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Acta Universitatis Carolinae. Mathematica et Physica, Vol. 32 (1991), No. 2, 61--64

Persistent URL: http://dml.cz/dmlcz/701969

1991 ACTA UNIVERSITATIS CAROLINAE—MATHEMATICA ET PHYSICA VOL. 32. NO. 2

Strong Monotonicity and Lipschitz-Continuity of the Duality Mapping

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Received 11 March 1991

Introduction

Various kinds of differentiability of the norm and various kinds of smoothness and convexity properties of normed spaces can be described by means of the duality mapping (eg. [1], [3], [5], [7]). In the present paper, characterizations of another two geometric properties of normed spaces in terms of the duality mapping are given.

Definitions and notation

Let X be a real normed linear space, X^* its dual space, S the unit sphere in X, S^* the unit sphere in X^*. The value of $f \in X^*$ at $x \in X$ is denoted by $f(x)$ or (f, x). By J the duality mapping of X into 2^{X^*} is denoted. J is defined by $J(x) = \{f \in X^* : \|f\| = \|x\|, f(x) = \|x\|^2\}$. For $x \in X$, by f_x any element of $J(x)$ is denoted. We say that J is Lipschitz-continuous if J is singlevalued and the mapping $x \to f_x$ is Lipschitz-continuous. We say that J is strongly monotone if there exists $b > 0$ such that $(f_x - f_y, x - y) \geqq b\|x - y\|^2$ for each $x, y \in X$, $f_x \in J(x)$, $f_y \in J(y)$. By J^* we denote the duality mapping of X^*.

According to [2], X is said to satisfy Lindenstrauss convexity condition (X is (LC)), if

$$\exists d > 0 \forall x, y \in S : 2 - \|x + y\| \geqq d\|x - y\|^2,$$

and X is said to satisfy Lindenstrauss smoothness condition (X is (LS)), if

$$\exists k > 0 \forall x \in S \forall y \in X : \|x + y\| + \|x - y\| \leqq 2 + k\|y\|^2.$$

We say that X satisfies the differentiability condition (δ) (X is (δ)), if

$$\exists c > 0 \forall x \in S \forall y \in X \forall f_x \in J(x) : \|x + y\| - \|x\| - f_x(y) \leqq c\|y\|^2.$$

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In [2], [6] the (LC) and (LS) conditions were defined in terms of the modulus of convexity δ and the modulus of smoothness ϱ as follows: X is (LC) if $\delta(t) \geqq \alpha t^2$ for some $\alpha > 0$, and X is (LS) if $\varrho(t) \leqq \beta t^2$ for some $\beta > 0$.

Theorem. *For a normed linear space X, the following conditions are equivalent.*

(a) *J is Lipschitz-continuous,*

(b) *X is (δ),*

(c) *X is (LS),*

(d) *X^* is (LC),*

(e) *J^* is strongly monotone.*

For a normed linear space X, the following conditions are equivalent.

(A) *J^* is Lipschitz-continuous,*

(B) *X^* is (δ),*

(C) *X^* is (LS),*

(D) *X is (LC),*

(E) *J is strongly monotone.*

Proof. **(a) ⇒ (b).** $f_x(y) = f_x(x + y) - \|x\| \leqq \|x + y\| - \|x\| = (\|x + y\|^2 - - \|x\| \|x + y\|)/\|x + y\| \leqq (f_{x+y}(x + y) - f_{x+y}(x))/\|x + y\| = ((f_{x+y}/\|z + y\|), y)$. So $\|x + y\| - \|x\| - f_x(y) \leqq ((f_{k+y}/\|x + y\|) - f_x, y) \leqq \|f_{x+y}((1/\|x + y\|) - 1) + + (f_{x+y} - f_x)\| \|y\| \leqq (1 + L) \|y\|^2$, where L is the Lipschitz constant in (a).

(b) ⇒ (c). Follows immediately by adding the inequality in the definition of (δ) to itself with y replaced by $-y$.

