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On the Thue-Morse Measure

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We obtain a precise description of the Thue Morse measure and the Fibonacci measure. More generally, we prove an equipartition result for ergodic measures on substitution dynamical systems generated by substitutions of constant length.

Let $x = x_0x_1x_2 \dots$ be the Thue Morse sequence, i.e. x is defined by

(1) $$x_0 = 0\,; \quad x_{2n} = x_n\,, \quad x_{2n+1} = 1 - x_n \quad \text{for} \quad n \geqq 0\,.$$

The sequence x has been studied by many authors in many different contexts. It is well known (see e.g.[6]) that x determines a shift invariant probability measure μ on the Borel sets of $X = \{0, 1\}^{\mathrm{N}}$ by defining μ on cylinder sets

$$[w] = \{x \in X : x_0 = w_0, \dots, x_n = w_n\}$$

with $w = w_0 \dots w_n \in \{0, 1\}^{n+1}$ by

(2) $$\mu([w]) = \lim_{N \to \infty} \frac{N_w(x_0 \dots x_N)}{N}\,.$$

Here $N_w(v)$ denotes the number of sequences of a word w in a word v, i.e. $v, w \in \in \bigcup_{n=0}^{\infty} \{0, 1\}^n$. Hence the measure of a cylinder $[w]$ simply equals the relative frequency of the word w in the Thue Morse sequence x.

Recently it has been proved that μ can occur as the unique shift invariant probability measure supported by the ground state configurations of a lattice gas model ([3]). This rises the question of a precise description of μ. The measure μ is rather different from a Gibbs measure, as there are very few cylinders of positive μ-measure. Theorem 1 implies however that μ satisfies a strong equipartition property. There exist constants $c_2 < c_1 < 0$ such that

(3) $$c_1 \leqq n\mu([w_1 \dots w_n]) \leqq c_2$$

for all $n \geqq 1$, and all cylinders $[w_1 \dots w_n]$ of length μ with positive μ-measure.

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It is in fact easy to derive from Theorem 1 that one can take $c_1 = \frac{1}{6}$, $c_2 = \frac{2}{3}$ and that these are the best possible constants.

Theorem 1. *Let μ be the Thue Morse measure. Let $[w] = [w_1 \ldots w_n]$ be a cylinder of length $n \geqq 2$ with positive μ-measure. Then*

$$\mu([w]) = \tfrac{1}{6}2^{-m} \quad \text{or} \quad \mu([w]) = \tfrac{1}{3}2^{-m},$$

where m is defined by $2^m < n \leqq 2^{m+1}$.

More precisely, for $n \geqq 3$ let $n = 2^m + r + 1$, where $0 < r < 2^m$, and let $\alpha_n = \mathrm{Card}\,\{w : \mu([w]) = \frac{1}{6}2^{-m}\}$, $\beta_n = \mathrm{Card}\,\{w : \mu([w]) = \frac{1}{3}2^{-m}\}$. Then

$$\alpha_n = 8r\,, \quad \beta_n = 3.2^m - 4r \quad \text{if} \quad 0 \leqq r < 2^{m-1}\,, \tag{4}$$

$$\alpha_n = 2^{m+1} + 4r\,, \quad \beta_n = 2^{m+1} - 2r \quad \text{if} \quad 2^{m-1} \leqq r < 2^m\,. \tag{5}$$

Proof. We first show that (4) holds for $r = 0$, i.e. that all cylinders of length $2^m + 1$ have μ-measure $\frac{1}{3} . 2^{-m}$. It is an easy exercise to show that all cylinders of length 3 of positive μ-measure have measure $\frac{1}{6}$ (cf. [6], p. 103). We proceed by induction. Suppose that (4) holds for $n = 2^m + 1$. Then consider a word w of length $2^{m+1} + 1$. Since $x_{2j}x_{2j+1}$ either equals 01 or 10 for all j, occurrence of w uniquely determines a word $\tilde{w}$ (obtained by extending w one letter to the left or the right) of length $2^{m+1} + 2$, occurring at say $2k$ in x. But then by (1) $\tilde{w}$, and hence w, determines a unique word v of length $2^m + 1$ occurring at place k in x. Conversely any occurrence of v leads to an occurrence of w. Hence for all N we have

$$N_w(x_0 \ldots x_{2N-1}) = N_v(x_0 \ldots x_{N-1})\,, \tag{6}$$

which by (2) implies $\mu([w]) = \frac{1}{2}\mu([v])$. Therefore $\mu([w]) = \frac{1}{3}2^{-m-1}$, by the induction hypothesis.

