergence of types theorem holds, idempotence of the type implies the existence of one-parameter groups $(a_t) \subseteq \Gamma$ such that $\mu$ is stable w.r.t. $(a_t)$. Especially $\mu$ is embeddable into a continuous convolution semigroup ($\mu_t$ = Exp $tA$) with generating distribution $A$ fulfilling $a_t A = tA$, $t > 0$. We show in § 1 that under fullness conditions $A$ has idempotent type, i.e. $\Gamma(A) + \Gamma(A) = \Gamma(A)$ (improving [Ha1]). Then it is known that the structure of $A$ (and hence of $\mu$) is completely determined by the structure of the corresponding generating distribution $\mathring{A}$ on the tangent space $\mathfrak{G}$.

Hence, in § 2, we collect more or less known results on vector spaces in order to apply these in § 3:

A measure on a vector space $E$ is called completely stable if it has idempotent type for $\Gamma = GL(E)$. It is known that, at least for finite dimensional vector spaces, completely stable measures are just the symmetric, full Gaussian measures.

Our goal is to obtain similar characterizations for groups. But (§ 3, § 4) it turns out that for general nilpotent groups our knowledge of the automorphism group

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Aut($G$) is not satisfactory. First we have to find a „correct" generalization of the notion of complete stability. In non-abelian groups full measures with idempotent Aut($G$)-type need not to exist.

Then we show by examples that there are large classes of groups containing the Heisenberg groups for which the above characterization of full Gauß-measures hold, but that there also exist groups, the automorphism groups of which do not admit full completely stable measures. And moreover there exist groups $G$ with „small" automorphismgroups admitting non-Gaussian full completely stable probabilities.

In § 5 we give a short survey on corresponding results for infinite dimensional vector spaces.

It should be noted that at least the first part of the investigations is of purely analytic nature: Consider $S := M^1(G)$ as topological semigroup then $\Gamma \times S \to S$ is a jointly continuous action of a transformation group acting as semigroup homomorphisms on S. The definition of fullness, the convergence of types theorem and the definitions of stability and of idempotent $\Gamma$-type could be formulated in the context of topological semigroups. And most of the proofs in § 1 also work in this more general setup.

## § 1 Idempotent $\Gamma$-types

Let $G$ be a nilpotent simply connected (real) Lie group. Let $\Gamma$ be a subgroup of Aut($G$), the group of topological automorphisms. W.l.o.g. we assume $\Gamma$ to be closed. $\Gamma$ acts as transformation group on the topological semigroup of probability measures $M^1(G)$ endowed with the convolution product. The topology on $M^1(G)$ is always understood as the topology $\sigma(M^b, C_0)$ of weak convergence. An important tool for the following considerations is a suitable concept of fullness and of convergence of types. We use a slight generalization of the usual fullness concept ([HN]). First we define:

**1.0 Definition.** Let $[G, G]$ be the commutator subgroup. $\mathfrak{m} := G/[G, G]$ is abelian, hence a vector space. Let $\pi : G \to G/[G, G]$ be the canonical projection. $\pi$ induces a convolution homomorphism $M^1(G) \to M^1(\mathfrak{m})$ which is again denoted by $\pi$.

Since $[G, G]$ is characteristic $\pi$ induces a canonical homomorphism $\pi : a \mapsto \bar{a}$ from $\mathrm{Aut}(G) \to \mathrm{Gl}(\mathfrak{m})$. Let $\bar{\Gamma}$ be the subgroup of $\mathrm{Gl}(\mathfrak{m})$ corresponding to $\Gamma$. We define $\mathrm{Aut}(\mathfrak{m}) := \pi(\mathrm{Aut}(G))$.

**1.1 Definition.** A subset $\mathscr{F}_\Gamma \subseteq M^1(G)$ is called set of $\Gamma$-full measures if the pair $(\Gamma, \mathscr{F}_\Gamma)$ fulfils

(1.1) **the convergence of types theorem:**

Let $\mu_n, \mu, \lambda \in M^1(G)$, $a_n \in \Gamma$, $n \in \mathbb{N}$. If $\mu_n \to \mu$, $a_n\mu_n \to \lambda$ and if $\mu$ and $\lambda \in \mathscr{F}_\Gamma$ then

$\{a_n\}$ is relatively compact in $\Gamma$. And then for any accumulation point $a$ of $\{a_n\}$ we have $a\mu = \lambda$.

**1.2 Remark.** For $\Gamma = \mathrm{Aut}(G)$ the set $\mathscr{F} := \{\mu : \mu$ is not concentrated on a proper connected subgroup$\}$ is an appropriate class of full measures. (See [HN], [N] or the more general discussion in [D], [HA 2]). We call measures $\mu \in \mathscr{F}$ simply full in the sequel. So the existence of $\mathscr{F}_\Gamma$ is always guaranteed. Since we have to consider simultaneously probabilities on groups $G$ and on vector spaces $\mathfrak{m} = G/_{[G,G]}$ resp. $\mathfrak{G}$ we need a generalized fullness concept depending on the group of admissible automorphisms.

**1.3 Definition.** In the following we fix a subset $\mathscr{F}_\Gamma$ of $\Gamma$-full measures fulfilling

(1.3.a) $\mathscr{F}_\Gamma$ is open and $\Gamma$-invariant.

(1.3.b) There is an (open, $\bar{\Gamma}$-invariant) set of $\bar{\Gamma}$ full measures $\mathscr{F}_{\bar{\Gamma}} \subseteq M^1(\mathfrak{m})$ fulfilling (1.1) on the vector space $\mathfrak{m}$, such that $\mathscr{F}_\Gamma = \bar{\pi}^{-1}\mathscr{F}_{\bar{\Gamma}}$.

For sake of convenience we assume in addition (w.l.o.g.) that

(1.3.c) $\mathscr{F} \subseteq \mathscr{F}_\Gamma$ ($\mathscr{F}$ being the set of $\mathrm{Aut}(G)$ – full measures defined in 1.2).

If $\mathscr{F}_\Gamma$ is fixed we define the set of $S$-full measures $\mathscr{F}_\Gamma^S := \{\mu \in M^1(G) : \mu * \tilde{\mu} \in \mathscr{F}_\Gamma\}$, where $\tilde{\mu}$ is defined by $\tilde{\mu}(R) = \mu(E^{-1})$. (Remark, if $G$ is a vector space then usually $S$-full measures are called full, cf. e.g. [Sh], [S], [M1].)

**1.4 Remark.** Since $\mathscr{F}_\Gamma$ is open we can write the convergence of types theorem (1.1) in the following equivalent form:

Let $\mathscr{A}', \mathscr{C}' \subseteq \mathscr{F}_\Gamma$ and $\mathscr{B}' \subseteq \bar{\Gamma}$. Assume $\mu \mapsto a_\mu$ to be a surjective map $\mathscr{A}' \to \mathscr{B}'$, and assume $\mathscr{C}' := \{a_\mu(\mu) : \mu \in \mathscr{A}'\}$.

Let $\mathscr{A}, \mathscr{C}$ resp. $\mathscr{B}$ be the closures in $\mathscr{F}_\Gamma$ resp. $\Gamma$. Then

(1.4) the compactness of two sets of $\mathscr{A}, \mathscr{B}, \mathscr{C}$ implies compactness of the third one.

**1.5. Definition.** Let $\mu \in M^1(G)$. The $\Gamma$-type of $\mu$ is the orbit $\Gamma(\mu) := \{\gamma\mu : \gamma \in \Gamma\}$. $\mu$ has idempotent $\Gamma$-type if

(1.5) $\Gamma(\mu) * \Gamma(\mu) = \Gamma(\mu)$.

**1.6. Remark.** The relation $\Gamma(\mu) * \Gamma(\mu) = \Gamma(\mu)$ can be written in the following form:

There exists a selection function $\Phi : \Gamma \times \Gamma \to \Gamma$ such that for $a, b \in \Gamma$, $c := \Phi(a, b)$

(1.5′) $a(\mu) * b(\mu) = c(\mu)$.

**1.7 Remark.** In case of vector spaces usually affine transformations $\bar{\Gamma}$ are used for the definition. Such measures are called often $\Gamma$-stable then. (See [S], [SP], [P], [M1], [Si2]).

Let $\mathfrak{J} = \mathfrak{J}(\mu) := \{a \in \Gamma : a(\mu) = \mu\}$ be the invariance group of $\mu$. Then obviously for $a, b \in \Gamma$, $\alpha, \beta \in \mathfrak{J}$ $\Phi(a\alpha, b\beta) = \Phi(a, b) \cdot \gamma$ for some $\gamma \in \mathfrak{J}$.

Let $X := \Gamma/\mathfrak{J}$ be the coset space. Then $\Phi$ may be considered as selection function $\Phi : X \times X \to X$. $\Phi$ is uniquely determined then.

⟦Assume $a(\mu) * b(\mu) = c(\mu)$ and $= d(\mu)$. Then $d^{-1}c \in \mathfrak{J}(\mu)$ and vice versa.⟧

The convergence of types theorem immediately implies that for $\Gamma$-full measures $\mathfrak{J}(\mu)$ is a compact subgroup.

Let $(\mu_t)_{t \geqq 0}$ be a continuous convolution semigroup in $M^1(G)$. Let $A \in \mathfrak{E}(G)'$ be the generating distribution. (I.e. $A \in \mathfrak{E}(G)'$, such that $\langle A, f \rangle = \mathrm{d}^+/\mathrm{d}t \, \langle \mu_t, f \rangle|_{t=0}$, $f \in \mathfrak{E}(G)$. See e.g. [He], [Ha3] for details). We use the abbreviation $(\mu t = \operatorname{Exp} tA)_{t \geqq 0}$, being justified by the fact that the convolution operator $f \mapsto A * f$ represents the infinitesimal generator of the operator semigroup $(f \mapsto \mu t * f)_{t \geqq 0}$ on $C_0(G)$.

Let $\mathfrak{B}(G)$ be the cone of generating distributions (resp. of infinitesimal generators). $\operatorname{Aut}(G)$, hence $\Gamma$, act in a natural way on $\mathfrak{B}(G)$ : If $\mu_t = \operatorname{Exp} tA$, $t \geqq 0$, then $(a(\mu_t) = = \operatorname{Exp} ta(A))_{t \geqq 0}$.

Let $(a_t)_{t>0} \subseteq \Gamma$ be a continuous one-parameter group, let $(\mu_t = \operatorname{Exp} tA)$ be a continuous convolution semigroup. $(\mu_t)$ resp. A is stable w.r.t. $(a_t)$ if $a_t \mu_1 = \mu_t$, $t > 0$ resp. $a_t A = tA$, $t > 0$. (See e.g. [Ha4], [Ha5], [Ha6] for details).

**1.8 Definition.** $\Gamma(A) : = \{\gamma(A) : \gamma \in \Gamma\}$ is called the $\Gamma$-type of $A$. $A$ (resp. $(\mu_t = = \operatorname{Exp} tA)_{t \geqq 0})$ has idempotent (infinitesimal) $\Gamma$-type if

$$\Gamma(A) + \Gamma(A) = \Gamma(A) . \tag{1.8}$$

This is again equivalent to the existence of a selection function $\Psi : \Gamma \times \Gamma \to \Gamma$, such that for $a, b \in \Gamma$, $c = \Psi(a, b)$

$$a(A) + b(A) = c(A) . \tag{1.8'}$$

Again, let $\mathfrak{J} = \mathfrak{J}(A) : = \{a \in \Gamma : a(A) = A\}$ and $X : = \Gamma/\mathfrak{J}$, then $\Psi$ can be considered as a function $\Psi : X \times X \to X$. And again $\Psi$ is uniquely determined then.