(c) ⇒ (d). Given $f, g \in S^*$ and $\lambda \in (0, 1)$, there exists $z \in X$ such that $\|z\| = \lambda/2k \|f - g\|$ and $(f - g, z) \geqq \lambda\|f - g\| \|z\|$. Now $\|f + g\| = \sup\{(f + g, x) : x \in S\} = = \sup\{(f, x + z) + (g, x - z) - (f - g, z) : x \in S\} \leqq \sup\{\|x + z\| + \|x - z\| : : x \in S\} - \lambda\|f - g\| \|z\| \leqq 2 + k\|z\|^2 - \lambda\|f - g\| \|z\| = 2 - \lambda^2/4k\|f - g\|^2$. Thus $2 - \|f + g\| \geqq 1/4k\|f - g\|^2$.

(A) ⇒ (B). Follows from (a) ⇒ (b).

(B) ⇒ (C). Follows from (b) ⇒ (c).

(C) ⇒ (D). Follows from (c) ⇒ (d).

(D) ⇒ (E). For $x, y \in S$, we have $(f_x - f_y, x - y) = 2 - f_x(x + y) + 2 - - f_y(x + y) \geqq 2(2 - \|x + y\|) \geqq 2d\|x - y\|^2$. This proves the strong monotonicity of J on S and yields $f_x(y) + f_y(x) \leqq 2(1 - d\|x - y\|^2)$. Now for $\zeta \in [0, 1]$ we have $(\zeta f_x - f_y, \zeta x - y) = \zeta^2 - \zeta(f_x(y) + f_y(x)) + 1 \geqq (\zeta - 1)^2 + 2d\zeta\|x - y\|^2 \geqq 2d \,. \, [(\zeta - 1)^2 + \zeta^2\|x - y\|^2]$, since $\zeta \leqq 1$ and $d \leqq \frac{1}{2}$ (otherwise the inequality in the definition of (LC) does not hold for $y = -x$). Since $\|\zeta x - y\| \leqq \zeta\|x - y\| + 1 - \zeta$

and $\frac{1}{2}(a + b)^2 \leqq a^2 + b^2$ for any $a, b \in R$, we obtain $\frac{1}{2}\|\zeta x - y\|^2 \leqq \zeta^2\|x - y\|^2 + (1 - \zeta)^2$. Therefore $(\zeta f_x - f_y, \zeta x - y) \geqq d\|\zeta x - y\|^2$. Thus $(f_x - f_y, x - y) \geqq d\|x - y\|^2$ holds for any x, y such that $\|x\| \leqq 1$ and $\|y\| = 1$, and therefore for every $x, y \in X$.

(d) ⇒ (e). Follows from (D) ⇒ (E).

(e) ⇒ (a). Since $f_x \in J(x) \Leftrightarrow \hat{x} \in J^*(f_x)$ (where by $\hat{x}$ the canonical image of x in X^{**} is denoted), we have $\|f_x - f_y\| \|x - y\| \geqq (f_x - f_y, x - y) = (\hat{x} - \hat{y}, f_x - f_y) \geqq b\|f_x - f_y\|^2$.

(E) ⇒ (D). We shall prove two lemmas first.

Lemma 1. *Let $(x_n), (y_n) \subset S$, $(\lambda_n) \subset (0, \infty)$ be sequences such that $\lambda_n \to 0$ and $2 - \|x_n + y_n\| \leqq \lambda_n\|x_n - y_n\|^2$. Then (for sufficiently large n)*

$$\|z_n - x_n\| \leqq \tfrac{1}{4}\|y_n - x_n\|$$

$$16\lambda_n\|z_n - x_n\|^2 \geqq 1 - f_{z_n}(x_n),$$

where $z_n = (x_n + y_n)/\|x_n + y_n\|$ and $f_{z_n} \in J(z_n)$.