We next prove the first statement of the theorem, again by induction on m. Let $n = 2^{m+1} + r + 1$ with $0 < r < 2^{m+1}$. First suppose $r = 2s$ is even. Then as in the argument above, occurrence of a word w of length $n = 2^{m+1} + 2s + 1$ implies occurrence of a unique word v of length $2^m + s + 1$ in x, and inversely. Hence we find in the same way that $\mu([w]) = \frac{1}{2}\mu([v]) = \frac{1}{6}2^{-m-1}$ or $\frac{1}{3}2^{-m-1}$ (by the induction hypothesis). Now suppose $r = 2s + 1$ with $0 \leqq s < 2^m$ is odd. Then w has even length $n = 2^{m+1} + 2s + 2$, and depending on whether w occurs at an even or odd place in x, it determines a unique word v of length $\frac{1}{2}n = 2^m + s + 1$ or $\frac{1}{2}(n + 2) = 2^m + s + 2$ in x. Now for $0 \leqq s \leqq 2^m - 2$, the claim will follow as before by induction. In the case $s = 2^m - 1$, $\frac{1}{2}n = 2^{m+1}$ and $\frac{1}{2}(n + 2) = 2^{m+1} + 1$. In the first case the claim follows as before, in the second case because by the first part of the proof $\mu([v]) = \frac{1}{3}2^{-m-1}$ for all cylinders of length $2^{m+1} + 1$, and hence $\mu([w]) = \frac{1}{2}\mu([v]) = \frac{1}{6}2^{-m-1}$.

We proceed with the proof of (4) and (5). Let $P(n)$ be the total number of cylinders of length n with positive μ-measure. It follows as in the arguments above that $P(n)$ satisfies for $n \geqq 2$

$$P(2n + 1) = 2P(n + 1)\,, \quad P(2n) = P(n) + P(n + 1)\,.$$

It follows from this by induction that $P(n)$ is given by

$$P(n) = \begin{cases} 3.2^m + 4r & \text{if} \quad 0 \leqq r < 2^{m-1}, \\ 4.2^m + 2r & \text{if} \quad 2^{m-1} \leqq r < 2^m, \end{cases} \tag{7}$$

where $n = 2^m + r + 1$. (This has been proved before in [2] and in [4]). Obviously α_n and β_n satisfy

$$\alpha_n + \beta_n = P(n), \quad \tfrac{1}{6}\alpha_n 2^{-m} + \tfrac{1}{3}\beta_n 2^{-m} = 1.$$

Solving for α_n and β_n yields (4) and (5). □

The Thue Morse measure is an example of a uniquely ergodic measure on a substitution dynamical system ([6]). Let υ be such a measure. It is well known that the number of cylinders of length n of positive υ-measure $P(n)$ grows at most linearly (see e.g. [6], Prop.V. 19). Explicit expressions for $P(n)$ like (7) are difficult to obtain in general (cf. [5]). However, it is possible to show that the equipartition property (3) holds for all systems generated by substitutions of constant length l, i.e. substitutions σ defined on an alphabet A such that the length of $\sigma(a)$ equals l for all $a \in A$. A substitution σ is called primitive if there exist n such that all letters occur in each $\sigma^n(a)$, $a \in A$. This property ensures unique ergodicity of the associated dynamical system.

Theorem 2. *Let v be the unique shift invariant measure on a substitution dynamical system generated by a primitive substitution of constant length. Then there exist $c_2 > c_1 > 0$ such that*

$$c_1 \leqq nv([w_1 \dots w_n]) \leqq c_2 \tag{8}$$

for all $n \geqq 1$, and all $[w_1 \dots w_n]$ with $v([w_1 \dots w_n]) < 0$.