⟦$a(A) + b(A) = c(A)$ and $= d(A)$ holds iff $d^{-1}c \in \mathfrak{J}(A)$.⟧

**1.9 Remarks.** a) We have $\mathfrak{J}(A) = \bigcap_{t>0} \mathfrak{J}(\mu_t) \subseteq \mathfrak{J}(\mu)$. Hence $\mathfrak{J}(A)$ is a compact subgroup if $\mu = \mu_1$ is $\Gamma$-full.

b) If the semigroups $(\operatorname{Exp} ta(A) = a(\mu t))_{t \geqq 0}$, $a \in \Gamma$, commute, then (1.2′) is equivalent to

$$a(\mu_t) * b(\mu_t) = c(\mu_t) , \quad t \geqq 0 . \tag{1.8''}$$

Especially, if $G$ is a vector space, then the conditions (1.5), (1.5′) (1.8), (1.8′) and (1.8″) are equivalent.

**1.10 Proposition.** *Let $\mu$ have idempotent $\Gamma$-type. Then $\mu$ is B-stable, i.e. for $n \in \mathbb{N}$ there exist $\tau_n \in \Gamma$ such that $\tau_n \mu = \mu^n$,*

$$\textit{resp.} \quad (\tau_n^{-1} \mu)^n = \mu . \tag{1.10}$$

Hence $\mu$ is infinitely divisible and therefore embeddable into a continuous convolution semigroup $(\mu_t = \operatorname{Exp} tA)$.

⟦See [Ha1] : $\tau_1 = id$; $\tau_2 = \Phi(id, id)$, i.e. $\mu * \mu = \tau_2(\mu)$, $\tau_3 = \Phi(\tau_2, id)$ etc.⟧

**1.11 Corollary.** *Assume $\mu$ to have idempotent $\Gamma$-type. Let $(\mu_t = \operatorname{Exp} tA)_{t \geqq 0}$ the corresponding c.c.s. with $\mu_1 = \mu$.*

*a) If $\mu$ is $\Gamma$-full then $(\mu_t)$ is uninuely by $\mu = \mu_1$, and is stable w.r.t. some continuous one-parameter group $(a_t)_{t>0} \subseteq \Gamma$.*

*b) Therefore $\mathfrak{I} = \mathfrak{I}(A) = \mathfrak{I}(\mu)$ if $\mu$ is $\Gamma$-full.*

⟦*See* [*N*] *for* $\Gamma = \operatorname{Aut}(G)$⟧.

The next theorem is an essential improvement of the results given in [Ha1]:

**1.12 Theorem.** *Let $\mu$ be a $\Gamma$-full measure with idempotent $\Gamma$-type. Let $(\mu_t = \operatorname{Exp} tA)$ be the corresponding stable convolution semigroup. Then $A$ has idempotent (infinitesimal) $\Gamma$-type.*

**Proof:** Choose $(a_t) \subseteq \Gamma$ such that $a_t\mu = \mu_t$, $t > 0$. Let $a, b \in \Gamma$, put $\gamma_t := \Phi(aa_t, ba_t)$ and $c_t := \gamma t a_t^{-1}$. Then $c_t(\mu_t) = \gamma_t(\mu) = aa_t(\mu) * ba_t(\mu) = a(\mu_t) * b(\mu_t)$ For any test function $f \in \mathfrak{E}(G)$

$$\left.\frac{d^+}{dt}\right|_{t=0} \langle a(\mu_t) * b(\mu_t), 'f\rangle = \langle a(A) + b(A), f\rangle .$$

Hence with $C := a(A) + b(A) \in \mathfrak{B}(G)$

$$\left.\frac{d^+}{dt}\right|_{t=0} \langle c_t(\mu_t), f\rangle = \langle C, f\rangle .$$

Let $A_t := 1/t(\mu_t - \varepsilon_e)$ be the Poisson generators approximating $A$: We have $A_t \xrightarrow[t\downarrow 0]{} A \sigma(\mathfrak{E}', \mathfrak{E})$ and hence ([Ha3] *I* § 2) for $s \geqq 0$ $\operatorname{Exp} sA_t \to \operatorname{Exp} sA$.

On the other hand we have $c_t(A_t) = 1/t(c_t(\mu_t) - \varepsilon_e) \xrightarrow[t\downarrow 0]{} C\ \sigma(\mathfrak{E}', \mathfrak{E})$, therefore as above $\operatorname{Exp} sc_t(A_t) = c_t(\operatorname{Exp} sA_t) \xrightarrow[t\downarrow 0]{} \operatorname{Exp} sC$. The limits $\mu$ and $\operatorname{Exp} sA$, $s > 0$, are $\Gamma$-full. If the limits $\operatorname{Exp} sC$, $s > 0$, are also $\Gamma$-full the convergence of types theorem (1.1) yields the relative compactness of $(c_t)_{t>0}$ in $\Gamma$. If this is the case let $c$ be an accumulation point of $\{c_t\}$. Then $c(\operatorname{Exp} sA) = c(\mu_s) = \operatorname{Exp} sC$, $s > 0$, equivalently, $c(A) = C = a(A) + b(A)$.

It remains to prove the fullness of $\operatorname{Exp} sC$: We have $C = a(A) + b(A)$ hence, $G/_{[G,G]}$ being abelian, $\pi(\operatorname{Exp} sC) = \pi[(\operatorname{Exp} sA) * (\operatorname{Exp} sB)] = \pi[(a(\mu_s) * b(\mu_s)] = \pi(c_s(\mu_s))$.The $\bar{\Gamma}$-fullness of $\pi(\mu_s)$, hence of $\pi(c_s(\mu_s))$ implies by assumption (1.3.b) that $\operatorname{Exp} sC \in \mathscr{F}_\Gamma$. □

Let $\mathfrak{Z}(A) = \{\tau \in \Gamma : \tau(A) = tA \text{ for some } t \in \mathbb{R}_+^*\}$ be the decomposability group (cf. [Ha5], [N]). If $\mu$ is $\Gamma$-full and stable w.r.t. $(a_t) \subseteq \Gamma$, then $\mathfrak{Z}(A)$ is a semidirect product of $(a_t)_{t>0} \cong \mathbb{R}$ and the compact group $\mathfrak{I}$. Moreover we can choose $(a_t)$ in such a way that $\mathfrak{Z}$ splits into a direct product of $(a_t)$ and $\mathfrak{I}$ (e.g. [Ha5]). Hence $\operatorname{Aut}(G) \supseteq \Gamma \supseteq \mathfrak{Z}(A) \cong \mathbb{R} \otimes \mathfrak{I}$ if $\mu$ has idempotent $\Gamma$-type. Then under weak conditions we can prove that $\mathfrak{I}$ is a maximal compact subgroup of $\Gamma$. In fact, we have

**1.13 Theorem.** *Let $\mu$ be $\Gamma$-full with idempotent $\Gamma$-type, let $(\mu_t = \operatorname{Exp} tA)$ be the* *corresponding stable semigroup. Let $K \subseteq \Gamma$ be a compact group with Haar measure $\omega_K$, and assume that $B_K = B := \int_K a(A)\, d\omega_K(a) \in \mathfrak{B}(G)$ generates a semigroup of $\Gamma$-full measures.*

*a) Then there exist $b = b_K \in \Gamma$ such that $B = b(A)$, i.e. $B$ belongs to the $\Gamma$-type of $A$.*

*b) Moreover $b^{-1}Kb \subseteq \mathfrak{J}$.*

*c) If $B = B_K$ generates $\Gamma$-full measures for any compact subgroup $K \subseteq \Gamma$ then $\mathfrak{J}$ is a maximal compact subgroup of $\Gamma$.*

**Proof:** Let $(x_i)_{i=1}^{\infty}$ be equidistributed w.r.t. $\omega_K$, i.e. $\langle 1/N \sum_1^N \varepsilon_{x_k}, g\rangle \xrightarrow[N\to\infty]{} \langle \omega_K, g\rangle$, for $g \in C(K)$. For $f \in \mathfrak{E}(G)$ define $g \in C(K) : g(x) := \langle x(A), f\rangle$. For any $N \in \mathbb{N}$ there exist $c_N \in \Gamma$ such that $\sum_1^N x_k(A) = c_N(A)$. Therefore

$$\left\langle \frac{1}{N} \sum_1^N x_k(A), f \right\rangle = \frac{1}{N} \langle c_N(A), f\rangle \xrightarrow[N\to\infty]{} \langle B, f\rangle := \int_K \langle x(A), f\rangle\, d\omega_K(x)\,.$$

Now we have: $\operatorname{Exp} s/N\, c_N(A) = c_N(\mu_{s/N}) = c_N a_{1/N}(\mu_s)$, $s \geqq 0$ and

$$\frac{1}{N} c_N(A) = c_N a_{1/N}(A) \to B\,, \quad \text{hence} \quad c_N a_{1/N}(\mu_s) \to \operatorname{Exp} sB\,.$$

$\mu_s$ and $\operatorname{Exp} sB$ are $\Gamma$-full by assumption, hence again the convergence of types theorem applies and yields the relative compactness of $(c_N a_{1/N})_{N \geqq 1}$. And for any limit point $b$ we have $b(A) = B$. The invariance of $\omega_K$ implies $x(B) = B$, $x \in K$, whence $K \subseteq \mathfrak{J}(B) = \mathfrak{J}(b(A))$ resp. $b^{-1}Kb \subseteq \mathfrak{J} = \mathfrak{J}(A)$ follows. To prove c) assume $\mathfrak{J} \subset K$ for some compact group $K \subset \Gamma$. Then $b^{-1}Kb \subseteq \mathfrak{J} \subseteq K$. Since $\Gamma$ is a Lie group this implies $b^{-1}Kb = K$, hence $\mathfrak{J} = K$. □

**1.14 Remarks.**

1. In the vector space-case this result is known, see [S] Lemma 4.4.

2. If $K$ is finite the assertion holds without fullness conditions. Therefore for any measure with idempotent $\Gamma$-type, for any finite subgroup $K \subseteq \Gamma$ $bKb^{-1} \subseteq \mathfrak{J}$ for some $b \in \Gamma$. (Cf. [P], [M1] for vector spaces).

3. Under weak conditions $\Gamma$-fullness of $\mu$ implies $\Gamma$-fullness of $B_K$. For example, if $\mu = \operatorname{Exp} A \in \mathscr{F}^S$ then the mixed generating distribution $B_K = \int_K a(A)\, d\omega_K(a)$ generates a semigroup in $\mathscr{F}^S$, hence in $\mathscr{F}_\Gamma$. (This is easily seen considering the projections $\pi(\mu)$ onto $G/_{[G,G]}$).

4. Theorem 1.12 shows that the existence of $\Gamma$-full measures with idempotent $\Gamma$--types has strong influence on the structure of $\Gamma$. Especially, $\Gamma$ has maximal compact subgroups (if $\mu \in \mathscr{F}^S$). Moreover, $\Gamma$ is not to small, i.e. $\Gamma \supseteq \mathfrak{Z}(A) \cong \mathbb{R} \otimes \mathfrak{J}$.

5. Assume $\Gamma$ to have an Iwasawa decomposition $\Gamma = \text{NAK}$ with maximal compact $K$, nilpotent $N$ and abelian $A$. Then according to 1.6 we may assume $K = \mathfrak{J}$.