Proof. Since $\|x_n + y_n\| \to 2$, z_n is defined if n is great enough. If $\lambda_n \leqq \frac{1}{4}$ then $\|z_n - x_n\| = \|(y_n - x_n)/\|x_n + y_n\| + (2/\|x_n + y_n\| - 1)\, x_n\| \geqq \|x_n - y_n\|/\|x_n + y_n\| - (2 - \|x_n + y_n\|)/\|x_n + y_n\| \geqq \|x_n - y_n\|/\|x_n + y_n\| (1 - \lambda_n\|x_n - y_n\|) \geqq \frac{1}{4}\|x_n - y_n\|$ and $1 - f_{z_n}(x_n) \leqq 1 - f_{z_n}(x_n) + 1 - f_{z_n}(y_n) = 2 - \|x_n + y_n\| \leqq \lambda_n\|x_n - y_n\|^2 \leqq 16\lambda_n\|x_n - z_n\|^2$.

Lemma 2. *Let $x, y \in S$, $x \neq -y$, $z = (x + y)/\|x + y\|$. Then $\|x + z\| \geqq \|x + y\|$.*

Proof. Let $v = (x + z)/\|x + y\|$, then $v = (x + y + (x + y)/\|x + y\|)/\|x + y\| - y/\|x + y\| = (1 + \|x + y\|)/\|x + y\|\, z - 1/\|x + y\|\, y$, so $\|v\| \geqq (1 + \|x + y\|)/\|x + y\| - 1/\|x + y\| = 1$.

Proof of (E) ⇒ (D). Suppose (D) does not hold. Then there exist sequcenes (x_n), (y_n), (λ_n) satisfying conditions of Lemma 1. Therefore

$$16\|x_n - z_n\|^2 \geqq \|x_n - y_n\|^2,$$

$$16\lambda_n\|x_n - z_n\|^2 \geqq 1 - f_{z_n}(x_n),$$

where z_n and f_{z_n} are as in Lemma 1. Using Lemma 2 and writing $u_n = (x_n + z_n)/\|x_n + z_n\|$, $f_{u_n} \in J(u_n)$, we obtain $2 - \|x_n + z_n\| \leqq 2 - \|x_n + y_n\| \leqq \lambda_n\|x_n - y_n\|^2 \leqq 16\lambda_n\|x_n - z_n\|^2$. So the sequence (x_n), (z_n), $(16\lambda_n)$ satisfy conditions of Lemma 1. Thus

$$16\|u_n - z_n\|^2 \geqq \|x_n - z_n\|^2,$$

$$16^2\lambda_n\|u_n - z_n\|^2 \geqq 1 - f_{u_n}(z_n).$$

Now, since $f_{z_n}(u_n) = f_{z_n}((x_n + z_n)/\|x_n + z_n\|) \geqq f_{z_n}((x_n + z_n)/2) \geqq f_{z_n}(x_n)$, we have $1 - f_{z_n}(u_n) \leqq 1 - f_{z_n}(x_n) \leqq 16\lambda_n\|x_n - z_n\|^2 \leqq 16^2\lambda_n\|u_n - z_n\|^2$. It follows that $(f_{z_n} - f_{u_n}, z_n - u_n) = 1 - f_{u_n}(z_n) + 1 - f_{z_n}(u_n) \leqq 512\lambda_n\|z_n - u_n\|^2$. J is not strongly monotone.

(D) ⇒ (A). If X is (LC), then it is uniformly rotund, and so X^* is uniformly smooth. Therefore X^* is reflexive, which is equivalent to $\hat{X}$ being dense in X^{**}. Since the inequality $2 - \|F + G\| \geqq d\|F - G\|^2$ holds for all $F, G \in \hat{S}$, it holds for all $F, G \in S^{**}$. So X^{**} is (LC), too. Now by (D) ⇒ (E), it follows that J^{**} is strongly monotone, and by (e) ⇒ (a), J^* is Lipschitz-continuous.

Remark. In fact, the equivalence (c) ⇔ (d) has been proved both in [6] and in [2], where it was formulated and dealt with in terms of moduli of smoothness and convexity (and not using the duality mapping). Moreover, in [2] it was shown that the spaces satysfying (c) [(d)] are just those satisfying the upper [lower] weak parallelogram law. The equivalences of (a) ⇔ (b) ⇔ (c) (with the mapping $x \to f_x$ defined in a slightly different way) were also given in [4].

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