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A Space by W. Gowers and New Results on Norm and Numerical Radius Attaining Operators

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We use a Banach space recently considered by W. Gowers to improve some results on norm attaining operators. In fact, we show that the norm attaining operators from this space to a strictly convex Banach space are finite-rank. The same Banach space is also used to get a new example of a space which does not satisfy the denseness of the numerical radius attaining operators. This new counterexample improves and simplifies the one previously obtained by R. Payá, who answered an open question raised by B. Sims in 1972.

Introduction

This paper deals with two parallel optimization problems concerning operators in Banach spaces. First we make some remarks on norm attaining operators, then we discuss a new counterexample on numerical radius attaining operators.

Given two Banach spaces, X and Y, a (bounded and linear) operator $T \in L(X, Y)$ *attains its norm* if there is an element x_0 in the unit sphere of X such that $\|Tx_0\| = \|T\|$; we will denote by $NA(X, Y)$ the set of norm attaining operators from X to Y. The general question (sometimes called the „Bishop-Phelps problem") is whether or not $NA(X, Y)$ is dense in $L(X, Y)$ for the norm topology. In his pioneering paper on this question, J. Lindenstrauss [14] introduced the so-called properties „A" and „B". A Banach space X has property A (resp. B) if $NA(X, Y)$ (resp. $NA(Y, X)$) is dense in $L(X, Y)$ (resp. $L(Y, X)$) for any Banach space Y. Thus, Bishop-Phelps Theorem can be stated by saying that the scalar field has property B.

Whereas property A is fairly well-known, there are few results on property B. Lindenstrauss [14] gave a strong geometrical condition which is sufficient for property B. However, Partington [15] proved that every Banach space can be renormed to satisfy Lindenstrauss condition, hence also property B. By using techniques developped by Bourgain [10] to prove the most relevant result that the Radon-

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-Nikodym property implies A, Huff [13] showed that Banach spaces failing Radon-Nikodym property can be renormed to fail B. Nevertheless, no isomorphic condition is known to be sufficient for property B. Even the natural question if some of the most classical spaces have property B was open for many years. In 1990, W. Gowers got a very interesting result by finding a Banach space G such that $NA(G, l_p)$ is not dense in $L(G, l_p)$ for $1 < p < \infty$, hence l_p fails property B. In fact, getting the most out of Gowers arguments, one can prove that a strictly convex Banach space X which contains an isomorphic copy of l_p $(1 < p < \infty)$ cannot have property B. In the first part of this paper, we go a little bit farther and show that every norm attaining operator from G into a strictly convex Banach space has finite rank. Therefore, if X is strictly convex, and there is a noncompact operator from G to X, then X fails property B.

In § 2 the same Gowers space will allow us to get new information on the problem of numerical radius attaining operators. The *numerical radius* of an operator T on a Banach space X $(T \in L(X, X) = L(X))$ is defined by

$$v(T) = \sup \{|x^*(Tx)| : (x, x^*) \in \Pi(X)\}, \qquad (*)$$

where $\Pi(X) = \{(x, x^*) \in X \times X^* : \|x\| = \|x^*\| = x^*(x) = 1\}$.

We refer to the books by F. Bonsall and J. Duncan [8, 9] for the theory of numerical ranges of operators on Banach spaces. It is said that the operator T *attains its numerical radius* when the supremum in $(*)$ is actually a maximum, that is, there exists $(x_0, x_0^*) \in \Pi(X)$ such that $|x_0^*(Tx_0)| = v(T)$. Let us denote by $R(X)$ the set of numerical radius attaining operators on X. Paralleling the Bishop-Phelps problem, B. Sims raised in 1972 the question if the set $R(X)$ is dense in $L(X)$. In [1] the reader may find the partial answers to Sims problem until 1990. Amongst them, we should mention the results by I. Berg and B. Sims [6], C. Cardassi [11] and those obtained in [2, 3]. For example, if X has the Radon-Nikodym property, then $R(X)$ is dense in $L(X)$ [3]. Recently, the first author [4, 5] showed that under weak hypothesis a Banach space can be renormed to satisfy the denseness of $R(X)$ in $L(X)$. The third author [16] proved that the answer to the general question posed by Sims is negative.