Hence $X = \Gamma/\mathfrak{I} \cong$ NA and the selection function $\Psi$ (resp. $\Phi$) in (1.2′) (resp. (1.1′)) is defined on $NA \times NA \to NA$, and is uniquely determined.

**1.15 Remark.** $G$ is nilpotent and simply connected, hence $C^\infty$-isomorphic to the Lie algebra $\mathfrak{G} \cong \mathbb{R}^d$. In this case there is a 1-1-correspondence to corresponding objects on $\mathfrak{G}$ (cf. [Ha4], [Ha6], [Ha1]): $\Gamma \subseteq \mathrm{Aut}(G) \leftrightarrow \mathring{\Gamma} \subseteq \mathrm{Aut}(\mathfrak{G}) \subseteq \mathrm{GL}(\mathfrak{G})$, where $\mathring{\tau} \in \mathrm{Aut}(\mathfrak{G})$ is the differential of $\tau \in \mathrm{Aut}(G)$. Furthermore $A \in \mathfrak{B}(G) \leftrightarrow \mathring{A} \in \mathfrak{B}(\mathfrak{G})$, $\mu \in \mathrm{M}^1(G) \leftrightarrow \mathring{\mu} \in \mathrm{M}^1(\mathfrak{G})$. $A$ has idempotent $\Gamma$-type iff $\mathring{A}$-has idempotent $\Gamma$-type on the vector space $\mathfrak{G}$. The fullness concept is translated in the following way: Define the class of $\mathring{\Gamma}$-full measures on $\mathfrak{G}$ to be $\mathscr{F}_{\mathring{\Gamma}}(\mathfrak{G}) := \{\mathring{\mu} : \mu \in \mathscr{F}_\Gamma(G)\}$. Then, since $\mathrm{Aut}(G) \leftrightarrow \mathrm{Aut}(\mathfrak{G})$, $\mathrm{M}^1(G) \leftrightarrow \mathrm{M}^1(\mathfrak{G})$ are topological isomorphisms, the convergence of types theorem holds for $\mathscr{F}_{\mathring{\Gamma}}(\mathfrak{G})$ iff it holds for $\mathscr{F}_\Gamma(G)$, i.e. $(\Gamma, \mathscr{F}_{\mathring{\Gamma}}(\mathfrak{G}))$ fulfil (1.3a,b,c) on $\mathfrak{G}$. (For $\Gamma = \mathrm{Aut}(G)$ see [HN]).

So we are well motivated to study $\Gamma$-full measures and $\Gamma$-types on finite dimensional vector spaces in the following § 2.

## § 2 Finite dimensional vector spaces

In the following we restrict our considerations to the case $G = E$, $E$ a finite dimensional vector space, hence the simplest class of nilpotent simply connected Lie groups. The results collected in the sequel serve as a toolbox for the general situation. The results are more or less known ([P], [SP], [S], [M1]), therefore it is sufficient to sketch the proofs.

2.1 Let $E = \mathbb{R}^d$, let $\Gamma \subseteq \mathrm{Aut}(E) = \mathrm{GL}(d, \mathbb{R})$ be a closed subgroup. Let $\mu \in \mathrm{M}^1(E)$ have idempotent $\Gamma$-type, i.e.

$$\Gamma\mu * \Gamma\mu = \Gamma\mu\,. \tag{2.1}$$

(Hence $\mu$ is $\Gamma$-stable in the sense of [P], [S], [M1].)

We fix according to 1.3 sets of $\Gamma$-full measures $\mathscr{F}_\Gamma$, $\mathscr{F}_\Gamma^S$. We call $\mu$ (simply) full if $\mu$ is $\mathrm{GL}(E)$-full, i.e. $\mu$ is not concentrated on a proper linear subspace of $E$.

**Problem A.** Under which conditions does (2.1) imply $\Gamma$-fullness?

**2.2 Proposition.** *Assume* (2.1) *to hold and assume further that*

(2.2.a) *$\Gamma$ acts irreducibly on E, i.e. the $\Gamma$-invariant subspaces are trivial, and*

$$\mu \neq \varepsilon_x\,, \quad x \in G\,. \tag{2.2.b}$$

*Then $\mu$ is full, hence $\Gamma$-full.*

⟦Let $E_1 := \langle \mathrm{supp}\,\mu \rangle$ be the linear subspace generated by supp $\mu$. By assumption for any $x \in E_1$, $x \neq 0$, there exists $b \in \Gamma$ such that $bx \notin E_1$.

1. Assume first $0 \in \mathrm{supp}\,\mu$. Assume further $x \in \mathrm{supp}\,\mu$, $x \neq 0$, $b \in \Gamma$, $bx \notin E_1$. Then $\mathrm{supp}\,(\mu * b(\mu)) \supseteq \mathrm{supp}\,\mu + \mathrm{supp}\,b(\mu) \supseteq \mathrm{supp}\,\mu \cup b(\mathrm{supp}\,\mu) \supseteq \mathrm{supp}\,\mu \cup \{x\}$. Hence $\dim \langle \mathrm{supp}\,(\mu * b(\mu)) \rangle > \dim E_1$. On the other hand $\mu * b(\mu) = c(\mu)$, where $c = \Phi(id, b) \in \Gamma$. Hence $\langle \mathrm{supp}\,(\mu * b(\mu)) \rangle = c(E_1)$, a contradiction.

2. If $0 \notin \operatorname{supp} \mu$ we consider the symmetrization $\nu = \mu * \tilde{\mu}$. We have $\nu \neq \varepsilon_0$ by assumption, and $0 \in \operatorname{supp} \nu$. Obviously $\nu$ has idempotent $\Gamma$-type (since $M^1(E)$ is abelian). Hence by 1. we obtain that $\nu$, and hence $\mu$, are full.]]

Obviously (2.2.b) can be replaced by

(2.2.c) $$\mu \neq \varepsilon_0$$

if $\mu$ is symmetric. Hence we obtain

**2.3 Proposition.** *Under the assumptions* (2.1), (2.2.a) (2.2.c.) *and* (2.2.d) $-id \in \Gamma$ *we obtain that $\mu$ is symmetric and full.*

**Problem B.** Given a symmetric Gauß measure $\mu$ with covariance operator $R$. For which groups $\Gamma$ is (2.1) fulfilled?

Remark that in this case (2.1) is equivalent to

(2.1′) For $a, b \in \Gamma$ there exist $c \in \Gamma$ such that $aRa^* + bRb^* = cRc^*$.

Furthermore $\mu$ is full iff $R \in \mathrm{GL}(E)$.

**2.4 Proposition.** *Let $R \in GL(E)$ as above. Let $a, b \in GL(E)$. Then there exists $c \in GL(E)$ satisfying* (2.1′) ([S]). *$c$ can be constructed as follows: Put $Q := aRa^* + bRb^*$. Then for any orthogonal $U$, $c = Q^{1/2}UR^{-1/2}$ is a solution. The invariance group $\mathfrak{J}(\mu) = R^{1/2}\mathfrak{O}(E)\,R^{-1/2}$ is conjugate to the group of orthogonal transformations $\mathfrak{O}(E)$. The solution $c$ of* (2.1′) *is unique up to a translate of $\mathfrak{J}(\mu)$.*

Let $\Gamma = \Delta^+(E)$ resp $\Delta^-(E)$ be the subgroups of upper resp. lower triangular matrices (with respect to a fixed orthonormal basis). Then, if $a, b \in \Delta^+(E)$ there exists a solution $c \in \Delta^+(E)$. Since $\mathrm{GL}(E)/\mathfrak{O}_0(E) \cong \Delta_1^+(E)$, and $[\mathfrak{O}(E) : \mathfrak{O}_0(E)] = 2$ this solution is unique up to $\mathfrak{O}(E) \cap \Delta^+(E)$.

Here $\Delta_1^+(E)$ is the subgroup of $\Delta^+$ with positive entries in the diagonal. Hence we obtain: Let $\mu$ a be full symmetric Gaussian measure with covariance operator $R \in \mathrm{GL}(E)$. Then Problem B has a solution in the following situations:

a) $\Gamma = \mathrm{GL}(E)$ ([S], [M1])

b) $\Gamma = \Delta^+(E)$ or $\Delta^-(E)$ ([S])

c) $\Gamma = \mathrm{D}(E)$ the group of diagonal matrices, more generally $\Gamma = \sum \oplus \mathrm{GL}(E_i)$, where $E = \sum \oplus E_i$ is a decomposition into (orthogonal) eigenspaces of $R$.

**Problem C.** Under which conditions does (2.1) imply that $\mu$ is Gaussian?

If $\mu$ satisfies (2.1) $\mu$ is infinitely divisible and hence embeddable into a continuous convolution semigroup. Let $Q$ be the symmetric Gaussian part of the generating distribution, let $R$ be the corresponding covariance operator, and let $\eta$ be the Lévy measure. We obtain immediately.

**2.5 Lemma.** *There exists a decomposition $E = E_1 \oplus E_2$ of $E$ such that $Q$ and $\eta$ are concentrated on $E_1$ resp. on $E_2$. Furthermore $Q$ and $\eta$ have indempotent $\Gamma$-type,* *precisely for $a, b \in \Gamma$ there exists $c \in \Gamma$ such that*

(2.5a) $$aRa^* + bRb^* = cRc^* \quad \textit{and}$$

(2.5.b) $$a(\eta) + b(\eta) = c(\eta) \quad \textit{hold}.$$

If $\mu$ has idempotent $\Gamma$-type $\Gamma$ is closely related to the decomposability semigroup $\mathrm{Dec}(\mu) := \{a \in \mathrm{End}(E) : \mu = a(\mu) * \nu$, for some $\nu = \nu_a \in \mathrm{M}^1(E)\}$ (see [MU], [M1]; see also [Ha1] for groups). Hence it is natural to consider the set of projectors in $\Gamma^h$, the closure of $\Gamma$ in the semigroup $\mathrm{End}(E)$.