Proof. We first recall that the substitution dynamical system is the closed orbit of a sequence x obtained by iterating σ starting with a letter $e \in A$ such that $\sigma(e)$ has first letter, e, i.e. for all N we have $x_1 \dots x_{\ell^N} = \sigma^N(e)$. By unique ergodicity and minimality, words w have positive v-measure iff they occur in x, and similarly to (2) we have

$$v([w]) = \lim_{N \to \infty} \frac{N_w(\sigma^N(e))}{l^N}.$$

Let $P = \{(a, b) \in A \times A : ab \text{ occurs in } x\}$. For a word $w = w_1 \dots w_n$ choose m such that $l^{m-1} < n \leqq l^m$. Then w occurs in x and only if w occurs in at least one word $\sigma^m(ab)$, $(a, b) \in P$. Hence

$$N_w(\sigma^N(e)) \geqq N_{\sigma^m(ab)}(\sigma^N(e)) = N_{ab}(\sigma^{N-m}(e)).$$

Dividing by l^N and leting $N \to \infty$, we obtain

$$v([w]) \geqq v([ab])\, l^{-m}.$$

Hence the lower bound of (8) follows with $c_1 = \inf_{(a,b) \in P} v([ab])/l$.

To obtain the upper bound, we first remark that we may assume that σ is one to one, i.e. that $\sigma(a) \neq \sigma(b)$ for $a \neq b$. (If σ is not one-to-one ν projects on the measure on the substitution dynamical generated by the substitution obtained by identifying (if necessary more than once) all a, b such that $\sigma(a) = \sigma(b)$). Next we prove the following claim.

Let σ be one-to-one, then for all letters $a, b \in A$, all $m \in \mathcal{N}$ and all words w of length $l^m - 1$

$$N_w(\sigma^m(ab)) \leqq l + 1 . \tag{9}$$

This is certainly true if $m = 1$, since $\sigma(ab)$ has length $2l$. We proceed by induction. Let w be a word of length $l^{m+1} - 1$ occurring at position j modulo l in $\sigma^{m+1}(ab)$. Then we can write $w = u\sigma(v)\,u'$ for some words u and u' of length $l - j$ and $j - 1$ (0 respectively $l - 1$ if $j = 0$), and v of length $l^m - 1$. As σ is one-to-one, v is unique. Hence w occurs at most as many times in $\sigma^{m+1}(ab)$ as v occurs in $\sigma^m(ab)$. The claim thus follows by induction. For words w of length $l^m - 1$ we find with (9)

$$N_w(\sigma^N(e)) \leqq \sum_{(a,b)\in P} N_{\sigma^m(ab)}(\sigma^N e)\, N_w(\sigma^m(ab)) \leqq (l + 1) \sum_{(a,b)\in P} N_{\sigma^m(ab)}(\sigma^N e) .$$

Dividing by l^N and letting $N \to \infty$, we obtain for such words

$$\nu([w]) \leqq (l + 1)\, l^{-m} \sum_{(a,b)\in P} \nu([ab]) = (l + 1)\, l^{-m} .$$

Finally let $w = w_1 \dots w_n$ be a word lf length n, and choose m as before. Then

$$n\nu([w_1 \dots w_n]) \leqq l^m \nu([w_1 \dots w_{\ell^{m-1}-1}] \leqq l(l + 1) .$$

Hence the upper bound of (8) follows with $c_2 = l(l + 1)$. □

We conjecture that (8) also holds for dynamical systems generated by substitutions of non-constant length. The lower bound will follow analogously to the proof above, but the problem will be to obtain an expression like (9).

Let τ be the substitution on $A = \{0, 1\}$ defined by $\tau(0) = 01$ and $\tau(1) = 0$. This substitution is known as the *Fibonacci* substitution. This is related to the fact that the length of $\tau^N(1)$ is equal to F_{N+1}, where $F_0 = 0$, $F_1 = 1$, $F_{N+1} = F_N + F_{N-1}$ are the Fibonacci numbers. Let in the sequel ν be the unique shift invariant measure of the associated dynamical system. We call ν the Fibonacci measure. This measure has properties which are remarkably similar to those of the Morse Thue measure. The proof of Thorem 3 is however rather different from the proof of Theorem 1.