Herein we use the Gowers space G to get a new counterexample. We show that $R(X)$ is not dense in $L(X)$ for $X = l_2 \oplus_\infty G$. This improves the earlier example in several aspects.

1. Gowers space and norm attaining operators

The next definition of the space G looks different from the one given by W. Gowers [12], but it is clearly equivalent.

1.1. Definitions and notation. For a scalar sequence x and $n \in \mathbb{N}$ let us write

$$\Phi_n(x) = \frac{1}{H_n} \sup \{\sum_{j \in J} |x(j)| : J \subset \mathbb{N}, |J| = n\}$$

where $|J|$ is the cardinality of the set J and $H_n = \sum_{k=1}^{n} k^{-1}$. We will denote by G the space of those sequences x such that

$$\lim_{n\to\infty} \Phi_n(x) = 0$$

with the norm given by

$$\|x\| = \sup\{\Phi_n(x) : n \in \mathbb{N}\} \quad (x \in G).$$

Note that every element in G is a sequence convergent to zero.

Apart from some obvious facts, the following lemma contains the most relevant property of the space G.

1.2. Lemma (Gowers [12]).

i) *G is a Banach space.*

ii) *The unit vector basis $\{e_n\}$ is a Schauder basis for G.*

iii) *For $1 < p < \infty$, G is contained in l_p and the formal identity from G into l_p is a bounded operator.*

Gowers also got another interesting property of G, its unit ball lacks extreme points. In fact, given an element x in the unit sphere of G, one can find a natural number k and $\delta > 0$ such that $\|x \pm \delta e_k\| = 1$. Something better can be shown:

1.3. Lemma. *For $x \in G$ with $\|x\| = 1$, there exist a natural number m and $\delta > 0$ such that*

$$\|x + \lambda e_k\| = 1$$

for $k \geqq m$ and any scalar λ with $|\lambda| \leqq \delta$.

Proof. By definition of G, there exists $N \in \mathbb{N}$ such that

$$\Phi_n(x) < \tfrac{1}{2} \quad \text{for} \quad n > N.$$

Since $\lim_{n\to\infty} x(n) = 0$, we can find $m \in \mathbb{N}$ satisfying

$$k \in \mathbb{N},\ k \geqq m \Rightarrow |x(k)| \leqq \frac{1}{2N}.$$

Now we choose $0 < \delta \leqq 1/2N$, $k \geqq m$, and take $y = x + \lambda e_k$, where $|\lambda| \leqq \delta$. We want to show that $\Phi_n(y) \leqq 1$ for all n.

If $n > N$ we simply have

$$\Phi_n(y) \leqq \Phi_n(x) + \delta < \tfrac{1}{2} + \delta \leqq 1.$$

So, let us take $J \subset \mathbb{N}$ with $|J| = n \leqq N$ to show that

$$\sum_{j\in J} |y(j)| \leqq H_n.$$

and we can clearly assume that $k \in J$. Then we get

$$\sum_{j \in J} |y(j)| = |y(k)| + \sum_{j \in J \setminus \{k\}} |y(j)| \leqq$$
$$\leqq \delta + |x(k)| + \sum_{j \in J \setminus \{k\}} |x(j)| \leqq$$
$$\leqq \frac{1}{2N} + \frac{1}{2N} + \sum_{j \in J \setminus \{k\}} |x(j)| \leqq \frac{1}{N} + H_{n-1} \leqq H_n . \qquad \square$$

We can now use a simple argument due to Lindenstrauss [14; Proposition 4] to get the following:

1.4. Theorem. *Let X be a strictly convex Banach space and $T \in NA(G, X)$. Then $T(e_k) = 0$ for large enough k, hence T is a finite-rank operator. As a consequence, if there is a noncompact operator from G to X, then $NA(G, X)$ is not dense in $L(G, X)$ and X fails property B.*

Proof. Assume without loss of generality that $\|T\| = 1$ and let $x \in G$ be such that

$$\|x\| = \|Tx\| = 1 .$$

By the previous lemma, we have $\|x \pm \delta e_k\| = 1$ for some positive δ and large enough k. Then

$$\|Tx \pm \delta T e_k\| \leqq 1 = \|Tx\|$$

and the strict convexity of X comes into play. □

By Lemma 1.2. iii) there is a noncompact operator from G to l_p $(1 < p < \infty)$. If a Banach space X contains a subspace isomorphic to l_p, there will be a noncompact operator from G to X as well, so we get:

1.5. Corollary (Gowers [12]). *If X is a strictly convex Banach space containing an isomorphic copy of l_p $(1 < p < \infty)$, then X fails property B.*

2. A new counterexample on numerical radius attaining operators

Using Gowers space we are now going to exhibit a new example of a Banach space X such that the set $R(X)$ of numerical radius attaining operators is not dense in the space $L(X)$ of all bounded linear operators on X. The first example of this kind was shown in [16] and had the form

$$X = Y \oplus_\infty Z \qquad (*)$$

where Z was the space c_0 with its usual norm, Y was a suitable equivalent renorming of c_0 and the symbol $\oplus_\infty$ means that we put the maximum norm on the direct sum:

$$\|y + z\| = \max\{\|y\|, \|x\|\} \quad (y \in Y, z \in Z) .$$

The new example which we are going to exhibit here will also have the form (*) but with a different choice of the spaces Y, Z. Therefore we can take advantage of some of the results obtained in [16]. To be precise, we will use the observations on the numerical radius of operators on a space of the form (*) contained in the following lemma. The closed unit ball and the unit sphere of a Banach space E will be denoted by B_E and S_E respectively.

2.1. Lemma [16; Lemma 1.2]. *Let Y, Z be Banach spaces, $X = Y \oplus_\infty Z$ and P, Q the projections from X onto Y, Z respectively. For $T \in L(X)$ we have*

i) $v(T) = \max\{v(PT), v(QT)\}$.

ii) *If $T \in B(X)$ and $v(PT) > v(QT)$, then $PT \in R(X)$.*

iii) $v(PT) = \sup\{|y^*(PT(y + z))| : (y, y^*) \in \Pi(Y), z \in B_Z\}$ *and $PT \in R(X)$ if and only if this supremum is attained.*

We will take $Y = l_2$ (any Hilbert space of infinite dimension could be used, separability is not required) and the set $\Pi(l_2)$ is particularly simple. The inner product of l_2 will be denoted by $(\cdot|\cdot)$ and for $y \in l_2$, the functional $(\cdot|y)$ will be denoted by y^*, that is, the mapping $y \to y^*$ is the canonical (conjugate linear in the complex case) identification of l_2 with its dual. It is clear that

$$\Pi(l_2) = \{(y, y^*) : y \in S_{l_2}\}.$$

Gowers space G, with which we are already acquainted, will play the role of Z. The fact that $R(l_2 \oplus_\infty G)$ is not dense in $L(l_2 \oplus_\infty G)$ will be an easy consequence of the following result, which keeps a strong parallelism with Theorem 1.4, although its proof is a bit more cumbersome.

2.2. Theorem. *Consider the Banach space $X = l_2 \oplus_\infty G$ and let $T \in L(X)$* be *defined by*

$$T(y + z) = Ay + Bz \quad (y \in l_2, z \in G)$$

where $A \in L(l_2)$ and $B \in L(G, l_2)$.

If $T \in R(X)$, then B is a finite-rank operator.