The following ideas are contained in [M1]:

**2.6 Proposition.** *a) Let $\mu$ be $\Gamma$-full with idempotent $\Gamma$-type. Let $\pi_i$, $i = 1, 2$, be projectors in $\Gamma^h \subseteq \mathrm{End}(E)$, such that $\pi_1 \cdot \pi_2 = 0$, $\pi_1 + \pi_2 = id$. Let $E_i = \pi_i(E)$. Then there exists $c \in \Gamma$ such that the Lévy measure $\eta$ is concentrated on $c(E_1) \cup c(E_2)$. b) If $E_1$ and $E_2$ are $\Gamma$-invariant $\eta$ is concentrated on $E_1 \cup E_2$. And $\mu = \nu_1 \otimes \nu_2$, where $\nu_i \in \mathrm{M}^1(E_i)$ have idempotent $\Gamma|_E$-type.*

⟦Cf. [M1]: Let $a_n, b_n \in \Gamma$ with $a_n \to \pi_1$, $b_n \to \pi_2$. Put $c_n := \Phi(a_n, b_n)$. Then $a_n(\mu) * b_n(\mu) \to \pi_1(\mu) * \pi_2(\mu) = \nu$, a full measure. On the other hand $a_n(\mu) * b_n(\mu) = c_n(\mu)$ with $c_n \in \Gamma$. The convergence of types theorem yields the relative compactness of $\{c_n\}$ with accumulation points $c \in \Gamma$ fulfilling $c\mu = \nu$, hence $\mu = c^{-1}\pi_1(\mu) * c^{-1}\pi_2\mu.$⟧

**2.7 Proposition.** *Let $\mu, \pi_1, \pi_2$ be as in 2.6. Assume there exists $a_0 \in \Gamma$ such that $a_0\pi_1(E) = \pi_1(E)$ and $a_0\pi_2(E) \cap \pi_2(E) = \{0\}$. Then the Lévy measure $\eta$ is concentrated on $c\pi_1(E) = c(E_1)$ for some $c \in \Gamma$.*

⟦Cf. [M1]: According to 2.6 $\eta$ is concentrated on $F_1 \cup F_2$, where $F_i = c\pi_i(E)$, $i = 1, 2$. Put $d := \Phi(\mathrm{id}, a_0)$. We obtain $\mu * a_0(\mu) = \mathrm{d}(\mu)$, hence $\eta + a_0(\eta) = \mathrm{d}(\eta)$. The measure on the left side is concentrated on $F_1 \cup F_2 \cup a_0(F_1) \cup a_0(F_2) = F_1 \cup F_2 \cup a_0(F_2)$ whereas the measure on the right side is concentrated on $\mathrm{d}(F_1) \cup \mathrm{d}(F_2)$. Hence we obtain $\eta(F_2) = 0.$⟧

**2.8 Corollary.** *Let $\mu, \pi_1, \pi_2, a_0$ be as in 2.7. Assume $\pi_1(E)$ to be $\Gamma$-invariant. Then $\eta$ is concentrated on $E_1 = \mu_1(E)$.*

**2.9. Proposition.** *Let $\mu$ be a symmetric (non-full) Gauß measure with idempotent $\Gamma$-type, let $R$ be the covaraince operator. Assume that the kernel $N(R) = E_1 := \{x \in E : Rx = 0\}$ is $\Gamma$-invariant and put $E_2 := R(E)$. Then $\Gamma \subseteq \mathrm{GL}(E_1) \oplus \mathrm{GL}(E_2)$, i.e. $E_2$ is $\Gamma$-invariant. Conversely, let $\mu$ be symmetric Gaussian with convariance operator $R$. Let again $E_1 = N(R)$ be the kernel and $E_2 = R(E)$. Assume $\Gamma = \mathrm{GL}(E_2) \oplus \mathrm{GL}(E_1)$. Then $\mu$ has idempotent $\Gamma$-type.*

⟦Let $a \in \Gamma$. Then $a$ has a representation $a = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}$, $\alpha \in \mathrm{GL}(E_2)$, $\gamma \in \mathrm{GL}(E_1)$, $\beta \in \mathrm{Hom}(E_2, E_1)$. We see immediately that the kernel of $aRa^*$ is $(\beta^* + \gamma^*)(E_1)$.

Put $c := \Phi(\mathrm{id}, a)$ with representation $\begin{pmatrix} \tilde{\alpha} & 0 \\ \tilde{\beta} & \tilde{\gamma} \end{pmatrix}$. Then $cRc^* = R + aRa^*$. Hence, $R$ being positively semidefinite, the kernel $N(cRc^*)$ is contained in the intersection $N(R) \cap N(aRa^*) = E_1 \cap (\beta^* + \gamma^*)(E_1)$. Therefore $(\tilde{\beta}^* + \tilde{\gamma}^*)(E_1) \subseteq E_1$, hence $\tilde{\beta}^* = 0$. Hence $N(cRc^*) = E_1$, this implies $(\beta^* + \gamma^*)(E_1) = E_1$, whence $\beta^* = 0$ follows. Therefore $\beta = 0$. The converse is obvious.]]

**2.10 Proposition.** *Let $\mu \in \mathrm{M}^1(E)$ fulfil* (2.1.) *Assume that, w.r.t. a fixed basis $(e_i)_{i=1}^d$, $\Gamma = \Delta^-(E)$, the group of lower triangular matrices. Then $\mu$ is concentrated on $E_d := \mathbb{R}e_d$ or $\mu$ is Gaussian.*

[[Cf. [S] 6.2. Assume that $\mu$ is not concentrated on the $\Gamma$-invariant subspace $E_d$. Let $E = F_1 \oplus F_2$ be a decomposition, $F_1$ supporting the Gaussian part $Q$, $F_2$ supporting the Lévy measure. According to corollary 2.8 $\eta$ is concentrated on $E_d$. Hence by assumption the Gaussian part is nontrivial. If $\eta \neq 0$, the kernel of the covariance operator contains $E_d$, hence $\ker R \neq \{0\}$. Therefore by proposition 2.9 $\Gamma \subset \mathrm{GL}(R(E))) \oplus \mathrm{GL}(\ker(R))$, a contradiction.]]

**2.11 Proposition.** *Let $\mu \in \mathrm{M}^1(E)$ fulfil* (2.1). *Assume $\Gamma = \mathrm{GL}(E)$, $\dim E \geqq 2$ and $\mu \neq \varepsilon_0$. Then $\mu$ is symmetric, full and Gaussian.*

[[According to proposition 2.2 and 2.3 $\mu$ is full and symmetric if $\mu \neq \varepsilon_0$. Since any projector is contained in $\Gamma^h$ we can apply proposition 2.6 and 2.7 to any decomposition $E = \pi_1(E) \oplus \pi_2(E)$. Therefore $\eta = 0$, i.e. $\mu$ is Gaussian. (Cf. [S], see also [P], [M1]).]]

The next example illustrating the problems A, B, C will be used in § 3 in connection with Heisenberg groups:

**2.12 Proposition.** *Let $n \geqq 1$. Let $E = \mathbb{R}^{2n}$. We use a vector space basis $(X_1, Y_1, \ldots, X_n, Y_n)$. Let $\Gamma := \{a = (a_{ij}) \in \mathrm{GL}(E)$ such that the subdeterminants $\omega_i(a) := a_{2i,2i}a_{2i+1,2i+1} - a_{2i,2i+1}a_{2i+1,2i}$ are independent of $i, 1 \leqq i \leqq n\}$. Assume $\mu \neq \delta_0$. Then, if $\mu$ has idempotent $\Gamma$-type, $\mu$ is full, symmetric and Gaussian.*

[[Let $E_i$ be the two dimensional subspace generated by $X_i, Y_i, 1 \leqq i \leqq n$. Let $\pi_i$ be the projections onto $E_i$. Let $\mu \neq \varepsilon_0$ fulfil (2.1.) Since $\Gamma$ acts irreducibly we obtain by proposition 2.3 that $\mu$ is full and symmetric. Since the projections $\pi_i$ belong to $\Gamma^h \subseteq \mathrm{End}(E)$ we obtain by proposition 2.6 that the Lévy measure $\eta$ is concentrated on a subset $c(E_1) \cup \ldots \cup c(E_n)$, with $c \in \Gamma$. W.l.o.g. we may assume $\mathrm{supp}(\eta) \subseteq \bigcup_1^n E_i$. Put $\eta_i := \eta|_{E_i}$, $1 \leqq i \leqq n$. For fixed $i_0$ let $\Gamma_{i_0} := \{a \in \Gamma : aE_i = E_i$, $1 \leqq i \leqq n$, and $a|_{E_i} = c\,\mathrm{id}_{E_i}, i \neq i_0$, where $c = c(a) \in \mathbb{R}^*\} \cong \mathrm{GL}(\mathbb{R}^2)$. Let $a, b \in \Gamma_{i_0}$. Then $a(\eta) + b(\eta) = c(\eta)$ implies $a(\eta_{i_0}) + b(\eta_{i_0}) = c(\eta_{i_0})$ for some $c \in \Gamma_{i_0}$, i.e. $\eta_{i_0}$ has idempotent $\Gamma_{i_0}$-type. Now by proposition 2.10 $\eta_{i_0} = 0$, $1 \leqq i_0 \leqq n$ hence we obtain $\eta = 0$.]]

In the last example we consider a special group $\Gamma \subseteq \mathrm{GL}(\mathbb{R}^2)$ admitting no full measure without idempotent $\Gamma$-type. The proof seems to be complicated at the first glance. But in § 4 examples of this type turn out ot be quite natural.

**2.13 Proposition.** *Let* $E = \mathbb{R}^2$,

$$let\ \Gamma := \left\{\begin{pmatrix} x & 0 \\ z & x^2 \end{pmatrix} : x \neq 0, z \in \mathbb{R}\right\} \subseteq \Delta^{-2}(E) \subseteq \mathrm{GL}(E).$$

*There is no full measure with idempotent $\Gamma$-type.*

**Proof:** Assume $\mu$ to be full with idempotent $\Gamma$-type. The Fourier transform is representable as $\hat{\mu} = e^{\psi}$ with conditionally positive definite and continuous $\psi$, such that $\psi(0,0) = 0$, $\psi(0,1)$ and $\psi(1,0) \neq 0$, and such that for $a, b \in \Gamma$ there exists $c \in \Gamma$ fulfilling $\psi \circ a^* + \psi \circ b^* = \psi \circ c^*$.

Any one-parameter group $(a_t)$ in $\Gamma$ is conjugate to $\left(\alpha_t = \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix}\right)_{t>0}$, hence w.l.o.g. we assume $\mu$ to be $(\alpha_t)$-stable. I.e. $\psi(t\xi_1, t^2\xi_2) = t^{\alpha}\psi(\xi_1, \xi_2)$, $(\xi_1, \xi_2) \in \mathbb{R}^2$, $t > 0$. Since $\mathbb{R}e_2$ is $\Gamma$-invariant and since $a = \begin{pmatrix} x & 0 \\ z & x^2 \end{pmatrix}$ acts on $\mathbb{R}e_1 = E/\mathbb{R}e_2$ by multiplication with $x(\neq 0)$, the projection of $\mu$ onto the first coordinate is symmetric. Hence $\psi(\xi_1, \cdot) = \psi(-\xi_1, \cdot)$.

The closure $\Gamma^h$ of $\Gamma$ in End $(E)$ contains the nilpotent endomorhphisms $\begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}$, $\sigma \in \mathbb{R}$. The convergence of types theorem easily yields the existence of $c = \begin{pmatrix} t & 0 \\ \varrho & t^2 \end{pmatrix}$, such that

$$(*) \qquad \psi(\xi_1, \xi_2) + \psi(\sigma\xi_2, 0) = \psi(t\xi_1 + \varrho\xi_2, t^2\xi_2).$$

Consider $\xi_2 = 0$, then $t = 1$ follows.

