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Perfect Sets of Independent Functions

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It is shown that there exists a perfect set $C \subseteq \mathrm{k}^N$ such that if $K \subseteq C$ is a perfect subset, then there is an infinite subset $X \subseteq \mathrm{N}$ for which $K_{|X} = \mathrm{k}^X$. A characterization of this property uses independent functions.

Notations. The set of all function from a set X into the subset $\{0, 1, \ldots, k - 1\} = \mathrm{k}$ of natural numbers is denoted by k^X. The set of all natural numbers $\{0, 1, \ldots\}$ is denoted by N. The set k^{N} with the product topology is homeomorphic with the Cantor set. If $X \subseteq \mathrm{N}$ and $f : \mathrm{N} \to \mathrm{k}$ is a function, then $f|_X$ denotes the restriction of f to X and if $C \subseteq k^{\mathrm{N}}$ then $C|_X$ denotes the set of all restrictions to X of functions from C.

Balcerzak's Question. For the case $k = 2$ the following question was raised by M. Balcerzak [1] (see also [3]).

Does there exist a perfect set $C \subseteq \mathrm{k}^{\mathrm{N}}$ such that whenever $K \subseteq C$ is a perfect subset, then there exists an infinite subset $X \subseteq \mathrm{N}$ for which $K|_X = \mathrm{k}^X$?

We show that this question has positive answer.

A family G is called a *k-partition* of a set Y if it consists of pairwise dsijoint k-element subsets of Y and the union of all members of G is equal to Y.

Lemma 1. *For every positive integer n there are a finite set I and a k^n-element family $T \subseteq \mathrm{k}^I$ such that if G is a k-partition of T, then there exists $g \in I$ such that for every $Y \in G$ the functions which belong to Y assume different values on g.*

Proof. Let I be the union of the set $\{0, 1, \ldots, n - 1\} = \mathrm{n}$ and the family of all k-partitions of the set k^{n}. We extend every function $f \in \mathrm{k}^{\mathrm{n}}$ to a function f^* defined on I as follows. For a k-partition g of k^{n} we choose the values $f^*(g)$ from the set k in such a way that for every $y \in g$ the set $\{f^*(g) : f \in y\}$ consists of k different numbers. Let T be the set of the defined above functions f^*. For every k-partition G of T we consider the restrictions to n of functions in each $Y \in G$. This yields a k-par-

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tition g of k^n because each function f has only one extension f^*. It is a desired element of I. □

Theorem 2. *There exists a perfect set $C \subset \mathrm{k}^{\mathrm{N}}$ such that for each perfect set $K \subseteq C$ there is an infinite set $X \subseteq \mathrm{N}$ for which $K|_X = \mathrm{k}^X$.*

Proof. For every natural number $n > 0$ we apply the lemma 1. We obtain a finite set I_n and a k^n-element family T_n of functions from I_n into k such that for every k-partition G of T_n there exists $g \in I_n$ such that $\{f(g) : f \in Y\} = \mathrm{k}$ for every $Y \in G$.

The sets $I_1, I_2, \ldots$ are finite. We identify them with pairwise disjoint subsets of natural numbers so that $I_1 \cup I_2 \cup \ldots = \mathrm{N}$.

We are going to define a sequence $C_1, C_2, \ldots$ of sets. Every set C_n will be contained in $k^{I_1 \cup \ldots \cup I_n}$ and consist k^n functions. The sets T_n and $C_n|_{I_n}$ will be equal. Every function from C_{n-1} will have exactly k extensions in C_n. If functions from C_n are different then their restriction to I_n will also be different.

We begin by putting $C_1 = T_1$. Let the set C_{n-1} be defined. According to the inductive hypothesis it contains k^{n-1} elements. Therefore there exists a k-to-1 function F from T_n onto C_{n-1}. We define C_n to be the set of all functions of the form $F(f) \cup f$, where $f \in T_n$. Now the sets $C_1, C_2, \ldots$ are as it was declared.

Let $C = \{f \in \mathrm{k}^{\mathrm{N}} : f|_{I_1 \cup \ldots \cup I_n} \in C_n, n \geqq 1\}$. Clearly, the set C is perfect.

Let $K \subset C$ be perfect set. We shall define a sequence $y_1, y_2, \ldots$ of natural numbers such that for every $n \geqq 1$ the sets $K|_{\{y_1,\ldots,y_n\}}$ and $\mathrm{k}^{\{y_1,\ldots,y_n\}}$ are equal.