Proof. Consider the projection P from X onto l_2. We have clearly $PT = T$ and by using Lemma 2.2. iii) we obtain:

a) *There exist $y_0 \in S_{l_2}$, $z_0 \in B_G$ such that*

$$|(Ay + Bz \mid y) \leqq |(Ay_0 + Bz_0 \mid y_0)|$$

for every $y \in S_{l_2}$, $z \in B_G$.

By rotating z we actually have

$$|(Ay \mid y)| + |(Bz \mid y)| \leqq |(Ay_0 \mid y_0)| + |(Bz_0 \mid y_0)|$$

equivalently

$$|(Ay \mid y)| + |[B^*y^*](z)| \leqq |(Ay_0 \mid y_0)| + |[B^*y_0^*](z_0)|.$$

By taking the supremum over $z \in B_G$ we get

$$|(Ay \mid y)| + \|B^* y^*\| \leqq |(Ay_0 \mid y_0)| + |[B^* y_0^*] (z_0)| ,$$

and this still holds for all $y \in S_{l_2}$. Now we take $y = y_0$ to get

$$\|B^* y_0^*\| \leqq |[B^* y_0^*] (z_0)| .$$

which shows that the functional $B^* y_0^* \in G^*$ attains its norm at z_0. So far we have proved

b) *There exists $y_0 \in S_{l_2}$ such that*

$$|(Ay \mid y)| + \|B^* y^*\| \leqq |(Ay_0 \mid y_0)| + \|B^* y_0^*\| \quad \forall y \in S_{l_2} .$$

Moreover, the functional $B^* y_0^*$ attains its norm at a point $z_0 \in S_G$.

The function of y appearing in the first member of the last inequality attains its maximum at y_0 and this leads us to consider a suitable derivative of such a function.

We fix $h \in S_{l_2}$ with $(h \mid y_0) = 0$ (a comfortable direction) and for $t \geqq 0$, we define:

$$y_t = y_0 + th , \quad \phi(t) = \|y_t\|$$

$$F_1(t) = \phi(t)^{-1} (Ay_t \mid y_t) , \quad H_1(t) = |F_1(t)|$$

$$F_2(t) = B^* y_t^* , \quad H_2(t) = \|F_2(t)\| .$$

By taking $y = \phi(t)^{-1} y_t$ in the inequality which appears in assertion b) we simply obtain:

$$H_1(t) + H_2(t) \leqq \phi(t) [H_1(0) + H_2(0)] \quad (t \geqq 0) \tag{1}$$

Note that

$$\phi(t) = (1 + t^2)^{1/2} \quad (t \geqq 0)$$

thus ϕ is differentiable at the origin with $\phi'(0) = 0$. Since

$$\phi(t) F_1(t) = (Ay_0 \mid y_0) + t[(Ay_0 \mid h) + (Ah \mid y_0)] + t^2(Ah \mid h)$$

we deduce that F_1 is also differentiable at the origin with

$$F_1'(0) = (Ay \mid h) + (Ah \mid y_0) . \tag{2}$$

If we now assume that

$$F_1'(0) = (Ay_0 \mid y_0) \neq 0 \tag{*}$$

we can conclude that the function H_1 is also differentiable at the origin with

$$H_1'(0) = \operatorname{Re}\left(\frac{|(Ay_0 \mid y_0)|}{(Ay_0 \mid y_0)} \cdot F_1'(0)\right) . \tag{3}$$

For the functions F_2 and H_2 the task is even simpler. It is clear that

$$F_2(t) = B^* y_0^* + tB^* h^* \quad (0 \leqq t)$$

so H_2 is a convex function and therefore it is differentiable (on the right) at the origin. From inequality (1) we conclude that:

$$H_1'(0) + H_2'(0) \leqq 0 \tag{4}$$

Note that the above arguments depend on a fixed vector $h \in S_{l_2}$ only satisfying $(h \mid y_0) = 0$, so let us see what happens if we replace h by $-h$.