Put $\xi_2 = \xi$, $\xi_1 = 0$ we obtain $\psi(0, 1/\xi) + \psi(\sigma, 0) = \psi(\varrho, 1/\xi)$. Passing to the limit for $\xi \to \infty$ this yields $|\sigma| = |\varrho|$. Put $\xi = \xi_2 > 0$, $\sigma\xi = \xi_1 > 0$, $\varrho = \sigma$, $t = 1$, $a := \psi(1,0)$, $b := \psi(0,1)$. We obtain by (*)

$$(**) \qquad \psi(\xi_1, \xi_2) = a\xi_1^{\alpha} + b\xi_2^{\alpha/2}.$$

But if $\mu$ has idempotent $\Gamma$-type we obtain e.g. $\psi(\xi_1, \xi_2) + \psi(\xi_1 + z\xi_2, \xi_2) = \psi(t\xi_1 + u\xi_2, t^2\xi_2)$. Consider $\xi_2 = 0$ then $t^{\alpha} = 2$, resp. $t = 2^{1/\alpha}$ follows. Furthermore put $\xi_1 = 0$, $\xi_2 = \xi > 0$ and let again $\xi \to \infty$, then $|u| = |z|/2^{1/\alpha}$ follows. Therefore for $\xi_1, \xi_2, z > 0$

$$(***) \qquad \psi(\xi_1, \xi_2) + \psi(\xi_1 + z\xi_2, \xi_2) = 2\psi(\xi_1 + z/2^{2/\alpha}, \xi_2).$$

But it is easily seen that $\psi$ fulfilling (**) cannot fulfil (***). □

## § 3 Completely stable measures

On a vector space $E = \mathbb{R}^d$, $1 \leq d < \infty$, $\mu$ is called completely stable (in the strict sense) if $\mu$ has idempotent $\Gamma$-type for $\Gamma = \mathrm{GL}(E)$. In this case (see § 2. 2.11) we have for $d \geqq 2$: Either $\mu = \varepsilon_0$ or $\mu$ is full and symmetric Gaussian. Our aim is to obtain similar characterizations of full stable Gauss measures for simply connected nilpotent groups. Remark that in contrast to the vector space case 1. not every symmetric Gauß-measure is stable and 2. fullness of Gauß-measures does not imply that the covariance operator is injective.

First step: Appropriate definitions of complete-stability. Let $G$ be a simply connected nilpotent Lie group. Let $\pi : G \to G/[G, G]$ be the canonical projection. The (abelian, simply connected nilpotent) group $G/[G, G] = : \mathfrak{m}$ is a vector space of $\dim \mathfrak{m} = \dim G$ if $G$ is a vector space ($=\mathfrak{m}$) or $\dim \mathfrak{m} \geqq 2$. $a \in \mathrm{Aut}(G)$ induces $\bar{a} \in \mathrm{GL}(\mathfrak{m})$. Let $\bar{\pi} : \mathrm{Aut}(G) \to \mathrm{GL}(\mathfrak{m})$ be the homomorphisms $a \mapsto \bar{a}$. If $\Gamma \subseteq \mathrm{Aut}(G)$ is a subgroup we define $\bar{\Gamma} : = \bar{\pi}(\Gamma) \subseteq \mathrm{GL}(\mathfrak{m})$. We define further $\mathrm{Aut}(\mathfrak{m}) : = \bar{\pi}(\mathrm{Aut}(G)) \subset \mathrm{GL}(\mathfrak{m})$.

**3.1 Definition.** Let $\mu \in \mathrm{M}^1(G)$ have idempotent $\Gamma$-type. $\mu$ is called completely stable if $\Gamma$ is sufficiently large, precisely if

$$\bar{\Gamma} : = \bar{\pi}(\Gamma) = \mathrm{Aut}(\mathfrak{m}) : = \bar{\pi}(\mathrm{Aut}(G)). \tag{3.1}$$

**3.2 Remark.** a) If $G$ is a vector space we have $G = \mathfrak{m}$, $\bar{\Gamma} = \Gamma$. Hence (3.1) is equivalent to $\Gamma = \mathrm{GL}(\mathfrak{m})$ resp. $\Gamma = \mathrm{Aut}(G)$. b) If $\Gamma = \mathrm{Aut}(G)$ then obviously (3.1) is fulfilled. But, as simplest examples show, often there are no measurres $\mu \neq e_e$ with idempotent $\mathrm{Aut}(G)$-type.

There is another vector space closely related to $G$ as pointed out in 1.15:

Let $\mathfrak{G}$ be the Lie algebra, let $\mathring{\mu}$ be the corresponding measure on $\mathfrak{G}$. To $a \in \mathrm{Aut}(G)$ there corresponds an automorphism $\mathring{a} \in \mathrm{Aut}(\mathfrak{G}) \subseteq \mathrm{GL}(\mathfrak{G})$. We choose $\mathscr{F}_\Gamma \subseteq \mathrm{M}^1(G)$ and $\mathscr{F}_{\mathring{\Gamma}} \subseteq \mathrm{M}^1(\mathfrak{G})$ such that we have: $\mu$ is $\Gamma$-full (on $G$) iff $\mathring{\mu}$ is $\mathring{\Gamma}$-full (on $\mathfrak{G}$). $\mathring{a} \in \mathrm{Aut}(\mathfrak{G})$ has a representation $\mathring{a} = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}$, where $\alpha \in \mathrm{Aut}(\mathfrak{m})$ ($\mathfrak{m} \cong \mathfrak{G}/[\mathfrak{G}, \mathfrak{G}] \cong G/[G, G]$), $\beta \in \mathrm{Hom}(\mathfrak{m}, \mathfrak{G}_1)$, $\gamma \in \mathrm{Aut}(\mathfrak{G}_1)$ (with $\mathfrak{G}_1 : = [\mathfrak{G}, \mathfrak{G}]$). We identify the images $\mathring{a} \mapsto \alpha$ and $a \mapsto \bar{a}$.

**3.3 Proposition.** *Assume* $\Gamma \subseteq \mathrm{Aut}(G)$ *to be a closed subgroup and* $\mu$ *to have idempotent* $\Gamma$*-type. Assme further that* (a) $\bar{\Gamma} = \{\bar{a} = \alpha \in \mathrm{GL}(\mathfrak{m}) : a \in \Gamma\}$ *acts irreducibly on* $\mathfrak{m}$. *Then* $\mu$ *is full, hence* $\Gamma$*-full, or* $\mu$ *is concentrated on a coset* $x[G,G]$. *If moreover* (b) $-\mathrm{id}_{\mathfrak{m}} \in \bar{\Gamma}$ *then* $\mu$ *is full or* $\mu$ *is concentrated on* $[G, G]$.

⟦Let $\pi$ be the natural projection $G \to G/_{[G,G]} = : \mathfrak{m}$. $\pi(\mu) \neq \varepsilon_{\bar{x}}$, $\bar{x} \in \mathfrak{m}$. $\bar{\Gamma}$ is irreducible by (a). Hence $\pi(\mu)$ is full on $\mathfrak{m}$ according to proposition 2.2. Therefore ([HN]) $\mu$ is full on $G$. hence $\Gamma$-full (cf. Definition 1.3). If moreover (b) holds proposition 2.3 yields that $\pi(\mu)$ is full provided $\pi(\mu) \neq \varepsilon_0$.⟧

A straightforward generalization is obtained in the following way:

**3.4 Proposition.** *Let* $H$ *be a* $\Gamma$*-invariant connected closed subgroup,* $H \supseteq [G, G]$. *Assume further that there is no proper* $\Gamma$*-invariant closed subgroup between* $H$

*and G. Then, if $\mu$ has idempotent $\Gamma$-type, $\mu$ is concentrated on a coset of H or the projection is full on $G/H$.*

⟦Let $\pi_1 : G \to G/H = : \mathfrak{n}$ be the canonical projection. Assume $\pi_1(\mu) \neq \varepsilon_{\bar{x}}$, $\bar{x} \in \mathfrak{n}$. $\Gamma$ acts irreducibly on $\mathfrak{n}$ by assumption. Hence $\pi_1(\mu)$ is full on $\mathfrak{n}$ by proposition 2.2.⟧

**3.5 Remark.** It is easily shown by examples that in general fullness on $G/H$ does not imply fullness on $G$ (if $H \neq [G, G]$).

**3.6 Corollary.** *Let $[G, G]$ be the largest proper characteristic closed connected subgroup of G. Then any completely stable measure $\mu$ is full on G or concentrated on a coset of $[G, G]$.*

⟦Follows immediately from 3.3⟧

**3.7 Proposition.** *Let $\mu$ be $\Gamma$-full with idempotent $\Gamma$-type. Let $(\mu_t = \operatorname{Exp} tA)$ be the corresponding c.c.s. Let H be a closed proper $\Gamma$-invariant connected subgroup $\supseteq [G, G]$, let $\mathfrak{H} \subseteq \mathfrak{G}$ be the corresponding ideal. Assume that, with respect to a fixed basis, the induced group $\bar{\Gamma}$ on $\mathfrak{n} : = \mathfrak{G}/\mathfrak{H}$ is the group of lower triangular matrices $\Delta^-(\mathfrak{n})$. Then we obtain a decomposition $A = A_1 + A_2$, where $A_1$ is Gaussian and the Lévy measure of $A_2$ is concentrated on H. Therefore, the projection of $\mu$ to $G/H \cong \mathfrak{n}$ is (full) Gaussian.*

**Proof:** $\mu$ is embeddable into a c.c.s. $(\operatorname{Expt} A = \mu_t)$, where $A$ has idempotent (infinitesimal) $\Gamma$-type (according to 1.11). Therefore, we consider the corresponding semigroup $(\mathring{\gamma}_t = \operatorname{Exp} t\mathring{A})$ on $\mathfrak{G}$. $\mathring{A}$ has idempotent $\mathring{\Gamma}$-type (1.4). Let $\pi_1 : \mathfrak{G} \to \mathfrak{n} : = \mathfrak{G}/\mathfrak{H}$ be the canonical projection. Proposition 2.10 implies that $\pi_1(\mathring{\gamma}_t)$ is full and Gaussian.

Let $A_1$ resp. $\mathring{A}_1$ be the Gaussian part of $A$ resp. $\mathring{A}$. Then we have $\pi_1(\operatorname{Exp} t\mathring{A}) = \pi_1(\operatorname{Exp} t\mathring{A}_1)$. Therefore the Lévy measure $\mathring{\eta}$ of $\mathring{A}$ is concentrated on $\mathfrak{H}$, hence the Lévy measure $\eta$ of $A$ is concentrated on $H$. □

**3.8 Corollary.** *Let $\mu$ be completely stable. Assume that $[G, G]$ is the largest proper closed characteristic connected subgroup. Assume $\mu$ is not concentrated on a coset of $[G, G]$. Then $\mu$ is full, hence embeddable into a stable c.c.s. $(\mu_t = \operatorname{Exp} tA)$, The Lévy measure $\eta_A$ of A is concentrated on $[G, G]$, the Gaussian part $A_1$ generates a full Gauß-semigroup on $G/[G, G]$.*

⟦Put $\Gamma \subseteq \operatorname{Aut}(G)$ such that $\bar{\Gamma} = \operatorname{Aut}(\mathfrak{m})$, $H : = [G, G]$. Proposition 3.2 resp. Corollary 3.6 yield fullness of $\mu$. By Proposition 3.7 we obtain the desired decomposition.⟧

In § 4 we will consider concrete examples and show that we have the means to determine the possible completely stable measures. For certain classes of groups, e.g. the Heisenberg groups we obtain the expected characterization: Full completely stable measures are (stable and) Gaussian. This rises the inverse question: Which (stable) Gaussian measures are completely stable? For example we obtain

**3.9 Proposition.** *Assume that to any complement* $\mathfrak{m}_1$ *of* $[\mathfrak{G}, \mathfrak{G}]$ *in* $\mathfrak{G}$ *there corresponds a subgroup* $\Gamma \subseteq \mathrm{Aut}(G)$, *hence* $\mathring{\Gamma} \subseteq \mathrm{Aut}(\mathfrak{G})$ *such that*

(3.9*a*) $\mathfrak{m}_1$ *is* $\mathring{\Gamma}$*-invariant and*

(3.9*b*) $\mathrm{Aut}(\mathfrak{m}) = \bar{\mathring{\Gamma}} - \Delta^-(\mathfrak{m})$ *with respect to some basis of* $\mathfrak{m} \simeq \mathfrak{m}_1$. *Then any full stable Gauß-measure* $\pi$ *on* $G$ *is completely stable.*

**Proof:** Let $(\mu_t = \mathrm{Exp}\, tA)$ be a full stable Gauß-semigroup on $G$. Then, with respect to some adapted basis $\{X_i\}$ of $\mathfrak{G}$, $\mathring{A} = \sum c_i X_i^2$. And with respect to a group $(\mathring{a}_t) \subseteq \mathrm{Aut}(\mathfrak{G})$, $\mathring{a}_t(\mathring{A}) = t\mathring{A}$. Since $[\mathfrak{G}, \mathfrak{G}]$ is characteristic, this implies that $c_i = 0$ for $X_i \in [\mathfrak{G}, \mathfrak{G}]$, hence $\mathring{A}$ is concentrated on a complement $\mathfrak{m}_1$ of $[\mathfrak{G}, \mathfrak{G}]$. Let $X_1, \ldots, X_s$ span $\mathfrak{m}_1$.

