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Hereditary Measurable Sets and Universal Measure

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All σ-fields of sets always contain all singletons if we do not say differently. By a nontrivial measure on a σ-field $\mathscr{A}$ of subsets of a set S is mean a countably additive real valued nonnegative finite function μ which vanishes on each of the singletons which are in $\mathscr{A}$ and such that $\mu(S) \neq 0$. If S is a set then $\mathscr{P}(S)$ denotes the power set of S and $[S]^{\leqq \aleph_0} : = \{X \subseteq S : \text{card}\, X \leqq \aleph_0\}$. If $\mathscr{F} \subseteq \mathscr{P}(S)$ and $X \subseteq S$ then $\mathscr{F} \cap X : = \{F \cap X : F \in \mathscr{F}\}$. If $\mathscr{A}$ is a σ-field of sets, then we denote by $I(\mathscr{A})$ the σ-ideal of all $A \in \mathscr{A}$ such that $\mathscr{P}(A) \subseteq \mathscr{A}$.

For an arbitrary set S consider the following properties:

(a) there is a nontrivial measure on $\mathscr{F}(S)$;

(b) there is a σ-field $\mathscr{A}$ on S such that there is a nontrivial complete measure on $\mathscr{A}$ and there is $X \in \mathscr{P}(S) \setminus \mathscr{A}$ with $\mathscr{A} \cap X = \mathscr{P}(X)$;

(c) there is a σ-field $\mathscr{A}$ on S such that there is a nontrivial nonatomic complete measure on $\mathscr{A}$ and there is $X \in \mathscr{P}(S) \setminus \mathscr{A}$ with $\mathscr{A} \cap X = \mathscr{P}(X)$.

It follows from Theorem 1 and Remark 5 in [1] the following

Theorem A. *For an arbitrary set S with card $S \geqq 2^{\aleph_0}$ there is a σ-field $\mathscr{A}$ on S such that there is a nontrivial nonatomic measure on $\mathscr{A}$ and there is $X \in \mathscr{P}(S) \setminus \mathscr{A}$ with $\mathscr{A} \cap X = \mathscr{P}(X)$. Additionally $\mathscr{A}$ can satisfy $I(\mathscr{A}) = [S]^{\leqq \aleph_0}$.*

Later we will prove the following easy:

Remark 1. For every σ-field $\mathscr{A}$ on S we have (α) if and only if (β), where

(α) there is $X \subseteq S$ with $I(\mathscr{A} \cap X) \neq I(\mathscr{A}) \cap X$;

(β) there is $X \subseteq S$ with $X \notin \mathscr{A}$ and $\mathscr{A} \cap X = \mathscr{P}(X)$.

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Remark 2. In Theorem A the measure μ cannot be complete even if we drop the assumption that μ is nonatomic.

In connection with Remark 2 it is worthwhile to mention that in Remark 1 in [1] we meant Theorem 1 there but without „$I(\mathscr{A}) = [S]^{\leqq \aleph_0}$" which was not written precisely. The proof of Remark 1 in [1] shows that assuming e.g. Continuum Hypothesis the real line does not have property (b).

The following theorems go further than the above remark about the real line.

Theorem B. *For S with card $S \leqq 2^{\aleph_0}$ or more generally for S on which there are no $0-1$ valued nontrivial measures we have that properties* (a), (b) *and* (c) *are all equivalent.*

Theorem C. *For an arbitrary set S we have* (a) *if ard only if* (b). *Of course* (c) *implies* (b).

It is worthwhile to observe the following

Lemma. *If non* (a) *for the real line R then non* (c) *for every set S.*

It follows from Theorem C and the Lemma the following

Corollary. *Assume that there is a nontrivial measure on $\mathscr{P}(S)$ but there is no such measure on $\mathscr{P}(R)$ (i.e. that S satisfies* (a) *but R does not). Then for such S we have* (b) *but non* (c).

Proofs.

Proof of Remark 1. First we prove that (β) implies (α). Let X be such that $X \notin \mathscr{A}$ and $\mathscr{A} \cap X = \mathscr{P}(X)$. Hence $X \in I(\mathscr{A} \cap X)$ and $X \notin I(\mathscr{A}) \cap X$ and so $I(\mathscr{A} \cap X) \neq \neq I(\mathscr{A}) \cap X$. Now we prove that (α) implies (β). For every $Y \subseteq S$ we have $I(\mathscr{A}) \cap \cap Y \subseteq I(\mathscr{A} \cap Y)$. Hence by (α) we have $I(\mathscr{A} \cap X) \nsubseteq I(\mathscr{A}) \cap X$. Therefore there is X^* such that $X^* \in I(\mathscr{A} \cap X^*)$ and $X^* \notin I(\mathscr{A}) \cap X^*$. The first property of X^* implies $\mathscr{A} \cap X^* = \mathscr{P}(X^*)$ which with the second property of X^* implies $X^* \notin \mathscr{A}$.