In view of (2), $F_1'(0)$ changes its sign, and (3) indicates that the same occurs with $H_1'(0)$. Therefore, if we add inequality (4) to the inequality which would be obtained by changing h for $-h$, we obtain:

$$\lim_{t \to 0} \frac{\|B^* y_0^* + tB^* h^*\| + \|B^* y_0^* - tB^* h^*\| - 2\|B^* y_0^*\|}{t} = 0 . \tag{5}$$

Actually we only obtain an inequality, but the other one is completely obvious. We must remember that this conclusion has been reached under the assumption $(*)$. If $(Ay_0 \mid y_0) = 0$, things are even simpler, for inequality (1) implies

$$H_2(t) \leqq \phi(t)\, H_2(0) ,$$

and instead of (4) we get $H_2'(0) \leqq 0$, which leads directly to (5).

Let us summarize again what has been demonstrated so far, bearing in mind the part of assertion b) which was not used in previous calculations:

c) *There exists $y_0 \in S_{l_2}$ such that (5) holds for all $h \in S_{l_2}$ with $(h \mid y_0) = 0$. Moreover, the functional $B^* y_0^*$ attains its norm at a point $z_0 \in S_G$*

It is worth explaining in advance how the previous assertion will allow us to deduce such a strong restriction on the operator B as to be of finite rank. Equality (5) gives us a certain „smoothness" of the norm of G^*, to be precise, this norm allows a directional derivative at $B^* y_0^*$ in a rather „general" direction $B^* h^*$. However we know that B_G lacks extreme points (Lemma 1.3) and therefore a functional which attains its norm in B_G can never be a smooth point. The only solution one can hope for consists of the image of B^* being very small so that $B^* h^*$ is a very „particular" direction, and that is what we are going to obtain.

To simplify the notation, we write

$$z_0^* = B^* y_0^*,\ z^* = B^* h^*$$

where $h \in S_{l_2}$ is fixed and always conditioned by $(h \mid y_0) = 0$. Given $\varepsilon > 0$ we use (5) to find $r > 0$ such that:

$$0 < t < r \Rightarrow \|z_0^* + tz^*\| + \|z_0^* - tz^*\| < 2\|z_0^*\| + t\varepsilon . \tag{6}$$

On the other hand we should not forget that z_0^* attains its norm at a point $z_0 \in S_G$ and there is no objection in assuming that $z_0^*(z_0) = \|z_0^*\|$. Lemma 1.3 provides us with a natural number m and a $\delta > 0$, such that:

$$k \in \mathbb{N},\ k > m,\ \mu \in \mathbb{K},\ |\mu| \leqq \delta \Rightarrow \|z_0 + \mu e_k\| = 1 . \tag{7}$$

For fixed $k > m$, we choose a scalar μ such that

$$|\mu| = \delta ,\quad \delta|z^*(e_k)| = z^*(\mu e_k) ,$$

and by using (6) and (7) we obtain, for $0 < t < r$,

$$2\|z_0^*\| + t\varepsilon > \|z_0^* + tz^*\| + \|z_0^* - tz^*\| \geqq$$

$$\geqq \mathrm{Re}\,([z_0^* + tz^*](z_0 + \mu e_k) + [z_0^* - tz^*](z_0 - \mu e_k)) =$$
$$= 2\|z_0^*\| + 2t\,\mathrm{Re}\,z^*(\mu e_k) = 2\|z_0^*\| + 2t\delta|z^*(e_k)|\,.$$

Subtracting $2\|z_0^*\|$, dividing by t and bearing in mind the arbitrariness of ε, we obtain $z^*(e_k) = 0$. But remember that $z^* = B^*h^*$. thus we have proved

$$(Be_k \mid h) = 0 \quad (\forall k > m)$$

and this must be true whenever $(h \mid y_0) = 0$, for we can already dispense with the normalization $\|h\| = 1$. It is crucial to observe that the natural number m is independent of h.