Consider the projection onto $\mathfrak{m} = G/[G, G] \cong \mathfrak{G}/[\mathfrak{G}, \mathfrak{G}] \cong \mathfrak{m}_1$:

Let $a, b \in \Gamma \subseteq \mathrm{Aut}(G)$, let $\bar{a}, \bar{b}$ be the corresponding automorphisms on $\mathfrak{m}$, hence $\bar{a}, \bar{b} \in \Delta^-(\mathfrak{m})$. According to proposition 2.4 there exists $\bar{c} \in \Delta^-(\mathfrak{m})$, such that $\bar{a}(\pi(A)) + \bar{b}(\pi(A)) = \bar{c}(\pi(A))$.

But $\bar{\Gamma} = \Delta^-(\mathfrak{m}) = \bar{\mathring{\Gamma}}$, hence there exist $\mathring{c} \in \mathring{\Gamma}$, such that $\bar{\mathring{c}}(\mathring{A}) = \mathring{\pi}\mathring{c}(\mathring{A})$. And since $\mathfrak{m}_1 \cap [\mathfrak{G}, \mathfrak{G}] = \{0\}$, and since $\mathring{A}$ is concentrated on $\mathfrak{m}_1$ and since $\mathfrak{m}_1$ is $\Gamma$-invariant we have $\mathring{a}(\mathring{A}) + \mathring{b}(\mathring{A}) = \mathring{c}(\mathring{A})$, hence $a(A) + b(A) = c(A)$ as asserted. □

Analogously we obtain

**3.10 Proposition.** *The assertion of* 3.9 *remains valid if we replace* (3.9*b*) *by*

$$\mathrm{Aut}(\mathfrak{m}) = \bar{\mathring{\Gamma}} = \mathrm{GL}(\mathfrak{m})\,. \tag{3.10}$$

## § 4 Some examples

In the following we discuss in details completely stable measures on some nilpotent Lie groups. Let $G$ be a nilpotent simply connected Lie group with Lie algebra $\mathfrak{G}$. Let $[G, G]$ resp. $[\mathfrak{G}, \mathfrak{G}]$ be the commutator and $\mathfrak{m} := G/[G, G] \cong \mathfrak{G}/[\mathfrak{G}, \mathfrak{G}]$. Let $\pi$ resp. $\mathring{\pi}$ be the canonical homomorphisms $\pi : G \to \mathfrak{m}$ resp. $\mathring{\pi} : \mathfrak{G} \to \mathfrak{m}$.

Let $\Gamma$ be a closed subgroup of $\mathrm{Aut}(G)$, let $\mathring{\Gamma}$ be the corresponding object in $\mathrm{Aut}(\mathfrak{G}) \subseteq \subseteq \mathrm{GL}(\mathfrak{G})$ and let $\bar{\Gamma} \subseteq \mathrm{GL}(\mathfrak{m})$ be the group induced on (the vector space) $\mathfrak{m}$. We assume throughout $\bar{\Gamma} = \mathrm{Aut}(\mathfrak{m})$, i.e. if $\bar{\pi} : \mathrm{Aut}(G) \to \mathrm{GL}(\mathfrak{m})$ is the canonical homomorphism, $\bar{\pi}(\Gamma) = \bar{\Gamma} = \bar{\pi}(\mathrm{Aut}(G))$ (hence $\cong \bar{\mathring{\pi}}(\mathring{\Gamma})$).

Let $\mu \in \mathrm{M}^1(G)$ have idempotent $\Gamma$-type. Then by 1.9 $\mu$ is embeddable into a continuous convolution semigroup $(\mu_t = \mathrm{Exp}\, tA)$. Let $(\mathring{\gamma}_t = \mathrm{Exp}\, t\mathring{A})$ be the corresponding convolution semigroup on (the vector space) $\mathfrak{G}$ (cf. 1.15, 3.2 ff).

### Example A. Heisenberg groups.

We start with $G = H_1$ the threedimensional Heisenberg group with Lie algebra $\mathfrak{G} = \mathfrak{H}_1$ generated by $\{X, Y, Z : [X, Y] = Z\}$.

A1. The automorphism group $\mathrm{Aut}(\mathfrak{H}_1)$.

With respect to the basis $\{X, Y, Z\}$ $\mathring{a} \in \mathrm{Aut}(\mathfrak{H}_1)$ has a matrix representation $\mathring{a} = \begin{pmatrix} \bar{a} & 0 \\ c & \gamma \end{pmatrix}$ where $\bar{a} \in \mathrm{GL}(\mathbb{R}^2)$, $c \in \mathbb{R}^2$ (representing a homomorphism $\mathbb{R}^2 \to \mathbb{R}$) and $\gamma = \det \bar{a} \in \mathbb{R}^*$. Note, in this case $\bar{\Gamma} \cong \mathrm{GL}(\mathbb{R}^2)$, $\mathfrak{m} \cong \mathbb{R}^2$.

A2. Either $\mu$ is full and $\pi(\mu)$ is symmetric Gaussian or $\mu$ is concentrated on the centre $[G, G] = \mathbb{R}Z$.

⟦Apply corollary 3.8.⟧

A3. The Lévy – measure $\eta$ of $A$ is concentrated on $[G, G]$.

⟦Again by corollary 3.8.⟧

A4. $\mu$ is either (1) a full stable Gaussian measure with generator concentrated on a complement of $[G, G]$ or (2) $\mu$ is a symmetric stable measure concentrated on the one-dimensional subgroup $[G, G]$.

⟦According to A.1 – A.3 the generator $A$ is representable as $A = Q + L$, where $Q$ is Gaussian, w.l.o.g. concentrated on $\mathfrak{m} = \{\mathbb{R}X + \mathbb{R}Y\}$ and $L$ is the corresponding term with Lévy measure $\eta$ concentrated on $[G, G]$. Let $a \in \mathrm{Aut}(G)$. Then the restriction $a|_{[G,G]}$ is the homothetical transformation $x \mapsto \gamma \cdot x$ with $\gamma = \det \bar{a}$. Since $\gamma$ runs through $\mathbb{R} \setminus \{0\} = \mathbb{R}^*$ and since $L$ has idempotent $\Gamma$-type, we obtain that $L$ generators on $[G, G] \cong \mathbb{R} \cdot Z \cong \mathbb{R}$ an one-dimensional symmetric stable measure. Let $c$ be the stability index.

Assume now that the Gaussian part $Q$ is non-trivial. We consider subsets of automorphisms

$$\mathring{u}(t) = \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ * & * & t \end{pmatrix}, \quad t > 0 \quad \text{and} \quad \mathring{v}(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ * & * & s \end{pmatrix}, \quad s > 0 .$$

W.l.o.g. we assume $Q = \sigma X^2 + \varrho Y^2$, $\sigma, \varrho > 0$. $Q$ has idempotent $\Gamma$-type therefore $\mathring{u}(t)(Q) + \mathring{v}(s)(Q) = \mathring{w}(s, t)(Q)$

$$\text{with} \quad \mathring{w}(s, t) = \begin{pmatrix} (t^2 + 1)^{1/2} & 0 & 0 \\ 0 & (s^2 + 1)^{1/2} & 0 \\ * & * & z \end{pmatrix}, \quad \text{where} \quad z = ((t^2 + 1)(s^2 + 1))^{1/2} .$$

On the other side, since $\mathring{u}(t)$, $\mathring{v}(s)$, $\mathring{w}(s, t)$ act on $(\mathfrak{G}, \mathfrak{G})$ by multiplication with $s$, $t$ resp. $z$, this yields:

If $Q \neq 0$ and $L \neq 0$ we obtain for $s > 0$, $t > 0$: $(s_c + t_c)^{1/2} = (t^2 + 1)^{1/2}$ . . $(s^2 + 1)^{1/2}$, a contradiction. But $L$ is non-Gaussian, since $A$ is stable and the Gaussian part $Q$ on $\mathfrak{m}$ is non-zero. Hence $L = 0$.⟧

A5. Let $\mu$ be full, hence Gaussian, and completely stable. If we assume w.l.o.g. that the Gauß generator on $\mathfrak{G}$ is supported by $\mathbb{R}X + \mathbb{R}Y = \mathfrak{m}$, then $\mathfrak{m}$ is $\mathring{\Gamma}$-invariant, hence $\mathring{a} \in \mathring{\Gamma}$ has a representation $\mathring{a} = \begin{pmatrix} \bar{a} & 0 \\ 0 & \gamma \end{pmatrix}$ (cf. proposition 2.9). This means $\Gamma$ acts without inner automorphisms.

So, even in the simplest non-vector space case there exist no full measures with idempotent $\mathrm{Aut}(G)$-type. This justifies our definition 3.1 of complete stability.

A6. Consider now the $2n+1$ – dimensional Heisenberg groups $H_n$ with Lie algebra $\mathfrak{H}_n$ generated by $\{X_i, Y_i, 1 \leqq i \leqq n, [X_i, Y_i] = Z\}$. Then, $\mathring{a} \in \mathrm{Aut}(\mathfrak{H}_n)$ has a matrix representation $\mathring{a} = \begin{pmatrix} \bar{a} & 0 \\ c & \gamma \end{pmatrix}$, where $\bar{a}$ runs through the subgroup of $\mathrm{GL}(\mathbb{R}^{2n})$ considered in proposition 2.12. This proposition yields again the dichotomy: either $\mu$ is full or concentrated on $[G, G]$. Now the same considerations as in A1–A5 yield:

Either $\mu$ is concentrated on $[G, G] = \mathbb{R}Z$ and symmetric stable or $\mu$ is full stable and Gaussian.

**B. Groups of Type H.** These are step-two nilpotent groups which can be represented as direct products of „baby-Heisenberg groups". For more information on the structure of type-$H$ algebras and their automorphisms see e.g. [CDKR], [Ko], [FKS]. Along the lines of example $A$ one can show that also in this case completely stable measures are either full Gaussian (and stable) or concentrated on the commutator subgroup $(G, G]$.

C. Next we consider an example with 2-dimensional centre: Let $G$ be the group with Lie algebra $\mathfrak{G} \cong \mathbb{R}^5$ generated by $\{X_1, X_2, X_3, Y_1, Y_2\}$ with $[X_1, X_2] = Y_1$, $[X_1, X_3] = Y_2$ (all other commutator relations zero).

Let $\mu$ be completely stable.