Theorem 3. *Let ν be the Fibonacci measure and let $p = \frac{1}{2}(\sqrt{5} - 1)$. Let $w = w_1 \dots w_n$ be a word of length $n \geqq 2$ such that $\nu([w]) > 0$. Then*

$$\nu([w]) \in \{p^{m-2}, p^{m-1}, p^m\} ,$$

where m is defined by $F_m \leqq n < F_{m+1} - 1$.

More precisely, let $n = F_m + r - 1$, where $1 \leqq r \leqq F_{m-1}$, and let

$\alpha_n = \text{Card}\,\{w : \nu([w]) = p^{m-2}\}$, $\beta_n = \text{Card}\,\{w : \nu([w]) = p^{m-1}\}$,
$\gamma_n = \text{Card}\,\{w : \nu([w]) = p^m\}$. *Then*

$$\alpha_n = F_{m-1} - r , \quad \beta_n = F_{m-2} + r , \quad \gamma_n = r . \tag{10}$$

Proof: With the Perron-Frobenius theorem it follows easily that $\nu([0]) = p$, and $\nu([1]) = 1 - p = p^2$. This implies that $\nu([01]) = \nu([1]) = p^2$ and $\nu([00]) = = \nu([0]) - \nu([01]) = p^3$. We first proceed by induction to prove that for any word w there exists an integer k such that $\nu([w]) = p^k$ (if $\nu([w]) \neq 0$). Any word w ending in 1 has a unique decomposition in words from the set $\{1, \tau(0), \tau(1)\}$, hence each occurrence of w in $\tau^N(0)$ determines an occurrence of a unique shorter (except if $w = 1$ or $w = 101$) word v in $\tau^{N-1}(0)$, and conversely. Hence

$$N_w(\tau^N(0)) = N_v(\tau^{N-1}(0)) .$$

Dividing by $((1+\sqrt{5})/2)^N = (1/p)^N$, and letting $N \to \infty$ yields that $\nu([w]) = p\nu([v])$. For words not ending in 1 we have $\nu([u10]) = v([u1])$ and $\nu([u00]) = \nu([u001])$, since 11 and 000 do not occur in the Fibonacci sequence.

To continue the proof we consider (following [7]) for fixed n the de Bruijn graph G_n of the words of length n occurring in the Fibonacci sequence, i.e., the nodes of G_n are words of length n, and $u \to v$ if there exist letters a and b, and a word w such that $u = bw$ and $v = wa$. It is well known that there occur $n + 1$ different words of length n in the Fibonacci sequence. Hence G_n has $n + 1$ nodes, hence consists of two cycles (cf. [7]). Let $u^+ = \{v \in G_n : u \to v\}$. Then $\operatorname{Card}(u^+) = 1$ implies $\nu([v]) \geqq \nu([u])$, where $u^+ = \{v\}$. One deduces from this and the cycle structure of G_n that $\nu([w])$ can take at most 3 different values if $\nu([w]) > 0$. Since we know already that these values are powers of p it also follows that these values are *consecutive* powers of p, say p^{j-1}, p^j, p^{j+1}. Then the equation $\alpha_n p^{j-1} + \beta_n p^j + \gamma_n p^{j+1} = 1$ implies

$$(\alpha_n + \gamma_n)\, p^{j-1} + (\beta_n + \gamma_n)\, p^j = 1 .$$

Equivalently, putting $\varphi = p^{-1}$, we have

$$(\alpha_n + \gamma_n)\, \varphi + \beta_n + \gamma_n = \varphi^j .$$

With induction one shows easily that the equation $x + \varphi y = \varphi^j$ has the solution $(x, y) = (F_{j-1}, F_j)$. Using this, one then proves that this equation has (F_{j-1}, F_j) as *unique* integer solution (again by induction). Hence necessarily $\alpha_n + \gamma_n = F_j$ and $\beta_n + \gamma_n = F_{j-1}$. Combining this with $\alpha_n + \beta_n + \gamma_n = n + 1 = F_m + r$, one obtains (10). $\square$

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