Therefore, we have $Be_k \in \{y_0\}^{\perp\perp} = \mathbb{K}y_0$ for $k > m$ and the range of B is contained in the subspace of l_2 generated by $\{y_0, Be_1, Be_2, \ldots, Be_m\}$. Thus B is a finite rank operator, as required. □

2.3. Remark. If we take $A = 0$ in the above theorem, then it is clear that $v(T) = \|B\|$ and T attains its numerical radius if and only if B attains its norm. Thus Theorem 1.4 is a special case of Theorem 2.2. How to dispose of the operator A was the main difficulty in the above proof.

We can already state the result which is the main objective of this section:

2.4. Corollary. *$R(l_2 \oplus_\infty G)$ is not dense in $L(l_2 \oplus_\infty G)$.*

Proof. Part iii) of Lemma 1.2 gives us a noncompact operator $I \in L(G, l_2)$. Let $S \in L(l_2 \oplus_\infty G)$ be defined by

$$S(y + z) = I(z) \quad (y \in l_2, z \in G)\,,$$

We are going to see that S cannot be the limit in the norm topology of operators which attain their numerical radius.

Assume, on the contrary, that there is a sequence $\{T_n\}$ satisfying

$$T_n \in R(l_2 \oplus_\infty G)\ \forall n \in \mathbb{N}\,, \quad \{\|T_n - S\|\} \to 0\,.$$

We have clearly $PS = S$, $QS = 0$ where P, Q are the projections from $l_2 \oplus_\infty G$ onto l_2 and G respectively, so $\{PT_n\} \to S$, $\{QT_n\} \to 0$ and the inequality $\|QT_n\| < \|PT_n\|$ will hold for large enough n. It follows from Lemma 2.1 that $PT_n \in R(l_2 \oplus_\infty G)$ also for large enough n. By deleting some terms of the sequence $\{T_n\}$ and replacing T_n by PT_n we may assume that

$$T_n \in R(l_2 \oplus_\infty G)\,, \quad PT_n = T_n\,. \quad \forall n \in \mathbb{N}\,, \quad \{\|T_n - S\|\} \to 0$$

For each $n \in \mathbb{N}$ we denote by A_n and B_n the restrictions of T_n to l_2 and G respectively. Hence we have

$$T_n(y + z) = A_n y + B_n z \quad (y \in l_2, z \in G)$$

where $A_n \in L(l_2)$ and $B_n \in L(G, l_2)$ for every natural n. Thus the assumptions of the above theorem are fulfilled and we conclude that B_n is a finite rank operator for all n.

This is a contradiction, for the sequence $\{B_n\}$ converges in norm to the noncompact operator I. □

2.5. Concluding remark. As we have already mentioned, the negative answer to the question posed by B. Sims in 1972 was already obtained in [16], with a similar example to the one given here and even using similar arguments in the proof. In our opinion, Corollary 2.4 improves on the conclusion of [16] in three ways, which we will try to explain. First, the counterexample given here is somewhow more „natural". There is nothing to be said against the naturality of the space l_2, whose role could be played by any Hilbert space of infinite dimension. The definition of G seems to be more contrived, but one should not forget that G can be considered in the orbit of „classical" Banach spaces. As said in [12], G^* is a Lorentz sequence space $d(\omega, 1)$ where the sequence of weights ω is $\{1/n\}$. Although $d(\omega, 1)$ has not a unique predual, G is an M-ideal in its bidual and is the only predual of $d(\omega, 1)$ with this property, a fact proved in [17] which shows that G is in a sense a „canonical" predual of $d(\omega, 1)$. Furthermore, the result proved here improves those in [16] because Theorem 2.2 gives a quite restrictive necessary condition for an operator to attain its numerical radius, while the results in [16] are far from being so explicit. Finally, although the arguments in the proof of Theorem 2.2 are essentially those used in [16], the simplicity of the geometry of l_2 makes the proof much easier and intuitive.

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