We obtain for $\mathring{a} \in \mathrm{Aut}(\mathfrak{G})$ a representation $\mathring{a} = \begin{pmatrix} \bar{a} & 0 \\ \gamma & \beta \end{pmatrix}$, where $\bar{a} = (a_{ij}) \in \mathrm{GL}(\mathbb{R}^3)$, $\beta = (b_{ij}) \in \mathrm{GL}(\mathbb{R}^2)$, $\gamma = (c_{ij}) \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R}^2)$, fulfilling the following conditions:

$$A_{33} = a_{11}a_{22} - a_{21}a_{12} = b_{11}, \quad A_{23} = a_{11}a_{32} - a_{12}a_{31} = b_{21},$$

$$A_{32} = b_{12}, \quad A_{22} = b_{22},$$

and (since $[aX_2, aX_3] = 0$) $A_{31} = A_{21} = 0$.

Here $A_{ij}$ are the 2-dimensional subdeterminants.

There, as in the preceeding examples. $\beta$ is represented as a function of $\bar{a}$. (This reflects the fact that $\mathfrak{G}$ is stratified.)

It is easily seen that $\bar{a} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & \alpha_1 & \end{pmatrix}$ corresponds to $\beta = \alpha_1$. Hence, if $\bar{a}$ runs through $\mathrm{Aut}(\mathfrak{m})$, $\beta$ runs through $\mathrm{GL}(\mathbb{R}^2)$. Note that $\mathrm{Aut}(\mathfrak{m}) = \{\bar{a}\} \supseteq \Delta^-(\mathbb{R}^3)$ and the permutation $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ exchanging $X_2$ and $X_3$. Hence $\mathrm{Aut}(\mathfrak{m})$ acts irreducibly on $\mathfrak{m}$.

Therefore we obtain

C1. $\mu$ is full or concentrated on $[G, G] \cong \mathbb{R}Y_1 + \mathbb{R}Y_2$.

⟦Since $\bar{\Gamma} = \operatorname{Aut}(\mathfrak{m})$ acts irreducibly on $\mathfrak{m} = \mathbb{R}^3$.⟧

C2. If $\mu$ is full, $\mu$ is Gaussian and symmetric.

⟦We obtain as before that $\pi(\mu)$ is full Gaussian on $\mathfrak{m}$, and that the Lévy measure $\eta$ is concentrated on $[G, G] \cong \mathbb{R}^2$. But $\eta$ has idempotent $\Gamma$-type (lemma 2.5) and $\Gamma|_{[G,G]} \cong \operatorname{GL}(\mathbb{R}^2)$. Hence $\eta = 0$ (proposition 2.11).⟧

C3. Let $\mu$ be full (hence Gaussian). If we assume w.l.g. that the generator is concentrated on $\mathbb{R}X_1 + \mathbb{R}X_2 + \mathbb{R}X_3 = \mathfrak{m}$. Then $\mathring{\Gamma} = \left\{\mathring{a} = \begin{pmatrix} \bar{a} & 0 \\ 0 & \beta \end{pmatrix} : \bar{a} \in \operatorname{Aut}(\mathfrak{m})\right\}$.

⟦Follows immediately from proposition 2.9.⟧

C4. Let $\mu$ be concentrated on $[G, G]$ and assume $\mu \neq \varepsilon_0$. Then $\mu$ is symmetric and Gaussian on $[G, G]$.

⟦$\mu$ has idempotent $\Gamma$-type and $\Gamma|_{[G,G]} \cong \operatorname{GL}(\mathbb{R}^2)$. Apply proposition 2.11.⟧

D. In the next example we consider a stratified Lie algebra with non-trivial proper characteristic ideal $\mathfrak{H} \supset [\mathfrak{G}, \mathfrak{G}]$: $G := H_1 \oplus \mathbb{R}$, $\mathfrak{G} = \mathbb{R}^4$ with generators $X, Y, Z, U$ such that $[X, Y] = Z$, all other commutators zero. We have $[\mathfrak{C}, \mathfrak{C}] = \mathbb{R}Z \subsetneqq \mathfrak{C}(\mathfrak{G}) = \mathbb{R}Z + \mathbb{R}U \subset \mathfrak{G}$. Therefore $\mathfrak{m} = \mathbb{R}X + \mathbb{R}Y + \mathbb{R}U$. Hence $\operatorname{Aut}(\mathfrak{G}) = \left\{\mathring{a} = \begin{pmatrix} \alpha & 0 \\ \gamma & \beta \end{pmatrix} : \alpha \in \operatorname{GL}(\mathfrak{m}), \gamma \in \operatorname{Hom}(\mathfrak{m}, \mathfrak{Z}), \beta \in \operatorname{GL}(\mathfrak{Z}) : \beta = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} : \right.$ $\left. : b_{22} = \det \alpha\right\}$. Let $\pi : \mathfrak{G} \to \mathfrak{G}/[\mathfrak{G}, \mathfrak{G}] = \mathfrak{m} = \mathbb{R}X + \mathbb{R}Y + \mathbb{R}U$ as before, and $\pi^* : \mathfrak{G} \to \mathfrak{G}/\mathfrak{Z}(\mathfrak{G}) = : \mathfrak{n} = \mathbb{R}X + \mathbb{R}Y$. We obtain $\operatorname{Aut}(\mathfrak{m}) = \left\{\bar{\pi}(\mathring{a}) = \begin{pmatrix} \alpha & 0 \\ c & b_1 \end{pmatrix}, \right.$ $\left. c = (\gamma_{11}, \gamma_{12}), b_{11} \in \mathbb{R} \setminus \{0\}, \alpha \in \operatorname{GL}(\mathbb{R}^2)\right\}$.

D1. $\pi^*(\mu) = \varepsilon_0$ or $\pi^*(\mu)$ is full symmetric Gaussian on $\mathfrak{n}$.

⟦As in the preceeding examples.⟧

D2. The Lévy measure $\eta$ of $\mu$ is concentrated on the centre $\mathfrak{Z}(G) \cong \mathbb{R}U + \mathbb{R}Z$

⟦As in the preceeding examples.⟧

D3. $\pi(\mu)$ is concentrated on $\mathfrak{Z} = \mathbb{R}U$ or $\pi(\mu)$ is full Gaussian.

⟦According to D1 the Lévy measure of $\pi(\mu)$ is concentrated on $\mathfrak{Z} = \mathbb{R}U$. Let $\pi(A) = : B$ be the generating distribution of $\pi(\mu)$, let $B = B_1 + B_2$ be the decomposition into the Gaussian part $B_1$ and the non Gaussian $B_2$. Assume that $\pi(\mu)$ is full. W.l.o.g. we assume further that $B_1$ is concentrated on $\mathfrak{n} = \mathbb{R}X + \mathbb{R}Y$. But $B_1$ has idempotent $\bar{\Gamma}$-type, hence (proposition 2.9) we obtain $\bar{\Gamma} \subseteq \left\{\bar{a} = \begin{pmatrix} \alpha & 0 \\ 0 & b_{11} \end{pmatrix}\right\}$, i.e.

$c = 0$, a contradiction to the assumption $\bar{\Gamma} = \mathrm{Aut}(\mathfrak{m})$. Hence $B_1$ is full Gaussian on $\mathfrak{m} \cong \mathbb{R}^3$, and therefore we obtain $\pi(\eta) = 0$.⟧

D4. Let $\mu$ be full. Then $\mu$ is Gaussian.

⟦According to D3 $\pi(\mu)$ is Gaussian, hence the Lévy measure $\eta$ is concentrated on $[G, G] \cong \mathbb{R}Z$. But $G$ contains the Heisenberg group $H_1$. Hence we repeat the proof of $A$ to obtain $\eta = 0$.⟧

E. Up to now our examples were step-two nilpotent groups resp. algebras. To obtain examples of groups with longer descending central series we consider a special class of groups the automorphism groups of which are not too complicated. A(nilpotent) Lie algebra $\mathfrak{G}$ (resp. the corresponding simply connected group $G$) is called threadlike (filiform) if there exists a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{G}$ such that

$$[X_1, X_i] = X_{i+1}, \quad 2 \leqq i \leqq n \quad (\text{put } X_j := 0, j > n)$$
$$[X_2, X_3] \in \Gamma_5 \quad \text{and}$$
$$[X_i, X_j] \in \Gamma_{i+j-1}, \quad 2 \leqq i < j \leqq n, \quad (i, j) \neq (2, 3),$$

hold. Here $\Gamma_k$ is the subspace generated by $\{X_k, \ldots, X_n\}$. Indeed, $\Gamma_k$ is an ideal then, and we have (cf. [MR] § 8):

1. The descending central series is given by
$$\mathfrak{G}_i = \Gamma_{i+2}, \quad i \geqq 1.$$
2. If $\dim \mathfrak{G} = n \geqq 4$
$\Gamma_2$ is also characteristic. Hence $\dot{a} \in \mathrm{Aut}(\mathfrak{G})$ has a matrix representation of lower triangular matrices, i.e.
$$\mathrm{Aut}(\mathfrak{G}) \subseteq \Delta^{-1}(\mathfrak{G}).$$
In the following we will always suppose $n = \dim \mathfrak{G} \geqq 4$.
3. Especially $\mathfrak{m} = \mathbb{R}X_1 + \mathbb{R}X_2$ is a complement of the commutator
$$[\mathfrak{G}, \mathfrak{G}] = \Gamma_3.$$
The essential group for our considerations is therefore
$$\bar{\Gamma} = \mathrm{Aut}(\mathfrak{m}) \subseteq \Delta^{-1}(\mathbb{R}^2).$$
4. Let $\dot{a} = (a_{ij}) \in \mathrm{Aut}(\mathfrak{G})$. The relations
$$[X_1, X_i] = X_{i+1} \quad \text{imply} \quad a_{11}a_{22} = a_{33}, \ldots, a_{11}a_{n-1,n-1} = a_{nn}.$$
Hence, put $a_{11} = x$, $a_{22} = y$, $a_{21} = z$, i.e. $\bar{a} = \begin{pmatrix} x & 0 \\ z & y \end{pmatrix}$, then the main diagonal of $\dot{a}$ is given by $x, y, xy, x^2y, \ldots x^{n-2}y$.
5. The relations $[X_1, X_i] = X_{i+1}$, $1 \leqq i < n$ imply further commutator relations, e.g.:
$$[X_2, X_4] = [X_1, [X_2, X_3]],$$
$$[X_2, X_5] = [X_1, [X_2, X_4]] - [[X_1, X_2], X_4] = [X_1, [X_2, X_4]] - [X_3 . X_4]$$
etc.

The algebra structure is therefore determined by

$$[X_2, X_3], \quad [X_3, X_4], \ldots .$$

6. $\Gamma_i = \mathfrak{G}_{i+2}$ are characteristic in $\mathfrak{G}$. Especially we have $\mathfrak{G}/\Gamma_4 = \mathfrak{G}/\mathfrak{G}_2 \cong \mathfrak{H}_1$, the Heisenberg algebra.
7. Let $\mathring{a} \in \mathrm{Aut}(\mathfrak{G})$. Since $\mathring{a}X_{i+1} = \mathring{a}[X_1, X_i] = [\mathring{a}X_1, \mathring{a}X_i] =$
$= [\mathring{a}X_1, \mathring{a}[X_1, X_{i-1}]] = \ldots$
the matrix $\mathring{a}$ is determined by the column vectors $(a_{i1})$ and $(a_{i2})$.