Proof of Remark 2. Let $\mathscr{A}$ be a σ-field on S such that $I(\mathscr{A}) = [S]^{\leqq \aleph_0}$ and there is $X \in \mathscr{P}(S) \setminus \mathscr{A}$ with $\mathscr{A} \cap X = \mathscr{P}(X)$. Suppose, a contrario, that there is a nontrivial complete measure μ on $\mathscr{A}$. We have $[S]^{\leqq \aleph_0} = \{A \in \mathscr{A} : \mu(A) = 0\}$. Let $\langle X_t \rangle_{t<\omega_1}$ be a pairvise disjoint family of uncountable subsets of X. Since $\mathscr{A} \cap X = \mathscr{P}(X)$ we have that for every $t < \omega_1$ there is $A_t \in \mathscr{A}$ such that $A_t \cap X = X_t$. For every $t < \omega_1$ define $A_t^* = A_t \setminus \bigcup\{A_s : s < t\}$. We have that $\langle A_t^* \rangle_{t<\omega_1}$ is a pairwise disjoint family of sets in A. Since for every $t \in T$, $X_t \subseteq A_t^*$ we have that for every $t < \omega_1$ A_t^* is uncountable and hence $\mu(A_t^*) > 0$. The existence of such family $\langle A_t^* \rangle_{t<\omega_1}$ contradicts with the assumption that μ is finite.

Proof of Theorem C. First observe that (b) implies (a). Let A be as in (b) and let X be such that $X \notin \mathscr{A}$ and $\mathscr{A} \cap X = \mathscr{P}(X)$. We have $\mu_e(X) > 0$, where μ_e is the outer measure induced by a nontrivial complete measure μ on $\mathscr{A}$. For every $Y \subseteq S$ define $\nu(Y) = \mu_e(Y \cap X)$. Since $\mathscr{A} \cap X = \mathscr{P}(X)$ we have that ν is a nontrivial measure on $\mathscr{P}(S)$. Now we prove that (a) implies (b). Assume that a set S satisfies (a). Let $\langle X_t : t \in T \rangle$ be a partition of S such that card $X_t = 2$ for every $t \in T$. Let $X \subseteq S$ be such that card$(X_t \cap X) = 1$ for every $t \in T$. Let $\mathscr{B} = \{\bigcup\{X_t : t \in T_1\} : T_1 \subseteq T\}$. Observe that $\mathscr{B}$ is a σ-field which does not contain singletons. Let ν be a nontrivial measure on $\mathscr{P}(X)$. For every $B \in \mathscr{B}$ define $\mu_1(B) = \nu(B \cap X)$. Let $\mathscr{A}$ be the family of all $A \subseteq S$ such that there are $B_1, B_2 \in \mathscr{B}$ with $B_1 \subseteq A \subseteq B_2$ and $\mu_1(B_1) = \mu_1(B_2)$. Let μ be the completion of μ_1. It is clear that μ is a nontrivial complete measure on $\mathscr{A}$. It is evident that for every nonempty $B \in \mathscr{B}$ we have $B \nsubseteq X$. We have also that for every $B \in \mathscr{B}$ if $B \supseteq X$ then $B = S$ and hence $\mu_1(B) = \mu_1(S) = \nu(X) > 0$. Hence $X \notin \mathscr{A}$. Since $\mathscr{A} \cap X \supseteq \mathscr{B} \cap X = \mathscr{P}(X)$ we have $\mathscr{A} \cap X = \mathscr{P}(X)$. We have proved that our $\mathscr{A}$, μ and X are as in (b) which ends the proof that (a) implies (b). It is evident that (c) implies property (b).

Proof of Theorem B. It follows from Theorem C that in order to prove Theorem B it is enough to prove only that (a) implies (c) for S with card $S \leqq 2^{\aleph_0}$. Let S be such that card $S \leqq 2^{\aleph_0}$. Let X, ν, μ_1, $\mathscr{B}$, $\mathscr{A}$ and μ be as in the proof that (a) implies (b) in the proof of Theorem C. Since as it is easy to see (and is well known, compare [2]) there are no nontrivial $0-1$ valued measures on $\mathscr{P}(X)$ for X with card $X \leqq 2^{\aleph_0}$ we have that the measure ν on $\mathscr{P}(X)$ is nonatomic. Since $\langle X, \mathscr{P}(X), \nu \rangle$ and $\langle S, \mathscr{B}, \mu_1 \rangle$ are isomorphic measure spaces (For every $x \in X$ let $f(\{x\}) = \{x, y\}$, where y is such that there is $t \in T$ with $\{x, y\} = X_t$. Then f is a measure preserving isomorphism.) the measure μ_1 is also nonatomic. Hence its completion μ is nonatomic.

Proof of Lemma. Assume that there are no nontrivial measures on $\mathscr{P}(R)$. Suppose, a contrario, that there is a set S such that there is a σ-field $\mathscr{A}$ on S such that there is a nontrivial nonatomic complete measure μ on $\mathscr{A}$ and there is $X \in \mathscr{P}(S) \setminus \mathscr{A}$ with $\mathscr{A} \cap X = \mathscr{P}(X)$. Let μ_e be the outer measure on S induced by μ. Let ν be the restriction of μ_e to the σ-field $\mathscr{A} \cap X$. Then it is easy to check that ν is a nontrivial nonatomic measure on $\mathscr{A} \cap X$ and hence on $\mathscr{P}(X)$. Hence, as Ulam has observed and proved, see [2], there is a nontrivial measure on $\mathscr{P}(R)$, a contradiction.

References

[1] Grzegorek E., Hereditary measurable sets, to appear in Proceedings of the Measure Theory Conference, Oberwolfach 1990, Rendiconti del Circolo Matematico di Palermo, supplementary volume.

[2] Ulam S , Zur Masstheorie in der allgemeinen Mengenlehre, Fund. Math. 16 (1930), 140—150.