Suppose $K|_{\{y_1,\ldots,y_{n-1}\}} = \mathrm{k}^{\{y_1,\ldots,y_{n-1}\}}$; it holds for $n = 1$ as $F|_\emptyset = \mathrm{k}^\emptyset$. Let m be a natural number such that $\{y_1, \ldots, y_{n-1}\} \subseteq I_1 \cup \ldots \cup I_{m-1}$ and each function from $K|_{\{y_1,\ldots,y_{n-1}\}}$ has at least k extensions in the set $K|_{I_1 \cup \ldots \cup I_{m-1} \cup I_m}$. This is possible by perfectness. We extend every function from $K|_{I_m}$ to the, unique by definition of C_m, function from $K|_{I_1 \cup \ldots \cup I_{m-1} \cup I_m}$. We restrict this extension to the set $\{y_1, \ldots$ $\ldots, y_{n-1}\}$. Thus we have defined a function $\phi : K|_{I_m} \to K|_{\{y_1,\ldots,y_{n-1}\}}$. The function ϕ is onto $K|_{\{y_1,\ldots,y_{n-1}\}}$. Every value h of ϕ is assumed at least k times on $K|_{I_m}$. We choose k elements from every preimage $\phi^{-1}(h)$ and extend it to a k-partition G of T_m. We take y_n as element of I_m such that for every $Y \in G$ the functions from Y assume different, exactly k, values on y_n. In consequence the functions from $\phi^{-1}(h)$, where $h \in K|_{\{y_1,\ldots,y_{n-1}\}}$, assume all possible values on y_n. This means that $K|_{\{y_1,\ldots,y_n\}}$ is equal to $\mathrm{k}^{\{y_1,\ldots,y_n\}}$.

Let $X = \{y_1, y_2, \ldots\}$. For every natural number $n > 0$ we have $K_{\{y_1,\ldots,y_n\}} =$ $= \mathrm{k}^{\{y_1,\ldots,y_n\}}$. Therefore $K|_X = \mathrm{k}^X$ since the set K is perfect □

The theorem 2 answers positively to M. Balcerzak's question. The same result can be obtained using the notion of independent family of functions, known also as families of large oscillation.

Independent family of functions. A set $F \subseteq \mathrm{k}^X$ is *independent* if for every sequence $f_1, \ldots, f_n$ of different functions from F and every sequence $x_1, \ldots, x_n$ of numbers from k the intersection $f_1^{-1}(x_1) \cap \ldots \cap f_n^{-1}(x_n)$ is non-empty.

Theorem 3. *If $F \subseteq k^{\mathrm{N}}$ is a perfect independent set, then there exists an infinite subset $X \subseteq N$ such that $F|_X = \mathrm{k}^X$.*

Proof. We define a sequence $y_1, y_2, \dots$ of natural numbers such that

$$F|_{\{y_1,\dots,y_n\}} = \mathrm{k}^{\{y_1,\dots,y_n\{}$$

for every $n \geqq 1$. Let $f_1, \dots, f_k$ be different functions from F. We take y_1 to be an element of the intersection $f_1^{-1}(0) \cap \dots \cap f_k^{-1}(k-1)$. We have $F|_{\{y_1\}} = \mathrm{k}^{\{y_1\}}$.

Suppose $F|_{\{y_1,\dots,y_{n-1}\}} = \mathrm{k}^{\{y_1,\dots,y_{n-1}\}}$. For every function $f \in \mathrm{k}^{\{y_1,\dots,y_{n-1}\}}$ we choose its different extentions $f_1, \dots, f_k$ which belongs to F. Such extensions exist because F is perfect. We take y_n from intersections of all the sets $f_1^{-1}(0) \cap \dots \cap f_k^{-1}(k-1)$, where f runs through $\mathrm{k}^{\{y_1,\dots,y_{n-1}\}}$. Clearly $y_n \notin \{y_1, \dots, y_{n-1}\}$ and we have $F|\{y_1, \dots, y_n\} = \mathrm{k}^{(y_1,\dots,y_n\}}$.

Let $X = \{y_1, y_2, \dots\}$. Therefore $F|_X = \mathrm{k}^X$ since the set F is closed □

The theorem 3 also provides the answer to Balcerzak's question, assuming the existence of a perfect independent set. This is well known fact following from many papers. The first such a paper due to G. Fichtenholz and L. Kantorovich [2]. It can also be obtained from slightly modified proof of the theorem 2.

Theorem 4. *Let $F \subseteq \mathrm{k}^{\mathrm{N}}$ be a perfect subset. There exists an infinite subset $X \subseteq N$ such that $F|_X = \mathrm{k}^X$ if and only if there exists a perfect independent subset of F.*

Proof. Suppose that $F|_X = \mathrm{k}^X$ for some infinite subset $X \subseteq \mathrm{N}$. Let P be a perfect independent subset contained in k^X. We take the minimal closed set $H \subseteq F$ such that $H|_X = P$. By minimality of H, it has no isolated points. Consequently it is prefect. The projection of H onto P is open on points from some dense G_δ set is the complement of the union of the preimages of the boundaries of the sets $V|_X$, where V runs on all closed-open subset of H. Such preimages have empty interiors. They consist of a countable family, as the family of closed-open subsets of H is countable. Thus the projection is one-to-one on a perfect set since any dense G_δ subset, contained in a perfect set contains a perfect subset. The image of this set is a perfect independent set.

The inverse implication was given in the theorem 3 □

References

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