**E1 Example.** (Free threadlike algebras).

$$[X_1, X_i] = X_{i+1}, \quad 2 \leqq i \leqq n-1,$$

(all other commutators zero). If $n = 7$ we obtain the algebra $G(0, 0, 0, 0)$ in the list [MR] § 8. The next examples also belong to this list:

**E2 Example.** $n = 7$ ($G(1, 0, 0, 0)$ in [MR] § 8).

$$[X_1, X_i] = X_{i+1}, \quad 2 \leqq i \leqq 6,$$

$$[X_2, X_3] := X_5, \quad \text{hence} \quad [X_2, X_4] = X_6,$$

$$[X_2, X_5] := 0, \quad [X_4, X_5] = 0,$$

(and all other commutators zero).

**E3 Example.** $n = 7$ ($G(0, 1, 0, 1)$ in [MR] § 8).

$$[X_1, X_i] = X_{i+1}, \quad 2 \leqq i \leqq 6,$$

$$[X_2, X_3] := X_5, \quad \text{hence} \quad [X_2, X_4] = X_7,$$

$$[X_3, X_4] := X_7, \quad \text{hence} \quad [X_4, X_5] = 0,$$

(and all other commutators zero).

Let $\mathring{a} = (a_{ij}) \in \mathrm{Aut}(\mathfrak{G})$ with $a_{11} = x, a_{22} = y, a_{21} = z,$ hence $a_{ii} = x^{i-2}y$, $3 \leqq i \leqq 6$.

**In Example E1** we obtain only further relations like $a_{11}a_{32} = a_{43}$, $a_{11}a_{42} = a_{53}, \ldots$, but no restrictions on $a_{11} = x$, $a_{22} = y$, $a_{21} = z$.

Therefore in this case

$$\mathrm{Aut}(\mathfrak{m}) = \Delta^-(\mathfrak{m}) = \Delta^-(\mathbb{R}^2).$$

**In Example E2** the additional relation $[X_2, X_3] = X_5$ implies $a_{22}a_{33} = a_{55}$, hence $xy^2 = x^3y$, resp. $y = x^2$. Therefore in this case we obtain

$$\mathrm{Aut}(\mathfrak{m}) = \left\{ \begin{pmatrix} x & 0 \\ z & x^2 \end{pmatrix}, \quad x \in \mathbb{R}^*, \quad z \in \mathbb{R} \right\}$$

and $a_{ii} = x^i$, $1 \leqq i \leqq 7$.

**In Example E3** the additional relations $[X_2, X_3] = X_6$ and $[X_3, X_4] = X_7$ imply $y = x^2$ as above and we obtain $x^2x^5 = x^6$, hence $a_{ii} = 1$, $1 \leqq i \leqq n$.

Therefore $\mathrm{Aut}(\mathfrak{G})$ and $\mathrm{Aut}(\mathfrak{m}) = \left\{\begin{pmatrix}1 & 0\\ z & 1\end{pmatrix}\right\}$ are groups of unipotent matrices and $\mathfrak{G}$. is not contractible ([MR] § 8).

Let now $\mu$ be full and completely stable on $G$. Let $\pi(\mu)$ be the projection onto $\mathfrak{m}$

**In Example E1** $\pi(\mu)$ has idempotent $\Delta^-(\mathfrak{m})$-type, therefore $\pi(\mu)$ is full, symmetric Gaussian on $\mathfrak{m}$. Hence the Lévy measure $\eta$ of $\mu$ is concentrated on $[G, G]$. On the other hand the Gaussian part of $\mu$ is concentrated on a complement $\mathfrak{m}_1$ of $\mathfrak{G}_1$. Assume w.l.o.g. that $\mathfrak{m}_1 = \mathfrak{m}$. Then according to proposition 2.9 $\mathfrak{m}$ is $\mathring{\Gamma}$ – invariant, therefore

$$\mathring{\Gamma} = \left\{\begin{pmatrix} x & 0 & & 0 \\ z & y & xy & \\ & & \ddots & \\ 0 & 0 & & x^{n-2}y \end{pmatrix}\right\}.$$

Consider now succesively the projections $\pi_i : \mathfrak{G} \mapsto \mathfrak{G}/\mathfrak{G}_i$, $i = 2, \ldots, n$, we obtain by the same arguments as in example $A$ that $\eta = 0$. (Note, e.g. that $\mathfrak{G}/\mathfrak{G}_2 = \mathfrak{H}_1$ is the Heisenberg algebra.)

**In Example E2** there is no full Gaussian measure on $\mathfrak{m}$ with idempotent $\mathrm{Aut}(\mathfrak{m})$--type. This follows immediately since the one-parameter groups in $\mathrm{Aut}(\mathfrak{m})$ are conjugate to $\begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix}$. But proposition 2.13 shows that there is no full measure with idempotent $\mathring{\Gamma}$-type on $\mathfrak{m}$. Indeed, if we don't suppose fullness, such measures are concentrated on $\mathbb{R}e_2$. Therefore we have: There is no full completely stable measure. The completely stable measures are concentrated on $\Gamma_2$ and are stable w.r.t. the dilation group $(a_t : X_i \mapsto t^i X_i,\ i = 1, \ldots 7)$ there.

**In Example E3.** $G$ being not contractible, there is no stable measure on $G$ (except $\varepsilon_e$) at all.

F. We show in the next example that also full completely stable non-Gaussian measures may exist. The underlying group $G$ (with Lie algebra $\mathfrak{G}$) is three-step nilpotent of dimension 4, a minimal example (cf. [$R$]): $\mathfrak{G}$ is a semidirect extension of a Heisenberg algebra $\mathfrak{H}_1$, i.e. $\mathfrak{G}$ is generated by $(X_i)_{1 \leq i \leq 4}$, such that $[X_1, X_2] = X_3$, $[X_2, X_3] = X_4$ (all other commutators zero). Let $\Gamma_i = \mathbb{R}X_i + \ldots + \mathbb{R}X_4$. $\Gamma_3 = [\mathfrak{G}, \mathfrak{G}]$ and $\Gamma_4 = \mathfrak{Z}(\mathfrak{G})$ are characteristic. Since $\Gamma_2 = ad_X^{-1}([\mathfrak{G}, \mathfrak{G}])$ we easily see that also $\Gamma_2$ is characteristic. Hence we obtain: $\mathfrak{m} = \mathbb{R}X_1 + \mathbb{R}X_2 \cong \mathbb{R}^2$, $\mathrm{Aut}(\mathfrak{G}) \subseteq \Delta^-(\mathfrak{G})$ and $\mathrm{Aut}(\mathfrak{m}) \subseteq \Delta^-(\mathfrak{m})$.

Let $\mathring{a} = (a_{ij}) \in \mathrm{Aut}(\mathfrak{G})$, put as before $x := a_{11}$, $y := a_{22}$, $z := a_{21}$. Then the commutator relations imply $a_{33} = xy$, $a_{44} = xy^2$, $zxy = 0$ and $a_{43} = -a_{31}y$. Therefore $\mathrm{Aut}(\mathfrak{m}) = \left\{\begin{pmatrix} x & 0 \\ 0 & y\end{pmatrix} : x, y \in \mathbb{R}^*\right\}$.

Let $\mu$ be full and completely stable. Then $\pi(\mu)$ splits $\pi(\mu) = \nu_1 \otimes \nu_2$ with symmetric stable one-dimensional marginals (corrolary 2.8).

Conversely, let $\nu_1, \nu_2$ be symmetric stable probabilities in $M^1(\mathbb{R})$ with generating distributions $A_i$, $i = 1, 2$. Let $j_i : \mathbb{R} \to \mathbb{R}X_i \subseteq \mathfrak{G}$ be the canonical injections and put $\mathring{A} := \mathring{A}_1 + \mathring{A}_2 \in \mathfrak{B}(\mathfrak{G})$ has idempotent $\mathring{\Gamma}$-type for

$$\mathring{\Gamma} := \left\{ \begin{pmatrix} x & & & 0 \\ & y & & \\ & & xy & \\ 0 & & & x^2y \end{pmatrix} : x, y \in \mathbb{R}^* \right\}.$$

Therefore the corresponding probabilities on $G$ are completely stable, and non-Gaussian if $\nu_i$ are non-Gaussian.

## § 5 Concluding remarks

We shall sketch briefly what is known for infinite dimensional vector spaces. Here the situation is more complicated. Especially a convergence of types theorem holds, but only under restrictive conditions on the operator norms (see [LS]; See also [Si1, 2, 3] for a recent survey on operator-stability).

Let $E$ be a Banach space, $\mu \in M^1(E)$ a probability measure.

5.1. Let now $\Gamma \subseteq \mathfrak{B}(E)$ be a **semi**group, closed with respect to the strong operator topology. Assume

$$\Gamma(\mu) * \Gamma(\mu) \subseteq \Gamma(\mu)\,. \tag{5.1}$$

Obviously then for any $n \in \mathbb{N}$ there exist $a_n \in \Gamma$ such that $a_n(\mu) = \mu^n$. But this equation does not even imply infinite divisibility (cf. [MU] example 2).

But if $\Gamma$ is a group, we obtain

$$\mu = (a_n^{-1}(\mu))^n\,, \tag{5.2}$$

i.e. especially $\mu$ is infinitely divisible. We call again measures fullfilling (5.2) $B$-stable (cf. (1.10)).

But, in contrast to the finite dimensional situation, $B$-stability does not imply stability. In [Si3] for a class of Banach spaces it is shown that any infinitely divisible probability measure is $B$-stable.

5.2. If we suppose suitable growth conditions on the norms of the operators $b_n$ (resp. $b_n^{-1}$, $b_n^{-1}b_k$) fulfilling (5.2) $b_n\mu^n = \mu$, we can apply the convergence of types theorem and find one-parameter (semi-)groups $(a_t)$ in $\Gamma$, such that $\mu$ is stable w.r.t. $(a_t)$, i.e. $\mu_t = a_t(\mu)$, $0 < t < 1$ (or $t \in \mathbb{R}_+^*$).

Hence, if we suppose (5.1) to hold with a selection function $\Phi : \Gamma \times \Gamma \to \Gamma$ fulfilling suitable boundedness conditions, again idempotence of $\Gamma$-type implies (operator-) stability. (See the definition of strong $\mathfrak{A}$-stability in [M2], see also [Si2].)

5.3. In [MU] and [M2] the analogs of completely stable measures are considered. Especially, if $\Gamma$ is the group of invertible bounded operators, the existence of $\mu$

with idempotent $\Gamma$-type and with continuous selection function („strong stability") imply that $\mu$ is full Gaussian.

And conversely, for full Gaussian measures on Hilbert spaces, the existence of a Gauß-measure with idempotent $\Gamma$-type is characterized by growth conditions on the eigenvalues of the covariance operator ([MU]).

5.4. As pointed out e.g. in [M2] and [MU] the existence of idempotent $\Gamma$-type is closely related to operator decomposability. (For groups see e.g. [Ha1]). Let $a \in \mathfrak{B}(E)$. $\mu$ is called $a$-decomposable if $\mu = a(\mu) * \nu$ for some $\nu \in \mathrm{M}^1(E)$ (called cofactor). Hence $a(\mu) * b(\mu) = c(\mu)$ implies $\mu = c^{-1}a(\mu) * c^{-1}b(\mu)$, i.e. $c^{-1}a \in \in \mathrm{Dec}(\mu)$ and $c^{-1}b(\mu)$ is a cofactor. E.g. in [Sie] the structure of the decomposability semigroup $\mathrm{Dec}(\mu)$ of full symmetric Gauss-measures is described. [Si4] contains further investigations on the structure of $\mathrm{Dec}(\mu)$, especially in connection with the existence of one-parameter semigroups.

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